Fields Institute Workshop
Galois Representations, Shimura Varieties and Automorphic Forms

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Lecture 1: Representations of $\mathrm{GL}_2(\mathbb{R})$.

1 From modular forms to automorphic forms

Recall first how to pass from Dirichlet characters to idele class characters. If χ is a character of $(\mathbb{Z}/N)^\times$ it corresponds to a unique (complex) character $\tilde\chi$ of $\mathbb{A}^\times/\mathbb{Q}^\times$. The character $\tilde\chi$ is invariant under $\mathbb{Q}^\times$, trivial on $\mathbb{R}_+^\times$, and coincides with χ on $\prod_p \mathbb{Z}_p^\times/\prod_p \mathbb{Z}_{p,N}^\times \simeq (\mathbb{Z}/N\mathbb{Z})^\times$, where $\mathbb{Z}_{p,N}^\times$ is the obvious congruence subgroup of $\mathbb{Z}_p^\times$.

Let f: holomorphic modular form of weight k or Maaß form (then[1] $k=0$), level N, nebentypus χ form. So, it satisfies:

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z)$$

for $\gamma \in \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \middle| c \equiv 0(N) \right\}$. The character χ is a Dirichlet character modulo N.

There is a compact open subgroup $K_0(N)$ of $G(\hat{\mathbb{Q}})$ (here: $G = \mathrm{GL}_2$ over $\mathbb{Q}$; later also SL_2; we denote $\hat{\mathbb{Q}} = \hat{\mathbf{Z}} \otimes \mathbb{Q}$, the finite adeles) corresponding to $\Gamma_0(N)$, namely:

$$K_0(N) = \prod_p \{ g \in \mathrm{GL}_2(\mathbb{Z}_p) | \, g \text{ is upper triangular modulo } p^{\mathrm{ord}_p(N)} \}$$

and then $G(\mathbb{Q}) \cap K_0(N)G(\mathbb{R})^+ = \Gamma_0(N)$. In other words, $\Gamma_0(N)$ is the stabilizer of the coset of 1 under the action of $G(\mathbb{R})^+$ on $G(\mathbb{Q})\backslash G(\mathbb{A})/K_0(N)$, and the *strong approximation theorem* says that there is only one orbit, i.e.:

$$G(\mathbb{A}) = G(\mathbb{Q})G(\mathbb{R})^+K_0(N).$$

For $g = \gamma g_\infty k_0 \in G(\mathbb{A})$ (the decomposition as above) with $g_\infty = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define:

$$\tilde f(g) = \det(g_\infty)^{\frac{k}{2}}(ci+d)^{-k}\tilde\chi^{-1}(k_0)f(g_\infty \cdot i),$$

where $\tilde\chi$ is the idele class character corresponding to χ, and by $\tilde\chi(k_0)$ we really mean $\tilde\chi$ applied to the lower right entry of $\prod_{p|N} k_{0,p}$.

This is a function on $G(\mathbb{Q})\backslash G(\mathbb{A})$, independent of choices, with *central character* $\tilde\chi^{-1}$, and right-invariant by $K_0(N)' := K_0(N) \cap \ker(\tilde\chi)$ (where $\tilde\chi$ is considered as a character of $K_0(N)$ as before). Moreover, if we set $K_\infty = \mathrm{SO}_2(\mathbb{R})$, the stabilizer of i in $\mathrm{SL}_2(\mathbb{R})$, then $\tilde f$ satisfies:

$$\tilde f(gk_\infty) = e^{ik\theta}\tilde f(g) \text{ for } k_\infty = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \in K_\infty.$$

To summarize:

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[1] Not all cuspidal automorphic representations are generated holomorphic modular forms or Maaß forms of weight zero, we also need to include those Maaß forms of weight one. They correspond to the representations later denoted by $\mathcal{I}^{-,it}$, $t \in \mathbb{R}$. For simplicity we omit them from the discussion at the beginning.

- The *level* of f, together with the conductor of χ, corresponds to the *maximal open subgroup of* $G(\widehat{\mathbb{Q}})$ by which $\tilde{f}$ is invariant.
- The *weight* of f corresponds to the K_∞*-type:= character (irreducible representation)* by which K_∞ acts on $\tilde{f}$.
- The *nebentypus* of f corresponds to the *central character* of $\tilde{f}$.

