Constraints on marginalised DAGs.

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Outline

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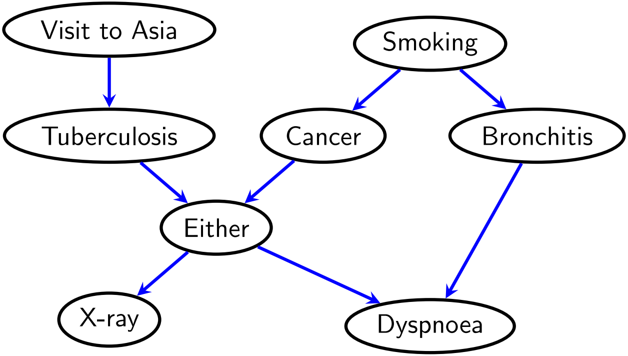
Implications of Models

It is common (especially in causal settings) to hypothesise models based on DAGs:



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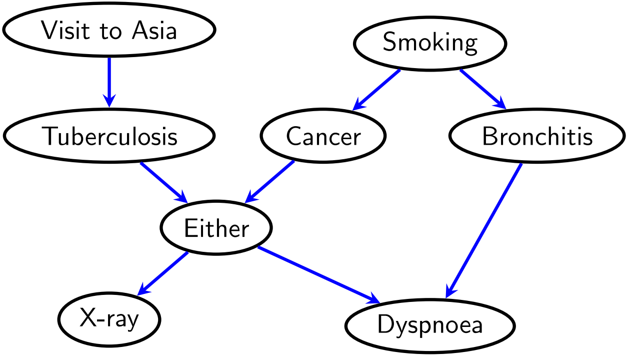

This encodes the assumption that the joint distribution factorises as:

$$p(A)\,p(T \mid A)\,p(S)\,p(C \mid S)\,p(B \mid S)\,p(E \mid T, C)\,p(X \mid E)\,p(D \mid E, B).$$

d-Separation

The factorisation criterion

$$p(x_V) = \prod_{v \in V} p(x_v \mid x_{\mathrm{pa}_{\mathcal{G}}(v)})$$

is equivalent to the global Markov property:

$$A \text{ d-separated from } B \text{ by } C \implies X_A \perp\!\!\!\perp X_B \mid X_C \, [P].$$

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In particular, all constraints on DAGs are conditional independences.

Marginalisation

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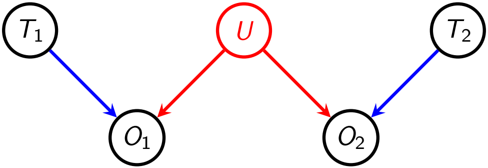

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O_i recorded outcome for sibling i

U **unmeasured** confounding (genetics etc.)

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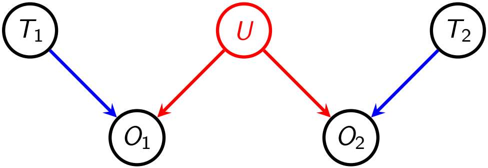

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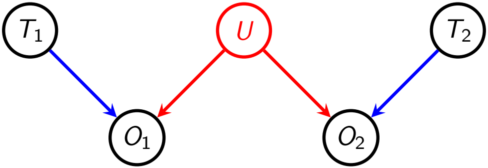

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Is this all?

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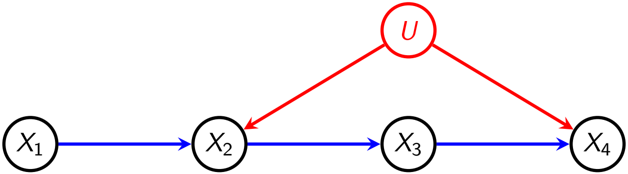
Example 1: Verma Graph

Consider the following DAG on 5 variables (Verma and Pearl, 1990).



