Triality over arbitrary fields and over $\mathbb{F}_1$

M-A. Knus, ETH Zürich

Outline

- Some history
- Triality over arbitrary fields (Chernousov, Tignol, K., 2011)
- Triality over $\mathbb{F}_1$ (Tignol, K., 2012)

I. Some history

Wikipedia:

"There is a geometrical version of triality, analogous to duality in projective geometry.

... one finds a curious phenomenon involving 1, 2, and 4 dimensional subspaces of 8-dimensional space ..."

Geometric triality

- (V, q) : Quadratic space of dimension 8 of maximal index. U_i : Set of isotropic subspaces of V of dimension i, $i \leq 4$.
- "Projective" terminology : $Q = \{q = 0\}$ defines a 6-dimensional quadric in $\mathbb{P}^7$, the elements of U_i, $i = 1, 2, 3, 4$, are **points, lines, planes and solids of Q**.
- Two solids are of the **same kind** if their intersection is of even dimension. Two solids are of the same kind if and only if one can be transformed in the other by a rotation.

  $\Rightarrow$ 2 kinds of solids !

Eduard Study

Grundlagen und Ziele der analytischen Kinematik, 1913

I The variety of solids of a fixed kind in Q^6 is isomorphic to a quadric Q^6.

II Any proposition in the geometry of Q^6 [about incidence relations] remains true if the concepts points, solids of one kind and solids of the other kind are cyclically permuted.

In other words, geometric triality is a geometric correspondence of order 3

Points	→	Solids 1	→	Solids 2	→	Points

which is compatible with incidence relations.

In analogy to **geometric duality** which is a geometric correspondence | Points | → | Hyperplanes |
in projective space.

The word **triality** goes back to Élie Cartan : "On peut dire que le *principe de dualité* de la géométrie projective est remplacé ici par un *principe de trialité*".

Élie Cartan

Le principe de dualité et la théorie des groupes simples et semi-simples, 1925

- The group PGO_8^+ admits a group of outer automorphisms isomorphic to S_3.
- Outer automorphisms are related to "**Cayley octaves**".

Outer automorphisms of order 3 will be called

trialitarian automorphisms.

Cayley octaves or Octonions

- Octonions are a 8-dimensional algebra $\mathbb{O}$ with unit, norm $\mathfrak{n}$ and conjugation $x \mapsto \overline{x}$ such that
- $\mathfrak{n}(x) = x \cdot \overline{x} = \overline{x} \cdot x, \quad \mathfrak{n}(x \cdot y) = \mathfrak{n}(x)\mathfrak{n}(y)$.
- Cartan :

  Given $A \in \mathrm{SO}(\mathfrak{n})$ there exist $B, C \in \mathrm{SO}(\mathfrak{n})$ such that

$$C(x \cdot y) = Ax \cdot By.$$

  $\sigma\colon A \mapsto B, \quad \tau\colon A \mapsto C$ induce $\hat{\sigma}, \hat{\tau} \in \mathrm{Aut}\left(\mathrm{PGO}^+(\mathfrak{n})\right)$ such that

$$\hat{\sigma}^3 = 1,\ \hat{\tau}^2 = 1,\ \langle \hat{\sigma}, \hat{\tau} \rangle = S_3 \text{ in } \mathrm{Aut}\left(\mathrm{PGO}^+(\mathfrak{n})\right).$$

The orthogonal projective group

- $\mathrm{PGO}(\mathfrak{n}) = \mathrm{GO}(\mathfrak{n})/F^\times$,
  $\mathrm{GO}(\mathfrak{n}) = \{f \in \mathrm{GL}(\mathbb{O}) \mid \mathfrak{n}\big(f(x)\big) = \mu(f)\mathfrak{n}(x)\}$, $\mu(f) \in F^\times$.
- $\mathrm{PGO}^+(\mathfrak{n}) = \mathrm{GO}^+/F^\times$, where $\mathrm{GO}^+(\mathfrak{n})$ is the subgroup of $\mathrm{GO}(\mathfrak{n})$ of direct similitudes (or projectively, of similitudes which respect the two kinds of solids).

