Around boundaries of exponential families

František Matúš

Institute of Information Theory and Automation
Academy of Sciences of the Czech Republic
E-mail: matus@utia.cas.cz

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(the smallest closed convex set with μ-negligible complement)

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$bar(C)$ ... the barrier cone of $C \subseteq \mathbb{R}^d$

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$\mathcal{E}_\mu = \mathcal{E}_\nu$ if and only if V_μ coincides with V_ν on a ball.

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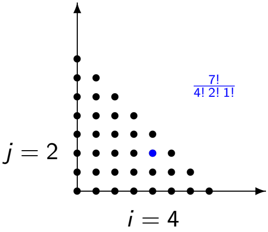

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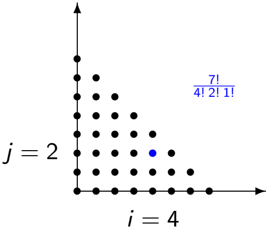

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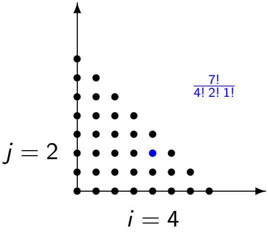

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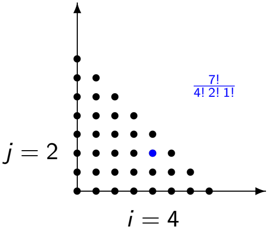

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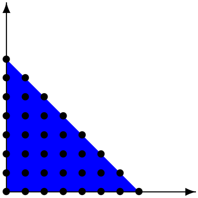

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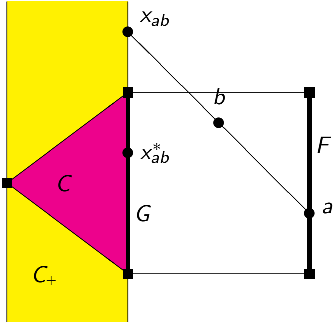
Matúš (2007) Λ^* when the support $s(\mu)$ of μ is finite

In the figure, μ is concentrated on the five black squares.



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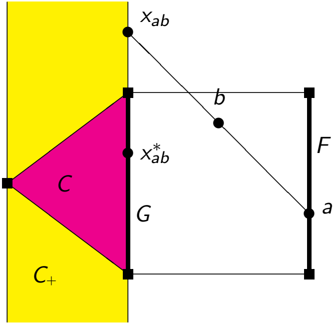
$cs(\mu)$ the pentagon



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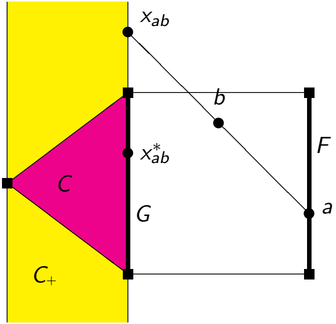

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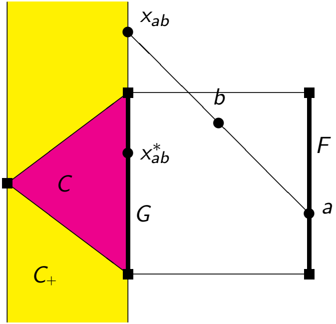

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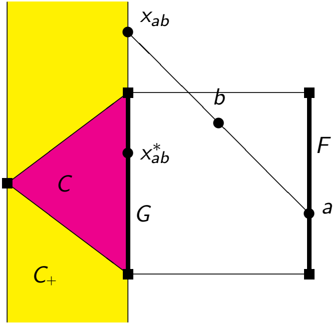

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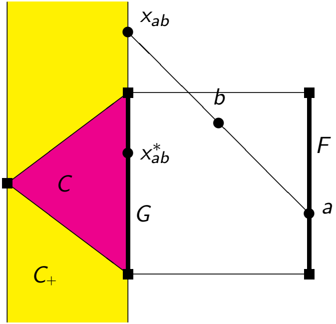

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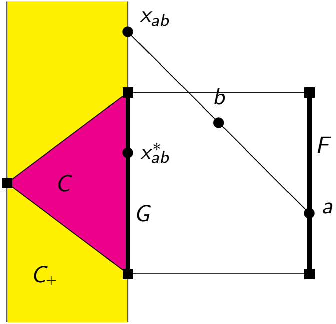

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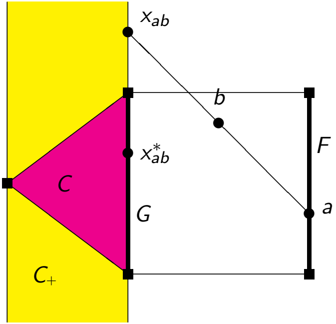

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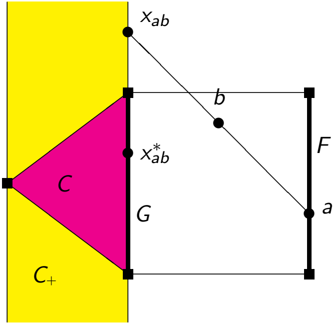

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x^*_{ab} … a special point inside G

Theorem (M07)

If μ is a pm on $\mathbb{R}^d$ concentrated on a finite set, $a \in ri(F)$ for a proper face F of $cs(\mu)$, $b \in int(cs(\mu))$ and $\varepsilon > 0$ then

$$\Lambda^*(a + \varepsilon(x_{ab} - a)) = \Lambda^*(a) + \varepsilon \ln \varepsilon + \varepsilon \left[\Psi^*_{C,\Xi}(x_{ab}) - 1 - \Lambda^*(a) \right] + o(\varepsilon)$$

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This approximation was applied to complete the first order conditions for a probability measure to be a maximizer of the divergence from an exponential family (Ay (2002)).

Theorem (unpubl)

Under the above assumptions,
if $x \in s(\mu)$ *then* $Q_{\psi(a+\varepsilon\,(x_{ab}-a))}(x)$ *equals*

$$\begin{cases} (1-\varepsilon)\cdot Q_{F,\psi_F(a)}(x) + \varepsilon \cdot \langle x_{ab} - x^*_{ab}, Q'_{F,\psi_F(a)}(x)\rangle\,, & x \in F\,, \\ \varepsilon \cdot Q_{G,\psi_G(x^*_{ab})}(x)\,, & x \in G\,, \\ 0\,, & \text{otherwise.} \end{cases}$$

up to $o(\varepsilon)$*-terms.*

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Theorem (Morris 82)

If the variance function of an exponential family $\mathcal{E}_\mu$ on $\mathbb{R}$ equals a quadratic polynomial on the open interval M_μ then $\mathcal{E}_\mu$ is one of the families $\mathcal{F}_{1,\cdot} - \mathcal{F}_{6,\cdot}$ up to an affine transform.

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... generalized hyperbolic secant

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