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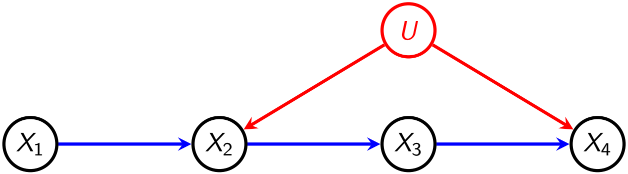

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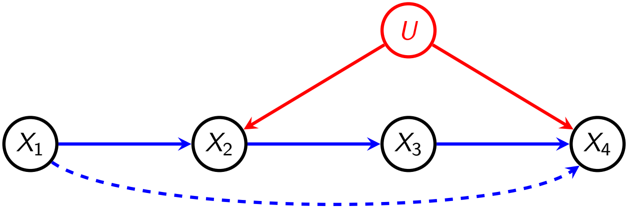

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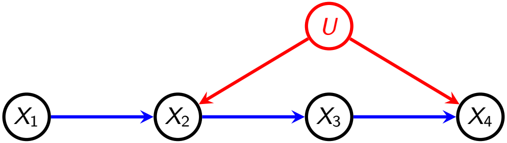
But if we add an arrow $X_1 \to X_4$, we still have $X_3 \perp\!\!\!\perp X_1 \mid X_2$.
So can we detect that $X_1 \not\to X_4$?

The Verma Constraint



$$f(x_1, x_2, x_3, x_4) = \int f(u)\, f(x_1)\, f(x_2 \mid x_1, u)\, f(x_3 \mid x_2)\, f(x_4 \mid x_3, u)\, du$$

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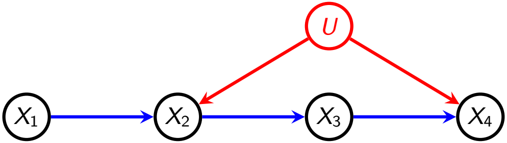

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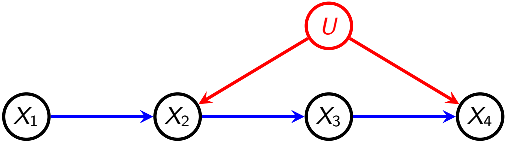

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Note that

$$\int f^*(x_2, x_4 \mid x_1, x_3)\, dx_2 = f(x_4 \mid x_3)$$

is independent of x_1, precisely because $X_1 \not\to X_4$.

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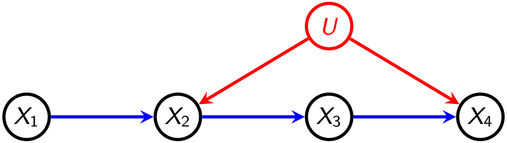

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This is the **Verma constraint**, and provides a non-parametric test for the presence of $X_1 \to X_4$.

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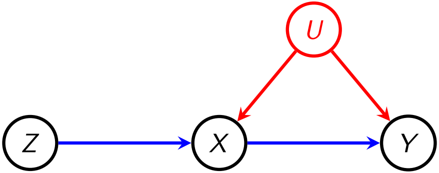

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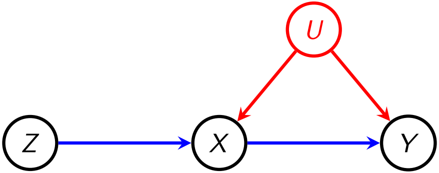

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$$\max_x \sum_y \max_z P(X = x,\ Y = y, \mid Z = z) \leq 1.$$

So, for example

$$P(X = 0,\ Y = 0 \mid Z = 0) + P(X = 0,\ Y = 1 \mid Z = 1) \leq 1.$$

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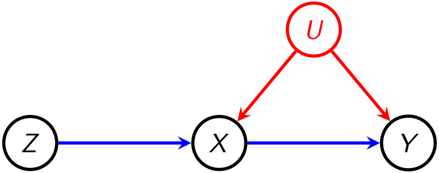

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Inequalities for the discrete IV model can be derived using linear programs (`Porta`, `cdd`).