Now one can interpret the Hecke operators in terms of the action of $G(\widehat{\mathbb{Q}})$ on $C^\infty(G(\mathbb{Q})\backslash G(\mathbb{A}))$. In this lecture we will only consider the action of $G(\mathbb{R})$, though, that is: we use the action of $G(\mathbb{R})$ on the coset represented by 1 to pull back $\tilde{f}$ to a function on:

$$\Gamma_0(N)\backslash \mathrm{GL}_2(\mathbb{R}).$$

We will discuss the representation of $\mathrm{GL}_2(\mathbb{R})$ generated by this function under right translations:

$$g \cdot \tilde{f}(x) := \tilde{f}(xg).$$

2 Smoothness properties

From now on $G = \mathrm{GL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{R})$, Γ is any lattice (= discrete subgroup, finite covolume). Also, K will be what was previously denoted by K_∞: a maximal connected compact subgroup. Usually, when discussing representations of G, there are two nice categories of representations that one considers:

1. *Unitary* representations, that is, continuous representations of G on Hilbert spaces, such as $L^2(\Gamma\backslash G)$.
2. *Smooth representations of moderate growth*, that is:
   - the representation is realized as a countable inverse limit of representations of Banach spaces (such an inverse limit V has the structure of a Fréchet space);
   - for every $X \in \mathfrak{g}$ the limit: $Xv := \lim_{t\to 0} \frac{\exp(tX)v - v}{t}$ exists for all $v \in V$ (hence we obtain an action of $\mathfrak{g}$ and, by extension, the universal enveloping algebra $U(\mathfrak{g})$), and moreover the seminorms $v \mapsto \rho(Xv)$, where ρ ranges over seminorms of V, are continuous (that is, they don't refine the topology).

   An example of such a representation is the space of smooth functions on $\Gamma\backslash G$, when $\Gamma\backslash G$ is compact, or the space of smooth functions of (fixed) "moderate growth" for a general $\Gamma\backslash G$. In particular, the functions obtained from modular forms are of this form.

There is a way to pass from a Hilbert representation V to a smooth representation, namely: take the space V^∞ of *smooth vectors*, i.e. those for which the above limit exists, with topology given by the seminorms $v \mapsto \|Dv\|$, where D ranges over $U(\mathfrak{g})$. Moreover, Harish-Chandra proved that that if the Hilbert representation V is *irreducible*, then the corresponding smooth representation V^∞ is *admissible*, which means, by definition:

Every irreducible representation of K appears with finite multiplicity.

Notice that K is a compact group, and by the Peter-Weyl theorem any Hilbert representation V decomposes into a direct sum of (finite-dimensional) irreducibles. Admissible representations (the word "admissible" is understood to imply "smooth") form an abelian category, hence kernels, cokernels etc are well-behaved. They are also reflexive with respect to the operation of taking smooth duals.

Not every irreducible admissible representation comes from an irreducible unitary representation. Compare: $e^{sx}(s \in \mathbb{C})$ vs $e^{itx}(t \in \mathbb{R})$ for the group $\mathbb{R}$. However, very often the study of admissible representations realized on function spaces (such as $C^\infty(G)$, locally, or $C^\infty(\Gamma\backslash G)$, globally) is often reduced via suitable "twists" to the study of the corresponding L^2 spaces.

The group G may have trivial connected center (e.g. when $G = \mathrm{SL}_2(\mathbb{R})$), but still the center of $U(\mathfrak{g})$ (denoted $\mathfrak{Z}(\mathfrak{g})$) will be nontrivial. In the case of SL_2 it is generated (freely) by the *Casimir operator* $C = \frac{1}{8}(4FE + H(H+2))$ (where (H, E, F) is the standard basis of the Lie algebra $\mathfrak{sl}_2$). You can check that the action of $-8C$ on $\tilde{f}$ corresponds to the following operator acting on modular functions of weight k:

$$\Delta f = -\left(4y^2\partial_z\partial_{\bar{z}} - 2iky\partial_{\bar{z}} + \frac{k}{2}\left(\frac{k}{2} - 1\right)\right) f.$$

Notes for self: in terms of the Iwasawa decomposition $\begin{pmatrix} 1 & x \\ & 1 \end{pmatrix}\begin{pmatrix} y^{\frac{1}{2}} & \\ & y^{-\frac{1}{2}} \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, $8C$ corresponds to the operator $y^2(\partial_x^2+\partial_y^2) - y\partial_x\partial_\theta$.