Notation : $f \in \mathrm{GO}(\mathfrak{n}) \mapsto [f] \in \mathrm{PGO}(\mathfrak{n})$

Octaves and geometric triality

Félix Vaney, Professeur au Collège cantonal, Lausanne,
PhD-Student of É. Cartan, 1929 :

I Solids are of the form

1. $K_a = \{x \in \mathbb{O} \mid a \cdot x = 0\}$ and 2. $R_a = \{x \in \mathbb{O} \mid x \cdot a = 0\}$.

II Geometric triality can be described as

$$a \mapsto K_a \mapsto R_a \mapsto a.$$

for all $a \in \mathbb{O}$ with $\mathfrak{n}(a) = 0$.

A selection of later works

E. A. Weiss (1938,1939) : More (classical) projective geometry

É. Cartan (1938) : Leçons sur la théorie des spineurs

N. Kuiper (1950) : Complex algebraic geometry

H. Freudenthal (1951) : Local and global triality

C. Chevalley (1954) : The algebraic theory of spinors

J. Tits (1958) : Triality for loops

J. Tits (1959) : Classification of geometric trialities over arbitrary fields

F. van der Blij, T. A. Springer (1960) : Octaves and triality

T. A. Springer (1963) : Octonions, Jordan algebras and exceptional groups

N. Jacobson (1964) : Triality for Lie algebras over arbitrary fields.

Books (Porteous, Lounesto, [KMRT], Springer-Veldkamp).

II. Triality over arbitrary fields

with V. Chernousov and J-P. Tignol

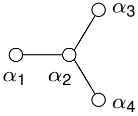
Simple groups with trialitarian automorphisms

G simple algebraic group with a trialitarian automorphism

$$\Rightarrow$$

G of type D_4

Reason D_4 is the only Dynkin diagram with an automorphism of order 3



Theorem G of classical type ${}^{1,2}D_4$ with a trialitarian automorphism

$\Rightarrow G = \mathrm{PGO}^+(\mathfrak{n})$ or $G = \mathrm{Spin}(\mathfrak{n})$, $\mathfrak{n}$ a 3-Pfister form.

Aim

- Classify all trialitarian automorphisms of $\mathrm{PGO}^+(\mathfrak{n})$, up to conjugacy.
- Classify all geometric trialities up to isomorphism.

Method Reduce to the (known) classification of a certain class of composition algebras.

Remark Similar results for $\mathrm{Spin}(\mathfrak{n})$.

M. Rost ($\sim$1994)

There is a class of composition algebras well suited for triality, which Rost called **symmetric compositions.**

Symmetric compositions

A **composition algebra** is a quadratic space $(S, \mathfrak{n})$ with a bilinear multiplication $\star$ such that the norm of multiplicative :

$$\boxed{\mathfrak{n}(x \star y) = \mathfrak{n}(x) \star \mathfrak{n}(y)}$$

They exist only in dimension 1, 2, 4 and 8 (Hurwitz).

A **symmetric composition** satisfies

$$\boxed{x \star (y \star x) = (x \star y) \star x = \mathfrak{n}(x)y} \quad \text{and} \quad \boxed{b(x \star y, z) = b(x, y \star z)}.$$

Remark For octonions the relations are

$$\boxed{\overline{x}(xy) = (yx)\overline{x} = \mathfrak{n}(x)y} \quad \text{and} \quad \boxed{b(xy, z) = b(x, z\overline{y})}.$$

Some history

Symmetric compositions existed already !

- **Petersson (1969) :** Einfach involutorische Algebren
  The product $x \star y = \overline{x}\,\overline{y}$ on an octonion algebra defines a symmetric composition (“**para-octonions**”).
- **Okubo (1978) : Pseudo-octonions algebras**
  $S = M_3(F)^0,\ x \star y = \frac{yx - \omega xy}{1-\omega} - \frac{1}{3}\,\mathrm{tr}(xy),\ \mathrm{Char}\,F \neq 3,\ \omega^3 = 1.$
- **Faulkner (1988) :** Trace zero elements in cubic separable alternative algebras.

Classification (Elduque-Myung, 1993) Over fields of characteristic different from 3 8-dimensional symmetric compositions are either para-octonions or Okubo algebras attached to central simple algebras of degree 3.

