Traceable regressions

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Set-up for sequences of regressions in vector variables $\mathbf{Y}_a \mathbf{Y}_b \ldots$

a	b	c	d	e
Y_a	Y_b	Y_c	Y_d	Y_e
primary responses	intermediate variables	intermediate variables	treatment variables	context variables

Main goal: understanding development with data from

– cohort studies, multi-wave panel data

– studies with randomized, sequential interventions

– cross-sectional and even retrospective studies

General motivation

- Trying to understand short- and long-term effects of risks or of interventions is motivating empirical research in many fields of science

- For this, the main purpose of statistical planning, analysis and interpretation is to capture and use potential data generating processes and to trace pathways of dependence

- Sequences of multivariate or univariate regressions, simplified by independences, provide a flexible framework; joint responses may be discrete, continuous or be mixed of both types

A regression graph, $G^{N}_{\mathbf{reg}}$, is traditionally the focus of interest

$G^{N}_{\mathbf{reg}}$ is a chain graph defined by node set N and three types of edge sets $E_{\prec\!\!-}$, E_{--} , and $E_{—}$

It has

– a split of $N = (u, v)$ with sequences of

– response nodes coupled as $\circ\!\!---\!\!\circ$ in u and

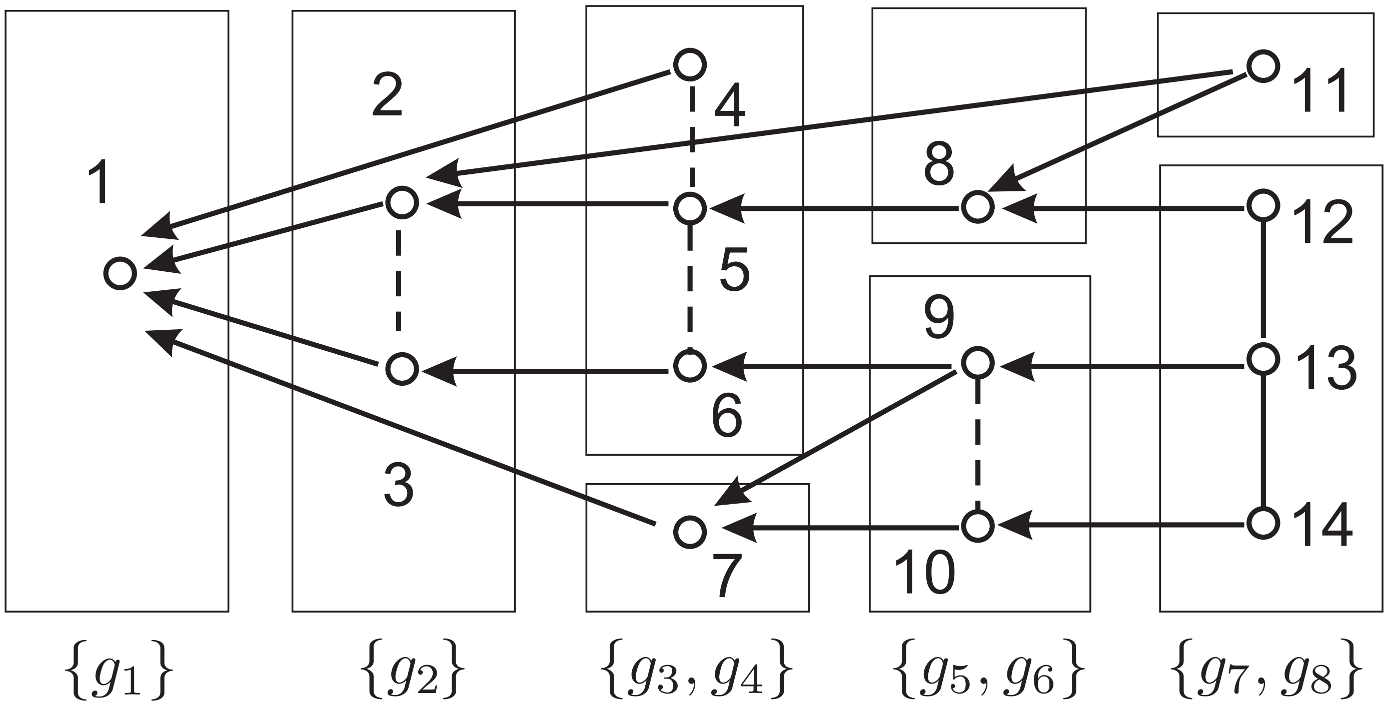
– context nodes coupled as $\circ$ —— $\circ$ in v