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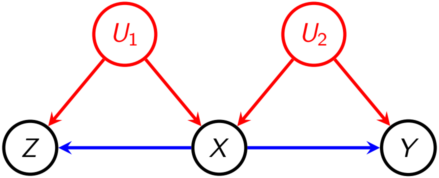

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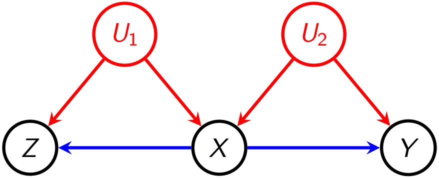

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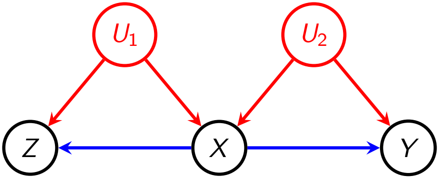

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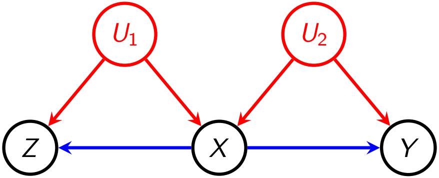

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However, there are inequalities (as we will see).

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Perhaps easier:

- can we find an equivalence class of these models?
- what graphs do we need to represent these models?

Prior Work

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Pearl (1995) first gave inequality constraints for IV model. Bonet (2001) used linear programming to derive tight bounds.

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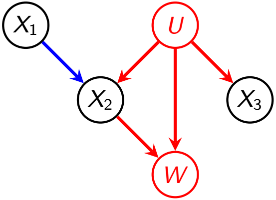

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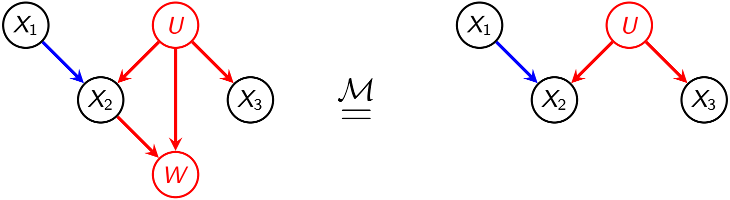

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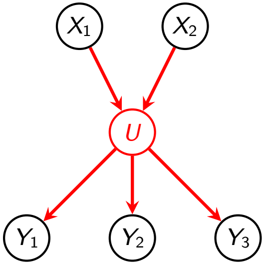

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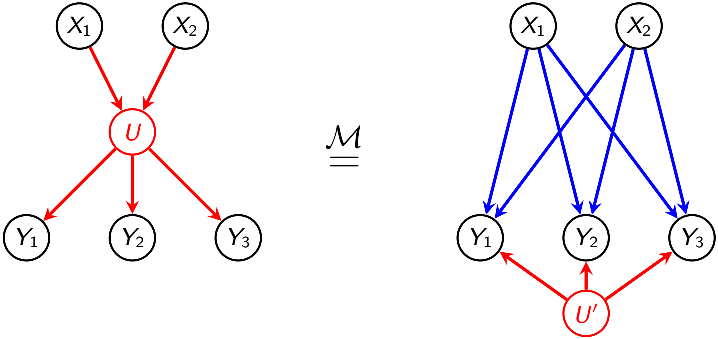

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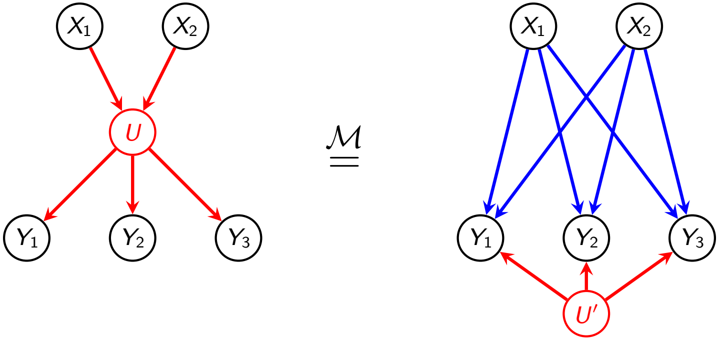

Hence we only need to consider latents with no parents.