Zorn matrices

The para-Zorn algebra $\mathfrak{Z} = \left\{ \begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \middle| \, \alpha, \beta \in F, \; a, b \in F^3 \right\}$

$$\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} * \begin{pmatrix} \gamma & c \\ d & \delta \end{pmatrix} = \begin{pmatrix} \beta\delta + a \bullet d & -\beta c - \gamma a - b \times d \\ -\delta b - \alpha d + a \times c & \alpha\gamma + b \bullet c \end{pmatrix},$$

The Petersson twist $x \star_\theta y = \theta(x) \star \theta^{-1}(y)$

$$\theta\left(\begin{pmatrix} \alpha & a \\ b & \beta \end{pmatrix} \right) = \begin{pmatrix} \alpha & a^\varphi \\ b^\varphi & \beta \end{pmatrix}, \; \varphi \colon (a_1, a_2, a_3) \mapsto (a_2, a_3, a_1)$$

Theorem (Petersson, Elduque-Perez) Symmetric compositions are forms of the para-Zorn algebra and its Petersson twist.

A variation (Chernousov, Tignol, K., 2011)

$(S, \mathfrak{n})$: 3-fold Pfister form ($\Leftrightarrow$ norm of an octonion algebra)

Symmetric composition : $\star : S \times S \to S$ such that

- $\mathfrak{n}(x \star y) = \lambda_\star \mathfrak{n}(x)\mathfrak{n}(y),\ \lambda_\star \in F^\times$ ($\lambda_\star$ is the **multiplier** of $\star$)
- $b(x \star y, z) = b(x, y \star z)$

Explanation This definition is more suited to deal with similitudes, $\lambda_\star = 1$, "normalized symmetric composition"

Symmetric compositions and triality

Theorem

$(S, \star, \mathfrak{n})$ a symmetric composition of dimension 8,

I Given $f \in \mathrm{GO}^+(\mathfrak{n})$, there exists $g, h \in \mathrm{GO}^+(\mathfrak{n})$, such that

$$f(x \star y) = g(x) \star h(y).$$

II the map $\rho_\star : [f] \mapsto [g]$ is an outer automorphism of order 3 of $\mathrm{PGO}^+(\mathfrak{n})$ and $\rho_\star^2[f] = [h]$.

Proof : With Clifford algebras, see [KMRT].

Remark : “Like” Cartan, but more symmetric !

More trialitarian automorphisms

There is a split exact sequence

$$1 \to \mathrm{PGO}^+(\mathfrak{n}) \to \mathrm{Aut}\big(\mathrm{PGO}^+(\mathfrak{n})\big) \to S_3 \to 1$$

Consequence

$\rho_\star$ a fixed trialitarian automorphism of $\mathrm{PGO}^+(\mathfrak{n})$

ρ any trialitarian automorphism of $\mathrm{PGO}^+(\mathfrak{n})$.

Then there exists $f \in GO^+(\mathfrak{n})$ such that

$$\boxed{\rho \text{ or } \rho^{-1} = \mathrm{Int}([f]^{-1}) \circ \rho_\star} \quad \text{and} \quad \boxed{f^{-1}\,\rho_\star(f^{-1})\,\rho_\star^2(f^{-1}) = 1}.$$

Theorem (CKT, 2011) : The rule $\star \mapsto \rho_\star$ defines a bijection

$$\boxed{\text{Sym. comp. on } (S, \mathfrak{n}) \text{ up to scalars}} \Leftrightarrow \boxed{\text{Trialit. aut. of } \mathrm{PGO}^+(\mathfrak{n})}$$

Proof of surjectivity

Given : ρ a trialitarian automorphism.

1) Choose a fixed symmetric composition $\star$.

2) Take $f \in GO^+(\mathfrak{n})$ such that ρ or $\rho^{-1} = \mathrm{Int}([f]^{-1}) \circ \rho_\star$ and $f^{-1}\rho_\star(f^{-1})\rho_\star^2(f^{-1}) = 1$ as above.

3) Pick $g \in \mathrm{PGO}^+(\mathfrak{n})$ such that $[g] = \rho_\star^2[f^{-1}]$.