– a unique set of the concurrent nodes in g_j for $j = 1, \ldots, J$

– in each compatible ordering of g_j, arrows, $\circ\!\!\prec\!\!\!-\!\!\circ$, never point to

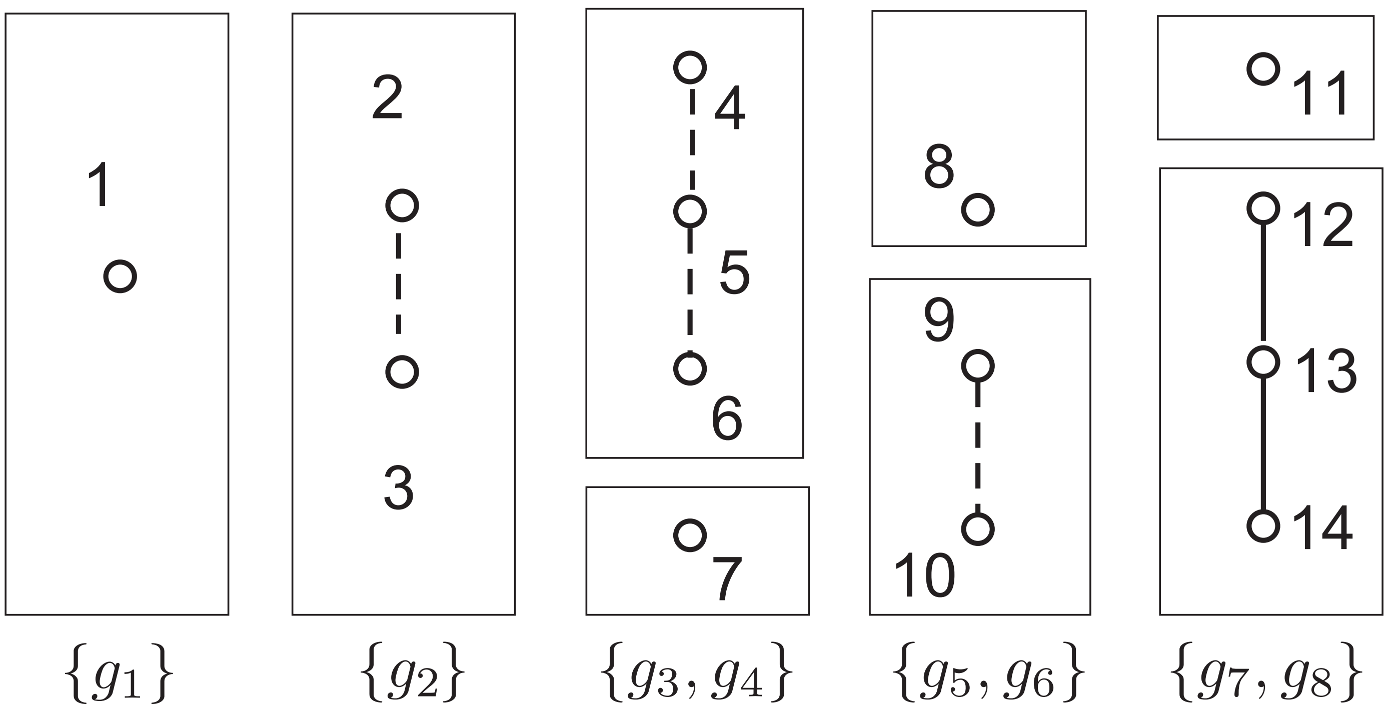
$$g_{>j} = g_{j+1} \cup g_{j+2}, \ldots, \cup g_J$$

