Mod p points on Shimura varieties

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We will again denote by $\mathrm{Sh}_K(G, X)$ this algebraic variety over $E(G, X)$.

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If $V_{\mathbb{Z}} \subset V$ is a $\mathbb{Z}$-lattice, and $h \in S^{\pm}$, then $V^{-1,0}/V_{\mathbb{Z}}$ is an abelian variety, which leads to an interpretation of $\mathrm{Sh}_K(\mathrm{GSp}, S^{\pm})$ as a moduli space for polarized abelian varieties.

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Almost all Shimura varieties where G is a classical group are quotients of ones of Hodge type. These quotients are called *abelian type*.

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Such subgroups exist if G is quasi-split at p and split over an unramified extension.

Example: Take $(G, X) = (\mathrm{GSp}, S^{\pm})$, defined by (V, ψ) as above. Let $V_{\mathbb{Z}} \subset V$ be a $\mathbb{Z}$-lattice, and K_p the stabilizer of $V_{\mathbb{Z}_p} = V_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Z}_p \subset V \otimes \mathbb{Q}_p$.

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K_p is hyperspecial if and only if a scalar multiple of ψ induces a perfect, $\mathbb{Z}_p$-valued pairing on $V_{\mathbb{Z}_p}$. Then $K_p = \mathrm{GSp}(V_{\mathbb{Z}_p}, \psi)(\mathbb{Z}_p)$.

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The choice of $V_{\mathbb{Z}}$ makes $\mathrm{Sh}_K(\mathrm{GSp}, S^{\pm})$ as a moduli space for polarized abelian varieties, which leads to a model $\mathscr{S}_K(\mathrm{GSp}, S^{\pm})$ over $\mathcal{O}_{(\lambda)}$.

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The $\mathscr{S}_K(\mathrm{GSp}, S^{\pm})$ are *smooth* over $\mathcal{O}_{(\lambda)}$ if and only if the degree of the polarization in the moduli problem is prime to p. This corresponds to the condition that ψ induces a perfect pairing on $V_{\mathbb{Z}_p}$.

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Conjecture. *(Langlands-Milne) Suppose that $K = K_p K^p \subset G(\mathbb{A}_f)$ is open compact and K_p is hyperspecial. Then for $\lambda | p$, The tower*

$$\mathrm{Sh}_{K_p}(G, X) = \lim_{\substack{\leftarrow \\ K^p}} \mathrm{Sh}_{K_p K^p}(G, X)$$

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In the case of Hodge type, $\mathscr{S}_{K_p}(G, X)$ is given by taking the normalization of the closure of

$$\mathrm{Sh}_{K_p}(G, X) \hookrightarrow \mathrm{Sh}_{K'_p}(\mathrm{GSp}, S^{\pm}) \hookrightarrow \mathscr{S}_{K'_p}(\mathrm{GSp}, S^{\pm})$$

into a suitable moduli space of polarized abelian varieties.

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The precise definition of the φ involves the *fundamental groupoid P* of the category of motives over $\bar{\mathbb{F}}_p$. Then $P_{\bar{\mathbb{Q}}}$ is a pro-torus. The φ run over representations $\varphi : P \to G$ satisfying certain conditions.

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• $b \in G(\mathrm{Fr}\, W(\mathbb{F}_{p^r}))$ is an element defined up to *Frobenius* conjugacy
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The data is required to satisfy certain conditions (corresponding to those on the φ).

Then we can define $S(\varphi)$.

$$S(\varphi) = \lim_{\leftarrow K^p} I(\mathbb{Q})\backslash(X_p(\varphi) \times G(\mathbb{A}_f^p))/K^p$$

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$$I(\mathbb{Q}) \longleftrightarrow \text{automorphisms of the AV+extra structure}$$

and I is a compact (mod center) form of the centralizer G_{γ_0}.