Then $x \diamond y = f(x) \star g(y)$ is such that ρ or $\rho^{-1} = \rho_\diamond$.

Trialititarian automorphisms up to conjugacy

Theorem (Chernousov, Tignol, K., 2011):

Isomorphism classes of symmetric compositions with norm $\mathfrak{n}$

$$\Leftrightarrow$$

Conjugacy classes of trialitarian automorphisms of $\mathrm{PGO}^+(\mathfrak{n})$

Consequences

1. The classification of 8-dimensional symmetric compositions (Elduque-Myung, 1993) yields the classification of conjugacy classes of trialitarian automorphisms of groups $\mathrm{PGO}^+(\mathfrak{n})$.
2. Conversely one can first classify conjugacy classes of trialitarian automorphisms of groups $\mathrm{PGO}^+(\mathfrak{n})$ (Chernousov, Tignol, K., 201?) and deduce from it the classification of 8-dimensional symmetric compositions.

Symmetric compositions and geometric triality

Theorem

Given : $(S, \star, \mathfrak{n})$ a 8-dimensional symmetric composition with hyperbolic norm.

Claim :

I All solids of one kind are of the form $x \star S$ and those of the other kind of the form $S \star y$, $x, y \in S$.

II The rule

$$\tau_\star \ : \ x \mapsto x \star S \mapsto S \star x \mapsto x$$

is a geometric triality.

III the rule $\star \mapsto \tau_\star$ defines a bijection

$$\boxed{\text{Sym. comp. on } (S, \mathfrak{n}) \text{ up to scal.}} \ \Leftrightarrow \ \boxed{\text{Geom. trialit. on } \{\mathfrak{n} = 0\}}$$

Automorphisms of symmetric compositions

Theorem : $\left[\mathrm{PGO}^+(\mathfrak{n})\right]^{\rho_\star} = \mathrm{Aut}(S, \star)$

- $(S, \star)$ para-octonions $\Rightarrow \left[\mathrm{PGO}^+(\mathfrak{n})\right]^{\rho_\star}$ of type G_2.
- $(S, \star)$ Okubo, Char $F \neq 3 \Rightarrow \left[\mathrm{PGO}^+(\mathfrak{n})\right]^{\rho_\star}$ of type A_2.
- $(S, \star)$ Okubo, Char $F = 3$, is still mysterious !

Groups with triality of outer type ${}^{3,6}D_4$

"Outer types" are related with

- Semilinear trialities (in projective geometry)
- Generalized hexagons (incidence geometry, Tits, Schellekens, ...)
- Twisted compositions (F_4, Springer)
- Trialitarian algebras (KMRT)

III. Triality over $\mathbb{F}_1$

(with J-P. Tignol, 2012)

Tits, le corps de caractéristique 1

Sur les analogues algébriques des groupes semi-simples complexes,1957

"Nous désignerons par $K = K_1$ le « corps de caractéristique 1» formé du seul élément $1 = 0$ ([19]). Il est naturel d'appeler *espace projectif à n dimensions sur* K, un ensemble $\mathcal{P}_n$ of $n+1$ points dont tous les sous-ensembles sont considérés comme des variétés linéaires {...}.

([19]) K_1 n'est généralement pas considéré comme un corps."

Vector spaces over $\mathbb{F}_1$

Since there is only one scalar, one has to work only with bases !

- n-dimensional vector space : $\mathcal{V} = \{x_1, \ldots, x_n, 0\}$
- $n-1$-dimensional projective space :
  $\mathbb{P}(\mathcal{V}) = \langle\mathcal{V}\rangle = \{x_1, \ldots, x_n\}$

$\Rightarrow \mathrm{Aut}(\mathcal{V}) = \mathrm{Aut}(\langle\mathcal{V}\rangle) = GL_n(\mathbb{F}_1) = \mathrm{PGL}_n(\mathbb{F}_1) = S_n$.

Tits' motivation There are algebraic (or geometric) objects whose automorphism groups are the simple algebraic groups. Tits wanted algebraic (or geometric) objects whose automorphism groups are the **Weyl groups** of these simple algebraic groups.