Since its action commutes with the action of G, a version of Schur's lemma says:

Lemma 2.1. *If V is an* irreducible, admissible *representation of G then $\mathfrak{z}(\mathfrak{g})$ acts on V by a character.*

This is the *infinitesimal character* of the irreducible representation. In our case it is enough to give its value on Δ. (For a general group, it is described via the *Harish-Chandra isomorphism* as a point on $\mathfrak{g}^*_{\mathbb{C}} /\!\!/ G$ – complexified coadjoint quotient.)

We see that this is indeed the case for holomorphic modular forms and for Maaß forms: on the former, $\partial_{\bar{z}}$ vanishes and we have an eigenvalue $\frac{k}{2}\left(\frac{k}{2}-1\right)$; on the latter, $k=0$ and they are required to be eigenfunctions of the *hyperbolic Laplacian* $-4y^2\partial_z\partial_{\bar{z}} = -y^2(\partial_x^2+\partial_y^2)$, by definition.

(In general, automorphic forms belong to smooth admissible representations of *finite length*, but not necessarily irreducible, and then they are annihilated by an ideal of $\mathfrak{Z}(\mathfrak{g})$ of *finite codimension*, not necessarily codimension one or zero.)

3 Generalized principal series and their subquotients

Proposition 3.1. *Every irreducible admissible representation of G is a* subquotient *of a* generalized principal series*.*

This is not true for p-adic groups.

A generalized principal series is a representation induced from *a character of a Borel subgroup* (e.g. the Borel subgroup of upper triangular matrices). We "twist" the induction in order to obtain the properties that follow:

$$I(\chi) := \{f \in C^\infty(G) | f(bg) = \chi\delta^{\frac{1}{2}}(b)g \text{ for all } b \in B, g \in G\}.$$

Here δ is the *modular character* of the Borel subgroup: $\delta\begin{pmatrix} a & b \\ & d \end{pmatrix} = \frac{|a|}{|d|}$. This twist will create problems when we discuss algebraicity. (For instance, for the analogous construction over p-adics, the values of δ are powers of $p^{\frac{1}{2}}$, and this creates problems if you want, e.g. to reduce modulo p.)

The representations $I(\chi)$, thus normalized, have the following properties:

1. There is a perfect duality: $I(\chi)\otimes I(\chi^{-1}) \to \mathbb{C}$ (depending on a choice of measure), given by:

$$(f_1, f_2) = \int_{B\backslash G} f_1 f_2(g) dg.$$

   In particular, if $0 \to A \to I(\chi) \to B \to 0$ is a short exact sequence, then $0 \to \tilde{B} \to I(\chi^{-1}) \to \tilde{A} \to 0$ will be a short exact sequence for the smooth duals.

2. In particular, if χ is unitary, $I(\chi)$ is also unitary. These are the *principal series*. (Remark: the word "generalized" is omitted in large part of the literature, hence the phrase "principal series" does not necessarily mean those induced from unitary characters there.)

3. The irreducible constituents of $I(\chi)$ and $I({}^w\chi)$ (here w denotes the non-trivial action of the Weyl group) coincide, while the irreducible constituents of $I(\chi_1)$ and $I(\chi_2)$ are disjoint if $\chi_2 \neq \chi_1, {}^w\chi_1$. From this point on we discuss $\mathrm{SL}_2(\mathbb{R})$, and will make some comments about $\mathrm{GL}_2(\mathbb{R})$ afterwards.

4. The characters of the Borel of $\mathrm{SL}_2(\mathbb{R})$ are given by:

$$\chi_{\epsilon,w}\begin{pmatrix} a & * \\ & a^{-1} \end{pmatrix} = \mathrm{sgn}(a)^\epsilon |a|^s,$$

   where $\epsilon \in \mathbb{Z}/2$ and $s \in \mathbb{C}$.

   We sometimes write $\epsilon = +$ or $\epsilon = -$. If we denote by $\mathcal{I}^{\pm,s}$ the corresponding principal series, then it is irreducible unless s is an integer of the opposite parity than ϵ.