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Let $\mathcal{O}_L = W(\bar{\mathbb{F}}_p)$ and $L = \operatorname{Fr} \mathcal{O}_L$. Then we have

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If the conjugacy class of μ is fixed by σ^s, then the p^s-Frobenius acts on $X_p(\varphi)$ by $\Phi_s(g) = (b\sigma)^s(g) = b\sigma(b)\ldots\sigma^{s-1}(b)\sigma^s(g)$

Conjecture.

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Remarks:

(1) Implicit in the statement is a generalization of Tate's theorem on the Tate conjecture for endomorphisms of AV's over finite fields, since one seeks to classify isogeny classes in terms of Frobenius conjugacy classes.

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(3) In the case of PEL type (A,C) this is due to Kottwitz and Zink. A subtle point is that Kottwitz doesn't quite construct a canonical bijection, and neither do we. (This seems to me to require a new idea.)

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Kottwitz has explained how one can use the Fundamental Lemma to stabilize this expression, and compare it with the stabilized geometric side of the trace formula. This should allow one to express the zeta function of $\mathrm{Sh}_{K_p}(G,X)$ in terms of automorphic L-functions.

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But it is not clear that $gx \in \mathscr{S}_{K_p}(G, X)(\bar{\mathbb{F}}_p)$ because this is defined as a closure and has no easy moduli theoretic description. So it isn't clear there is a map

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In the PEL case one can deduce this from Tate's theorem.

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We'll sketch this in Tate's *original* context of principally polarized AV's.

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However automorphisms in the definition of I_l commute with Frobenius, so the quotient on the left *is* finite. (First ingredient used by Tate !).

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Lemma. *Let I' be a connected algebraic group over $\mathbb{Q}_l$, whose reductive quotient is split. If $I \subset I'$ is a closed subgroup such that $I(\mathbb{Q}_l)\backslash I'(\mathbb{Q}_l)$ is compact, then I contains a Borel subgroup of I'.*

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By the lemma $I_l/I_{\mathbb{Q}_l}$ is projective. But since I is reductive the quotient is also affine, and connected, hence a point. □

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Theorem. *Every isogeny class in $\mathscr{S}_K(G, X)(\bar{\mathbb{F}}_p)$ contains a point which admits a special lifting.*

Using the theorem one can solve the problem 2) about existence of γ_0 and γ_l being stably conjugate, and hence get the indepenence of l.

Shimura varieties of Abelian type:

Recall that a Shimura datum (G_2, X_2) is called of *Abelian type* if there is a Shimura datum of Hodge type (G, X) and a central isogeny $G^{\mathrm{der}} \to G_2^{\mathrm{der}}$ which induces an isomorphism on adjoint Shimura data

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If $K_p = \mathcal{G}(\mathbb{Z}_p)$ is hyperspecial, then this induces an action of $\mathcal{G}(\mathbb{Z}_{(p)})^+$ on

$$\mathrm{Sh}_{K_p}(G, X) = \lim_{\leftarrow K^p} G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K^p K_p$$

which extends to an action on $\mathscr{S}_{K_p}(G, X)$.

The integral model $\mathscr{S}_{K_p}(G_2, X_2)$ is constructed from $\mathscr{S}_{K_p}(G, X)$ using this action; the geometrically connected components of the former are quotients of those of the latter. This is analogous to Deligne's construction of canonical models.

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It turns out that one can construct the analogous structures for $\coprod_\varphi S(\varphi)$; there is a $\mathcal{G}(\mathbb{Z}_{(p)})^+$-action and a notion of connected components. This bijection

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To do this it seems essential to work with the morphisms φ and not just the triples $(\gamma_0, (\gamma_l)_{l \neq p}, \delta)$.

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We can form an abelian scheme up to isogeny

$$\mathcal{A}_x^{\mathcal{P}} = (\mathcal{A}_x \otimes_{\mathbb{Q}} \mathcal{O}_{\mathcal{P}})^{Z_G}$$

where the RHS can be thought of in terms of fppf sheaves.