Quadratic spaces over $\mathbb{F}_1$

- A $2n$-dimensional quadratic space is a pair $\mathcal{Q} = (\mathcal{V}, \widetilde{\ })$ where $\mathcal{V}$ is a $2n$-dimensional vector space over $\mathbb{F}_1$ and $\widetilde{\ } : \mathcal{V} \to \mathcal{V}$ is a bijective self-map of order 2 such that $\widetilde{0} = 0$ and without other fixed points :
  $\mathcal{V} = \{x_1, \ldots, x_n, y_1, \ldots y_n, 0\}$, $\widetilde{x_i} = y_i$, $\widetilde{y_i} = x_i, \widetilde{0} = 0$.
- $\langle \mathcal{Q} \rangle = \mathcal{Q} \setminus \{0\}$ is the **quadric** associated to $\mathcal{Q}$.
- $\langle \mathcal{Q} \rangle$ is a double covering !

Example : (V, q) "classical" hyperbolic space with hyperbolic basis

$$\{e_i,\ f_i, i \leq i \leq n \mid q(e_i) = q(f_i) = 0,\ b(e_i, f_j) = \delta_{ij}\}.$$

Set $\widetilde{e_i} = f_i,\ \widetilde{f_i} = e_i$.

Let $\mathcal{Q} = (\mathcal{V}, \tilde{\ })$ be a $2n$-dimensional quadratic space over $\mathbb{F}_1$ and let $\mathcal{U}$ be a linear subspace of $\mathcal{V}$.

- $\mathcal{U}^{\perp} = \{x \in \mathcal{V} \mid \widetilde{x} \notin \mathcal{U}\} \sqcup \{0\}$;
- $\mathcal{U}$ is **isotropic** if $\mathcal{U} \subset \mathcal{U}^{\perp}$ and **maximal isotropic** if $\mathcal{U} = \mathcal{U}^{\perp}$;
- $\mathcal{U}$ isotropic $\Rightarrow \dim \mathcal{U} \leq n$;
- Two kinds of maximal isotropic spaces : two maximal isotropic spaces $\mathcal{U}$ and $\mathcal{U}'$ are of the **same kind** if $\dim(\mathcal{U} \cap \mathcal{U}')$ has the same parity as $\frac{\dim \mathcal{V}}{2}$;
- $\mathcal{U}$ maximal isotropic $\Leftrightarrow$ $\langle \mathcal{U} \rangle$ is a section of the double covering $\langle \mathcal{Q} \rangle$;

Orthogonal groups over $\mathbb{F}_1$

$$\mathrm{O}(\mathcal{Q}) = \mathrm{PGO}(\langle \mathcal{Q} \rangle) = \mathrm{PGO}_{2n}(\mathbb{F}_1) = S_2^n \rtimes S_n,$$

$$\mathrm{O}^+(\mathcal{Q}) = \mathrm{PGO}^+(\langle \mathcal{Q} \rangle) = \mathrm{PGO}_{2n}^+(\mathbb{F}_1) = S_2^{n-1} \rtimes S_n$$

Trialitarian automorphisms of $\mathrm{PGO}_8^+(\mathbb{F}_1)$

Known facts:

I The Weyl group $S_2^3 \rtimes S_4$ of type D_4 (which is $\mathrm{PGO}_8^+(\mathbb{F}_1)$) admits outer automorphisms of order 3.

II If α, β are trialitarian automorphisms of $\mathrm{PGO}_8^+(\mathbb{F}_1)$, then $\alpha \circ \beta^{-1}$ or $\alpha \circ \beta^{-2}$ is an inner automorphism.

Aim : Describe trialitarian automorphisms and geometric triality over $\mathbb{F}_1$ with symmetric compositions over $\mathbb{F}_1$!

Algebras over $\mathbb{F}_1$

A finite-dimensional **algebra** $(\mathcal{S}, \star)$ over $\mathbb{F}_1$ is a finite-dimensional $\mathbb{F}_1$-vector space $\mathcal{S}$ together with a map

$$\star\colon \mathcal{S} \times \mathcal{S} \to \mathcal{S}, \quad (x, y) \mapsto x \star y,$$

called the **multiplication**, such that $0 \star x = x \star 0 = 0$ for all $x \in \mathcal{S}$.