5. All $\mathcal{I}^{+,w}$ have the same restriction to K (hence contain the same K-types), and so do all $\mathcal{I}^{-,s}$ (this follows from the Iwasawa decomposition $G = BK$). The representations $\mathcal{I}^{+,s}$ contain all the *even* K-types with multiplicity one – in particular, they are spherical, i.e. contain a nonzero K-invariant vector – and the representations $\mathcal{I}^{-,s}$ contain all the *odd* K-types with multiplicity one.

6. The multiple $-8C$ of the Casimir operator acts on $\mathcal{I}^{\epsilon,s}$ by $\frac{1-s^2}{4}$.

7. In the reducible cases we have non-split short exact sequences with all components appearing irreducible:

$$0 \to \mathcal{D}_k^+ \oplus \mathcal{D}_k^- \to \mathcal{I}^{\epsilon,k-1} \to V_{k-2} \to 0 \text{ when } k \geqslant 2 \text{ is an integer of the same parity as } \epsilon.$$

   Here $\mathcal{D}_k^+$ and $\mathcal{D}_k^-$ are called the *holomorphic and anti-holomorphic discrete series* of weight k. The choice of notation and language is not a coincidence: they will correspond to modular forms of the same weight. The representation V_{k-2} is *the finite-dimensional representation of* SL_2 *of weight* $k-2$ (hence, isomorphic to $S^{k-2}V$, where V is the standard two-dimensional representation).

   When s is a negative integer we get a dual sequence. Finally, when $s = 0$, $\epsilon = -$ we have a decomposition:

$$\mathcal{I}^{-,0} \simeq \mathcal{D}_1^+ \oplus \mathcal{D}_1^-,$$

   and these are called the (holomorphic and anti-holomorphic) *limits of discrete series*. They correspond to modular forms of weight one.

For $\mathrm{GL}_2(\mathbb{R})$ the situation is only notationally more complicated, while it is representation-theoretically simpler: The complex manifold of characters of the Borel is now two-dimensional. In order not to repeat a case-by-case listing, here is the rule of thumb: by restriction to $\mathrm{SL}_2(\mathbb{R})$ we get a principal series for $\mathrm{SL}_2(\mathbb{R})$. The short exact sequences are similar to the above, except that $\mathcal{D}_k := \mathcal{D}_k^+ \oplus \mathcal{D}_k^-$ is irreducible as a $\mathrm{GL}_2(\mathbb{R})$ representation, for all $k \geqslant 0$. The results on unitarity and temperedness that follow hold for GL_2, as long as the central character is unitary.

We notice that the finite-dimensional representations and irreducible generalized principal series for $\epsilon = +$ are spherical. On the other hand, the "minimal" K-type that appears in $\mathcal{D}_k^{\pm}$ for $k \geqslant 1$ is $\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \mapsto e^{\pm ik\theta}$.

4 Unitary representations

Although the notion of "automorphic representation" does not necessarily imply "unitary", their study is essentially reduced to those that "appear" in $L^2(\Gamma\backslash G)$ in the sense of Plancherel decomposition. We won't discuss Plancherel decomposition, but the upshot is that in the end we are basically happy to classify the irreducible *unitary* representations of G that appear in the space of functions on $\Gamma\backslash G$. This is a hard (unsolved) problem, so a first step is to classify *all* irreducible unitary representations of G. A complete list of those is as follows (as in the end of the previous paragraph, here again $G = \mathrm{SL}_2(\mathbb{R})$):

- The irreducible generalized principal series $\mathcal{I}^{\epsilon,it}$, ($t \in \mathbb{R}$, $(\epsilon, t) \neq (-, 0)$).
- The limits of discrete series $\mathcal{D}_1^+, \mathcal{D}_1^-$. These have alternative realizations as the holomorphic, resp. anti-holomorphic, functions on the upper half plane, with group action:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot f = (cz+d)^{-1} f\left(\frac{az+b}{cz+d}\right)$$

  and norm:

$$\|f\|^2 = \sup_{y>0} \int_{-\infty}^{+\infty} |f(x+iy)|^2 dx.$$

- The discrete series $\mathcal{D}_k^+, \mathcal{D}_k^-$. Similarly, they can be realized as the holomorphic, resp. anti-holomorphic, functions on the upper half plane, with group action:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} \cdot f = (cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right)$$

and norm:

$$\|f\|^2 = \int |f(x+iy)|^2 y^k \frac{dxdy}{y^2}.$$

- The trivial representation. This is the only finite-dimensional unitary representation.
- The *complementary series* $I^{+,s}$ for $0 < s < 1$ (or, equivalently, $-1 < s < 0$). They admit a non-obvious invariant inner product, which we will not describe.