Example for a refined sets of concurrent nodes in $\boldsymbol{g_j}$ obtained by statistical analyses after a first ordering into five blocks



within a set of concurrent nodes, $\boldsymbol{g_j}$, each node can be reached via at least one undirected path, no order is implied by stacked boxes

Example continued: deleting all arrows gives uniquely the sets of concurrent responses and concurrent context variables, the chain components $\boldsymbol{g_j}$



A joint density $\boldsymbol{f_N}$ is said to be generated over $\boldsymbol{G^N_{\mathbf{reg}}}$

if it has the basic factorizations with regressions $\boldsymbol{f_{g_j|g_{>j}}}$ as

$$\boldsymbol{f_N = f_{u|v} f_v} \text{ with } \boldsymbol{f_{u|v} = \prod_{j \in u} f_{g_j|g_{>j}}} \text{ and } \boldsymbol{f_v = \prod_{j \in v} f_{g_j}}$$

and satisfies the independences implied for each missing edge

For $\boldsymbol{i, k}$ a node pair and $\boldsymbol{c \subset N \setminus \{i, k\}}$, we have in general

$$\boldsymbol{i \perp\!\!\!\perp k | c \iff (f_{i|kc} = f_{i|c}) \iff f_{ik|c} = (f_{i|c} f_{k|c})}$$

For tracing pathways of dependence, the variable pairs needed to generate f_N are instead the focus of interest and

the substantive context determines which variable pairs are modeled by a conditional independence and which **variable pairs are taken to be dependent**

Suppose one regressor is a risk factor for a response, then the prevention of the risk is generally judged to be of quite different importance if, for instance, the response is
– the occurrence of a common cold
– the infection with an HIV virus or
– an accident in a nuclear plant

We write $\boldsymbol{i \pitchfork k|c}$ for $\boldsymbol{Y_i, Y_k}$ **conditionally dependent given** $\boldsymbol{Y_c}$ for some $\boldsymbol{c \subset N \setminus \{i, k\}}$

A graph is **edge-minimal** for a distribution generated over it, if every missing edge in the graph corresponds to a conditional independence statement and every edge present to a dependence statement

A dependent variable pair $\boldsymbol{Y_i, Y_k}$ is one needed in the generating process of $\boldsymbol{f_N}$ and a family of densities $\boldsymbol{f_N}$ generated over an edge-minimal graph changes if any one edge is removed from the graph

Defining dependences and independences for an edge-minimal G_{reg}^N

Definition 1

An edge-minimal regression graph with $N = (u, v)$ and $g_1 < \cdots < g_J$ specifies a generating process for f_N, where

$$i \dashrightarrow k : \; i \pitchfork k | g_{>j} \quad \text{for } i, k \text{ concurrent response nodes in } g_j \text{ of } u$$

$$i \prec\!\!- k : \; i \pitchfork k | g_{>j} \setminus \{k\} \quad \text{for response } i \text{ in } g_j \text{ of } u \text{ and explanatory } k \text{ in } g_{>j}$$

$$i \,\text{---}\, k : \; i \pitchfork k | v \setminus \{i, k\} \quad \text{for } i, k \text{ concurrent context nodes in } g_j \text{ of } v$$

define **edges present** in G_{reg}^N

define **edges missing** in G_{reg}^N when the dependence sign $\pitchfork$ is replaced by $\perp\!\!\!\perp$

Thus, for an edge-minimal G^N_{reg}

– one fixed ordering of g_j is assumed, so that the density of variables in g_J is generated first,
the one of g_{J-1} given g_J next,
up to the density of g_1 given $g_{>1}$

– the graph implies for each variable pair either conditional dependence or independence given the same type of conditioning set