Symmetric compositions over $\mathbb{F}_1$

A **symmetric composition** is a quadratic space $(\mathcal{S}, \tilde{\ })$ with an algebra multiplication $\star$ satisfying the following properties for all $x, y \in \mathcal{S}$:

(SC1) $\widetilde{x \star y} = \tilde{x} \star \tilde{y}$.

(SC2) If $x, y \neq 0$, then

$$x \star y = 0 \iff x \star \tilde{y} \neq 0 \iff \tilde{x} \star y \neq 0 \iff \tilde{x} \star \tilde{y} = 0.$$

(SC3) If $x \star y \neq 0$, then $(x \star y) \star \tilde{x} = y$ and $\tilde{y} \star (x \star y) = x$.

(SC4) If $x \star y = 0$, then $(x^{\perp} \star y) \star x = y \star (x \star y^{\perp}) = \{0\}$; i.e., $(u \star y) \star x = y \star (x \star v) = 0$ for all $u \neq \tilde{x}$ and $v \neq \tilde{y}$.

Maximal isotropic spaces = solids

Theorem

I The sets $x \star \mathcal{S}$ and $\mathcal{S} \star y$, $x, y \in \mathcal{S}$ are solids of $\langle \mathcal{S} \rangle$ of different kinds;

II Any solid is of the form $x \star \mathcal{S}$ or $\mathcal{S} \star y$.

III $\dim \mathcal{S} = 2$, 4 or 8.

Proof of III : $2^n \leq 4n$, so $n \leq 4$!

Examples in dimension 8

We use a “monomial” multiplication table for a “classical symmetric composition” and forget scalars !

For para-octonions:

$*$	e_1	f_1	e_2	f_2	e_3	f_3	e_4	f_4
e_1	0	e_4	f_3	0	$-f_2$	0	$-e_1$	0
f_1	f_4	0	0	$-e_3$	0	e_2	0	$-f_1$
e_2	$-f_3$	0	0	e_4	f_1	0	$-e_2$	0
f_2	0	e_3	f_4	0	0	$-e_1$	0	$-f_2$
e_3	f_2	0	$-f_1$	0	0	e_4	$-e_3$	0
f_3	0	$-e_2$	0	e_1	f_4	0	0	$-f_3$
e_4	0	$-f_1$	0	$-f_2$	0	$-f_3$	f_4	0
f_4	$-e_1$	0	$-e_2$	0	$-e_3$	0	0	e_4

For the split Petersson algebra:

$\star$	e_1	f_1	e_2	f_2	e_3	f_3	e_4	f_4
e_1	f_1	0	$-f_3$	0	0	e_4	$-e_2$	0
f_1	0	$-e_1$	0	e_3	f_4	0	0	$-f_2$
e_2	0	e_4	f_2	0	$-f_1$	0	$-e_3$	0
f_2	f_4	0	0	$-e_2$	0	e_1	0	$-f_3$
e_3	$-f_2$	0	0	e_4	f_3	0	$-e_1$	0
f_3	0	e_2	f_4	0	0	$-e_3$	0	$-f_1$
e_4	0	$-f_3$	0	$-f_1$	0	$-f_2$	f_4	0
f_4	$-e_3$	0	$-e_1$	0	$-e_2$	0	0	e_4

Symmetric compositions, trialitarian automorphisms and geometric triality over $\mathbb{F}_1$

Theorem (Tignol, K., 2012) : The rules

$\star \mapsto \rho_\star, \; \rho_\star[f] = [g]$, if $f(x \star y) = g(x) \star h(y)$

and

$\star \mapsto \tau_\star$ where $\tau_\star : x \mapsto x \star \mathcal{S} \mapsto \mathcal{S} \star x \mapsto x$

define bijections

Trialit. aut. of $\mathrm{PGO}_8^+(\mathbb{F}_1)$ $\Leftrightarrow$ 8-dim. sym. comp.

$\Leftrightarrow$ Geom. trialities

Back to geometric trialities

Let $\langle \mathcal{Q} \rangle$ be the quadric associated to an 8-dimensional quadratic space $\mathcal{Q}$ over $\mathbb{F}_1$.