Notice that the eigenvalue $\frac{1-s^2}{4}$ of the operator $-8C$ is $\geqslant \frac{1}{4}$ for principal series, and $< \frac{1}{4}$ for complementary series. Thus, Selberg's conjecture (there are no Maaß cusp forms with $\Delta f < \frac{1}{4}$ when Γ is arithmetic) amounts to saying that there are no complementary series in the cuspidal automorphic spectrum.

5 Tempered representations and Langlands/Arthur parameters

A word about the relation of the irreducible unitary representations to the decomposition of $L^2(G)$ (local problem) and the Ramanujan conjectures (global problem).

For every Hilbert representation V there is a notion of *decomposition into a direct integral of irreducibles* (Plancherel decomposition) and the *representations that appear in* V (support of Plancherel measure). Compare: the decomposition of $L^2(\mathbb{R})$ into irreducibles spanned by the exponentials $e^{it}, t \in \mathbb{R}$. Of course, in the abelian case the representations which appear in $L^2(G)$ of the group (the *tempered representations*) are *all* unitary representations, but this is not the case for non-abelian G.

Returning to the above classification for $G = \mathrm{SL}_2(\mathbb{R})$, the tempered representations (i.e. those appearing in $L^2(G)$ – this is a "local question") are precisely the discrete series, the principal series and their subrepresentations (limits of discrete series). The trivial representation and the complementary series are not tempered.

Langlands attaches to *all* irreducible admissible representations (not necessarily unitary) a *Langlands parameter*, i.e. a map from the Weil group of $\mathbb{R}$ to the dual group $\mathrm{PGL}_2(\mathbb{C})$. We will talk about those in a moment. The *tempered* representations are those with *bounded* (equivalently: relatively compact) image.

Recall: $\mathcal{W}_\mathbb{R} = \mathbb{C}^\times \sqcup j\mathbb{C}^\times$ with $j^2 = -1$, $jzj^{-1} = \bar{z}$. Its abelianization is isomorphic to $\mathbb{R}^\times$ and its group of connected components isomorphic to $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$. Here are the Langlands parameters of the irreducible representations of $\mathrm{SL}_2(\mathbb{R})$:

- The irreducible quotients of $\mathcal{I}^{\epsilon,s}$ (irreducible generalized principal series, finite dimensional representations, limits of discrete series), with $\Re s \geqslant 0$, have parameter equal to that of the inducing character, that is:

  $$\mathcal{W}_\mathbb{R} \to \mathbb{R}^\times \xrightarrow{\mathrm{sgn}^\epsilon \cdot |\bullet|^s} \mathbb{C}^\times,$$

  where $z \in \mathbb{C}^\times$ is identified with the image of $\begin{pmatrix} z & \\ & 1 \end{pmatrix}$ in $\mathrm{PGL}_2(\mathbb{C})$. We notice that this has bounded image iff $\Re(s) = 0$.

- The discrete series $\mathcal{D}_k^\pm$ have parameter:

  $$\mathcal{W}_\mathbb{R} \supset \mathbb{C}^\times \ni z \mapsto \begin{pmatrix} z & \\ & \bar{z} \end{pmatrix}^{k-1} \in \mathrm{PGL}_2(\mathbb{C}),$$

  $$j \mapsto \begin{pmatrix} & (-1)^{k-1} \\ 1 & \end{pmatrix}.$$

  Again these have bounded image.

Which representations can appear in $L^2(\Gamma\backslash G)$? Clearly, tempered are not enough because the trivial representation appears. The trivial representation and, in the case of $\mathrm{GL}_2(\mathbb{R})$, all unitary characters of the determinant, form the non-cuspidal discrete spectrum of $L^2(\Gamma\backslash G)$. Selberg's conjecture (=the generalized Ramanujan conjecture at infinity) states that the cuspidal spectrum is tempered. Equivalently (since this is clearly true for holomorphic modular forms and non-spherical principal series), the eigenvalues of the hyperbolic Laplacian on Maaß cusp forms should be $\geqslant \frac{1}{4}$. Caution: this should only hold for arithmetic Γ, there are counterexamples for non-arithmetic discrete subgroups!