Of course this is not true if we assume, e.g. latents are binary!

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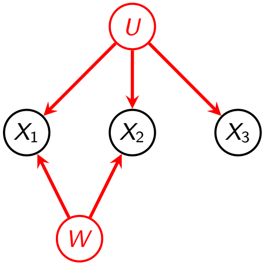
Simplification 3. If U, W are latent with $\text{ch}_{\mathcal{G}}(W) \subseteq \text{ch}_{\mathcal{G}}(U)$, then we don't need W.



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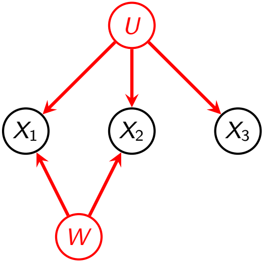

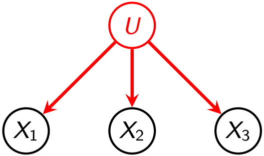

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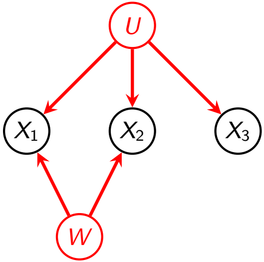
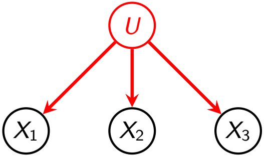


$$\overset{\mathcal{M}}{=}$$



mDAGs

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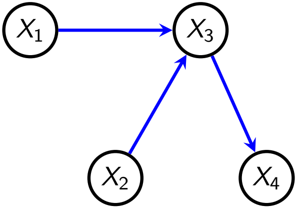

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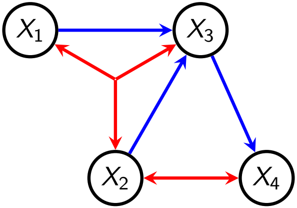

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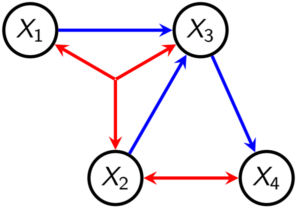

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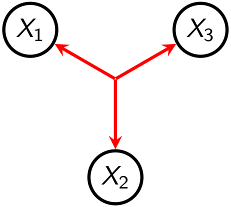
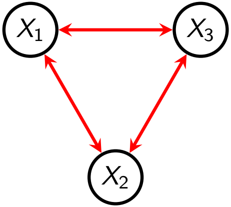
ADMGs are special case where B only contains subsets of size 2.

ADMGs are not sufficient

In general we need to distinguish between $\{1, 2, 3\}$ and $\{1, 2\}$, $\{1, 3\}$, $\{2, 3\}$.





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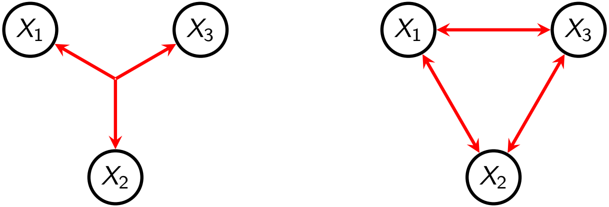

The model on the right is not saturated. Still true if we dichotomise.

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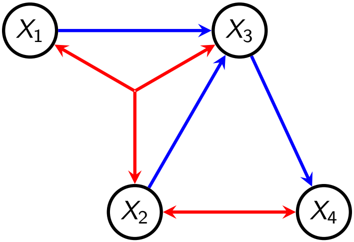
More precisely: with every mDAG $\mathcal{G}$ we can associate a DAG $\bar{\mathcal{G}}$ which includes those hidden variables U.