- $C = \{\text{solids of } \langle \mathcal{Q} \rangle\}$;
- The choice of a decomposition $C = C_1 \sqcup C_2$ into the two kinds of solids is an **orientation** of $\langle \mathcal{Q} \rangle$;

A **geometric triality** on $\langle \mathcal{Q} \rangle$ is a pair (τ, ∂), where ∂ is an orientation $C = C_1 \sqcup C_2$ of Z and τ is a map

$$\tau \colon Z \sqcup C_1 \sqcup C_2 \to Z \sqcup C_1 \sqcup C_2$$

with the following properties:

(GT1) τ commutes with the structure map $\widetilde{\ } \colon x \mapsto \widetilde{x}$;

(GT2) τ preserves the incidence relations;

(GT3) $\tau(\langle \mathcal{Q} \rangle) = C_1$, $\tau(C_1) = C_2$, and $\tau(C_2) = \langle \mathcal{Q} \rangle$;

(GT4) $\tau^3 = I$.

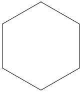
The image of a line under τ is again a line !

Absolute points

An **absolute point** of a geometric triality (τ, ∂) is a point $x \in \langle \mathcal{Q} \rangle$ such that $x \in \tau(x)$.

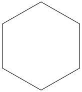
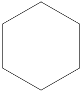
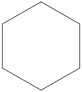
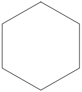
Theorem (Tignol, K.)

1) Suppose (τ, ∂) is a triality on $\langle \mathcal{Q} \rangle$ for which there exists an absolute point. Then the pair (V, E) where V is the set of absolute points of $\langle \mathcal{Q} \rangle$ and E is the set of lines fixed under τ is an hexagon:

(absolute points, fixed lines) = (V,E) = 

Moreover, for every hexagon (V, E) in $\langle \mathcal{Q} \rangle$ and any orientation ∂ there is a unique geometric triality (τ, ∂) on $\langle \mathcal{Q} \rangle$ such that V is the set of absolute points of τ and E is the set of fixed lines under τ.

2) Let (τ, ∂) be a geometric triality on $\langle \mathcal{Q} \rangle$ without absolute points. There are four hexagons (V_1, E_1), $\ldots$, (V_4, E_4) with disjoint edge sets such that each edge set E_i is preserved under τ and $E_1 \sqcup E_2 \sqcup E_3 \sqcup E_4$ is the set of all lines in $\langle \mathcal{Q} \rangle$.

$\{\text{lines}\} =$  $\sqcup$  $\sqcup$  $\sqcup$ 

Any one of these hexagons determines the triality uniquely if the order in which the edges are permuted is given. More precisely, given an orientation ∂ of $\langle \mathcal{Q} \rangle$, an hexagon (V, E) in $\langle \mathcal{Q} \rangle$ and an orientation of the circuit of edges of E, there is a unique triality (τ, ∂) on $\langle \mathcal{Q} \rangle$ without absolute points that permutes the edges in E in the prescribed direction.

All geometric trialities

Theorem Let ∂ be a fixed orientation of $\langle \mathcal{Q} \rangle$.

I There are 16 trialities (τ, ∂) with absolute points on $\langle \mathcal{Q} \rangle$. All these trialities are conjugate under $\mathrm{PGO}^+(\langle \mathcal{Q} \rangle)$.

II There are 8 geometric trialities (τ, ∂) on $\langle \mathcal{Q} \rangle$ without absolute points. These trialities are conjugate under the group $\mathrm{PGO}^+(\langle \mathcal{Q} \rangle)$.

Consequence :

- 2 isomorphism classes of geometric trialities;
- 2 isomorphism classes of 8-dimensional symmetric compositions;
- 2 conjugacy classes of trialitarian automorphisms;

Automorphisms

Theorem (τ, ∂) a geometric triality.

1) With absolute points.

$$\mathrm{Aut}(\tau, \partial) = D_{12} = S_2 \times S_3.$$

2) Without absolute points.

$$\mathrm{Aut}(\tau, \partial) = \widetilde{A}_4 (\simeq \mathrm{SL}_2(\mathbb{F}_3)).$$

Thank you for your attention !