It is good to know the modification proposed by Arthur to the Langlands parametrization. Arthur attaches to the "good" unitary representations (those which are expected to appear in the study of automorphic forms) a homomorphism:

$$\mathcal{W}_{\mathbb{R}} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C}),$$

where the map on $\mathcal{W}_{\mathbb{R}}$ has again bounded image (and the map on SL_2 is algebraic). If the map on SL_2 is trivial, we get back the parametrization of tempered representations by Langlands parameters. But for the trivial representation (or a character of $\mathrm{GL}_2(\mathbb{R})$) the Arthur parameter is injective on SL_2 and takes $\mathcal{W}_{\mathbb{R}}$ to the center of $\mathrm{GL}_2(\mathbb{C})$.

6 Algebraic automorphic forms and cohomological realizations

When the infinitesimal character of the component at infinity of a discrete automorphic representation of GL_2 is integral, that is: the representation is a subquotient of $I(\chi)$, where χ is up to a finite order character equal to an algebraic character of $B(\mathbb{R})$ (i.e. the diagonal (a_1, a_2) should be mapped by χ to some finite character times $a_1^{s_1} a_2^{s_2}$ with s_1, $s_2 \in \mathbb{Z}$), then one expects to be able to attach a two-dimensional Galois representation ρ with coefficients in $\overline{\mathbb{Q}_p}$ (any prime p) whose localizations (restrictions to decomposition groups) "match" the eigenvalues of Hecke operators on the given automorphic representation. This concerns cases which have the following restrictions π to SL_2:

- π is a discrete series;
- π is a limit of discrete series;
- π is the principal series $\mathcal{I}^{+,0}$.
- π is the trivial representation.

The Weil group ($\mathcal{W}_{\mathbb{R}}$) representations that we saw above are not quite contained in this Galois representation. Rather, the information about the automorphic representation at ∞ gets "distributed" between the restrictions of the Galois representation ρ to the decomposition groups at p and ∞. More precisely, the integers s_1, s_2 give the "Hodge-Tate weights" of the representation at p, while at ∞ one has a recipe for the conjugacy class of $\rho(c)$, where c is complex conjugation. In particular, $\rho(c)$ should have determinant -1 (*odd Galois representation*) for discrete series and limits of discrete series, and $+1$ (*even Galois representation*) for $\mathcal{I}^{+,0}$.

The existence of ρ is known in the first two cases. More precisely, for discrete series of weight $k \geqslant 2$ the theorem of Eichler, Shimura and Deligne states that there is an isomorphism of Hecke modules:

$$S_k(N) \oplus \overline{S_k}(N) \simeq H^1_!(Y_0(N)_{\overline{\mathbb{Q}}}, V_{k-2}) \otimes \overline{\mathbb{Q}_p},$$

where the left hand side is the sum of weight k holomorphic and antiholomorphic cusp forms for $\Gamma_0(N)$ and the right hand side is the image of compactly supported (etale) cohomology in etale cohomology of the local system on the modular curve defined by the $(k-2)$-nd symmetric power of the standard representation of SL_2. We have implicitly chosen an isomorphism: $\mathbb{C} \simeq \overline{\mathbb{Q}_p}$ since the two sides are vector spaces over these fields. But the right hand side comes with a Galois action, and this allows one to naturally attach a Galois representation to every weight-k discrete series automorphic representation of GL_2 when $k \geqslant 2$.

For modular forms of weight 1 it is relatively easy to prove that the eigenvalues of Hecke operators all live in a fixed number field, because the representation is realized on coherent cohomology and these are finite-dimensional rational vector spaces preserved by the Hecke action. It is harder to show the existence of a Galois representation, and this was done by Deligne and Serre using congruences with forms of weight $\geqslant 2$.

Finally, the case of $\mathcal{I}^{+,0}$ is completely open, since it can be realized neither in etale nor in coherent cohomology (hence, we do not even know how to prove algebraicity of the Hecke action).