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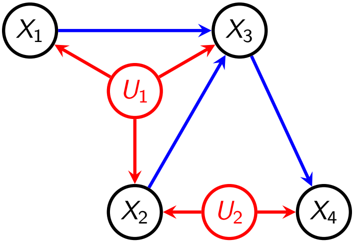

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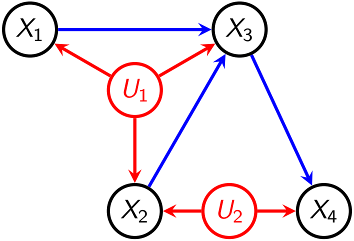
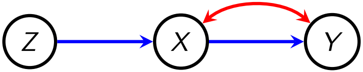


Then $P \in \mathcal{M}(\mathcal{G}, \mathfrak{X}_V)$ if there exists some product space $\mathfrak{X}_U$ (and σ-algebra) and some distribution $\bar{P} \in \mathcal{M}(\bar{\mathcal{G}}, \mathfrak{X}_V \times \mathfrak{X}_U)$ such that P is the V-margin of $\bar{P}$.

Example

Instrumental Variables mDAG $\mathcal{G}$:



Let $\mathfrak{X}_V = \{0,1\}^3$; then Pearl (1995) shows

$$\mathcal{M}(\mathcal{G}, \mathfrak{X}_V) \subseteq \left\{ P \,\middle|\, \max_x \sum_y \max_z P(x, y \mid z) \leq 1 \right\}.$$

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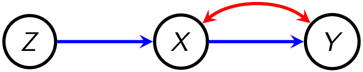

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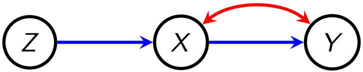
$$\mathcal{M}(\mathcal{G}, \mathfrak{X}_V) \subseteq \left\{ P \,\middle|\, \max_x \sum_y \max_z P(x, y \mid z) \leq 1 \right\}.$$

With linear programming one can show equality holds (Bonet, 2001).

In the general discrete case (especially for increasing statespace of Z) these inequalities are not sufficient.

Instrumental Inequality: Alternative Interpretation



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$$f(x, y \mid z) = \int f(u)\, f(x \mid z, u)\, f(y \mid x, u)\, du.$$

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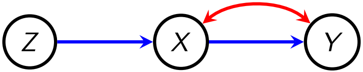

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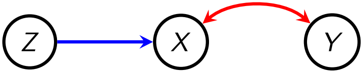

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so Y behaves *as if* $X = 0$.

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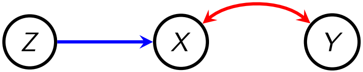

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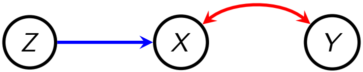

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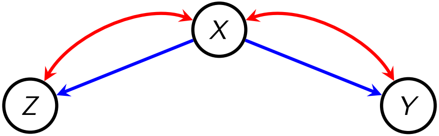
So $f(0, y \mid z)$ for y, z must be **compatible** with a distribution under which $Y \perp\!\!\!\perp Z$. This gives Pearl's instrumental inequality.

Applying to Other Graphs

Consider the 'unrelated confounding' model.



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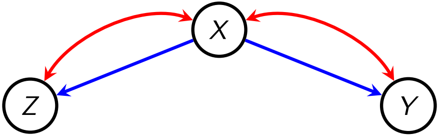

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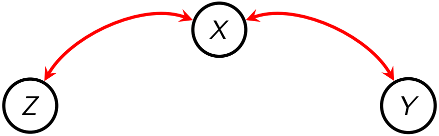

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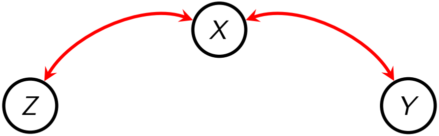

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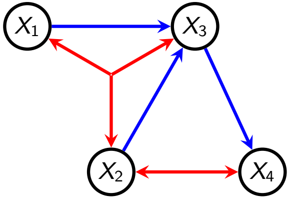
But by same argument, probabilities $f(0, y, z)$, must be compatible with $Y \perp\!\!\!\perp Z$.

So e.g.

$$(1 - f(0, y, z))^2 + (1 - f(0, 1 - y, 1 - z))^2 \geq 1.$$

Skeleton

Call the **skeleton** of an mDAG the undirected graph given by joining any vertices in the same edge.



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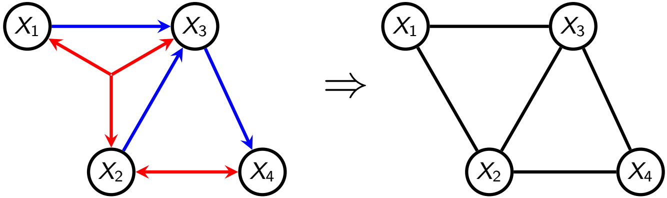

Distinction by Skeleton

The instrumental inequality can be generalised for discrete graphs.

Theorem

Let $\mathcal{G}$, $\mathcal{G}'$ be mDAGs and $\mathfrak{X}_V$ a discrete statespace. If

- $\mathcal{G}' \subseteq \mathcal{G}$; and
- $\mathcal{G}'$ and $\mathcal{G}$ have different skeletons,

then $\mathcal{M}(\mathcal{G}', \mathfrak{X}_V) \subsetneq \mathcal{M}(\mathcal{G}, \mathfrak{X}_V)$. In other words, a constraint is always induced.

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The proof of this result is constructive (in that it produces inequalities).

Causal Effects

Corollary

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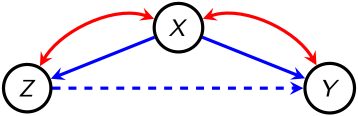

Unrelated confounding model, e.g.

$$\text{ACDE}(x) \leq \frac{1 + P(Y = 1, x, z) - P(x)}{P(x, z)} - \frac{P(Y = 1, x, 1 - z)}{1 - P(x, z)}$$

for each x, z.

Model Equality

Lemma

Let $\mathcal{G}'$ and $\mathcal{G}$ be ADMGs, with $\mathcal{G}'$ a subgraph of $\mathcal{G}$. Then $\mathcal{M}(\mathcal{G}') \subseteq \mathcal{M}(\mathcal{G})$

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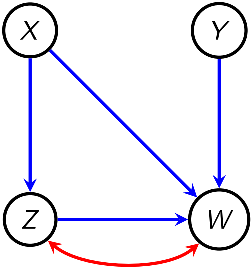
Theorem

Let $\mathcal{G}$ be an ADMG with an edge $a \leftrightarrow b$ such that

- $\text{pa}_{\mathcal{G}}(a) \subseteq \text{pa}_{\mathcal{G}}(b)$;
- $\text{sp}_{\mathcal{G}}(a) = \{b\}$.

Then if $\mathcal{G}'$ is equal to $\mathcal{G}$ except that $a \to b$ and $a \not\leftrightarrow b$, we have $\mathcal{M}(\mathcal{G}') = \mathcal{M}(\mathcal{G})$.

Examples (1)



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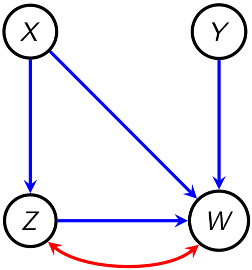

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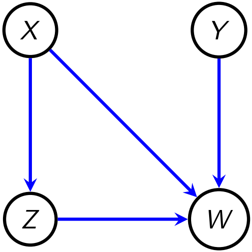
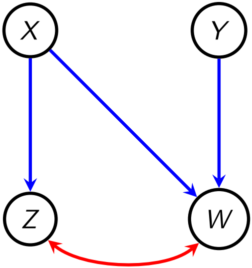


m-separation implies $X, Z \perp\!\!\!\perp Y$... but we get precisely this without the bidirected edge anyway.

Examples (2)



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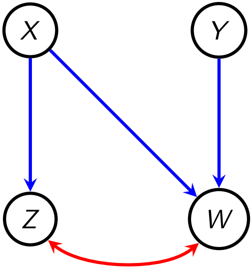

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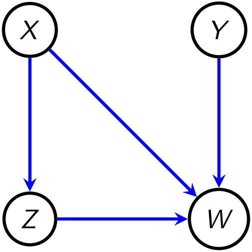

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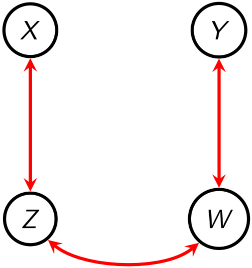

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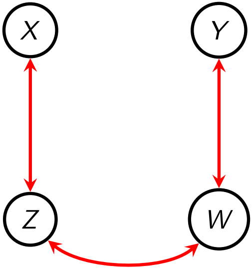

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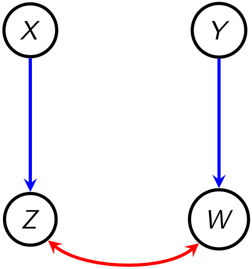

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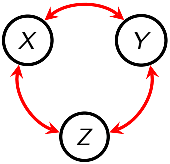
Here we have $X \perp\!\!\!\perp W, Y$ and $Y \perp\!\!\!\perp X, Z$ as well as Bell's inequalities.

Models on Three Observed Variables

40 unlabelled ADMGs on 3 variables (48 mDAGs).

$\perp\!\!\!\perp \{X, Y, Z\}$	complete independence	1
$X \perp\!\!\!\perp Y, Z$	joint independence	3
$X \perp\!\!\!\perp Y \mid Z$	conditional independence	5
$X \perp\!\!\!\perp Y$	marginal independence	6
$\mathrm{IV}(X, Y, Z)$	instrumental variable	3
$\mathrm{UC}(X, Y, Z)$	unrelated confounding	1
	unrestricted	20
	3-cycle	1



Larger Models

There are 1567 ADMGs over 4 variables. At least 509 are equivalent to a DAG.

After applying Theorem on equivalence, at most 671 distinct models not equivalent to DAGs.

Can reduce to 543 by splitting into districts.

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- this interpretation leads to constructive bounds for other models;
- the absence of an edge in *any* mDAG can (in principle) be refuted;
- consequently causal bounds can be constructed for any unconfounded variables.

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Thank you!

References

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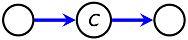

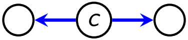

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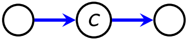

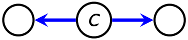

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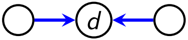

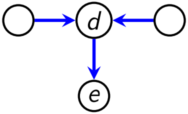

Two vertices v and w are **d-separated** given $C \subseteq V \setminus \{v, w\}$ if **all** paths are blocked.

Bonet's Inequalities

Suppose Z ternary and X, Y binary for IV model on Z, Y, X. Then

$$p(x_0, y_1 \mid z_1) + p(x_0, y_0 \mid z_2) + p(x_0, y_1 \mid z_0) + p(x_1, y_1 \mid z_1) + p(x_1, y_0 \mid z_0) \leq 2$$

ADMGs are not sufficient

Lemma

Let $\mathcal{F}$, $\mathcal{G}$, $\mathcal{H}$ be mutually independent σ-algebrae (so that $\mathcal{F} \perp\!\!\!\perp \mathcal{G} \vee \mathcal{H}$ and so on), and let X, Y and Z be random variables such that

(i) X is $\mathcal{F} \vee \mathcal{G}$-measureable;

(ii) Y is $\mathcal{G} \vee \mathcal{H}$-measureable;

(iii) Z is $\mathcal{F} \vee \mathcal{H}$-measureable.

Then $P(X = Y = Z) > 1 - \epsilon$ implies

$$\operatorname{Var} X < 3\epsilon.$$