– for each node i of g_j in u, nodes in
$g_{>j} = g_{j+1} \cup g_{j+2}, \ldots, g_{J-1} \cup g_J$ are in the ***past of*** g_j

Requirements for two results on the independence structure of G^N_{reg}

Let $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ denote disjoint subsets of N where only $\boldsymbol{d}$ may be empty and let any joint independence $\boldsymbol{b} \perp\!\!\!\perp \boldsymbol{ac}|\boldsymbol{d}$ have **three equivalent decompositions** as

$$
\begin{array}{ll}
(i) & (\boldsymbol{b} \perp\!\!\!\perp \boldsymbol{a}|\boldsymbol{cd} \text{ and } \boldsymbol{b} \perp\!\!\!\perp \boldsymbol{c}|\boldsymbol{d}) \\
(ii) & (\boldsymbol{b} \perp\!\!\!\perp \boldsymbol{a}|\boldsymbol{d} \text{ and } \boldsymbol{b} \perp\!\!\!\perp \boldsymbol{c}|\boldsymbol{d}) \\
(iii) & (\boldsymbol{b} \perp\!\!\!\perp \boldsymbol{a}|\boldsymbol{cd} \text{ and } \boldsymbol{b} \perp\!\!\!\perp \boldsymbol{c}|\boldsymbol{ad})
\end{array}
$$

then (i) named contraction, holds for all probability distributions

(ii) combines decomposition and composition, holds in a regression when there is also a main-effect for every higher-order interactive or nonlinear dependence

(iii) combines weak union and intersection, holds for all positive distributions

Given the three equivalent decompositions of any joint dependence, active paths in $G^N_{\mathbf{reg}}$ can be expressed in terms of anterior paths

An **anterior ik-path** is a descendant-ancestor iq-path with a context-nodes qk-path attached to it (or any subpath)

$$i \prec\!\!- \underbrace{\overbrace{\circ \prec\!\!- \circ, \ldots, \circ \prec\!\!- q}^{\text{ancestors of } i} \;—\!\!— \circ, \ldots, \circ \;—\!\!— k}_{\text{anteriors of } i}$$

Let $\{a, b, c, m\}$ partition N, where c denotes a conditioning set of interest for a, b and m the set of nodes to be ignored

A **path in $G^N_{\mathbf{reg}}$ is active given c** if of its inner nodes, every collision node is in $c \cup \mathbf{ant}_c$ and every transmitting node is in m

Lemma 1

Global Markov property of $G^N_{\mathbf{reg}}$ (Sadeghi, 2009) $G^N_{\mathbf{reg}}$ implies $a \perp\!\!\!\perp b | c$ if and only if there is no active path in $G^N_{\mathbf{reg}}$ between a and b given c

Lemma 2

Equivalence of the pairwise and the global Markov property

(Sadeghi and Lauritzen, 2012) The independence structure of $G^N_{\mathbf{reg}}$ is equivalently defined by its lists of the three types of missing edges and by its global Markov property.

Two-edge subgraphs induced by three nodes in $G^N_{\mathbf{reg}}$, named Vs

There are just two basic types of Vs in $G^N_{\mathbf{reg}}$

collision Vs:

$$i \;\text{- - -}\; \circ \prec\!\!\!- k, \quad i \;-\!\!\!\succ \circ \prec\!\!\!- k, \quad i \;\text{- - -}\; \circ \;\text{- - -}\; k,$$

and **transmitting** Vs:

$$i \prec\!\!\!- \circ \prec\!\!\!- k, \quad i \prec\!\!\!- \circ \;\text{——}\; k, \quad i \;\text{——}\; \circ \;\text{——}\; k, \quad i \prec\!\!\!- \circ -\!\!\!\succ k, \quad i \prec\!\!\!- \circ \;\text{- - -}\; k$$

Lemma 3

Markov equivalence (Wermuth and Sadeghi, 2012) Two regression graphs with the same skeleton are Markov equivalent if and only if their sets of collision Vs are identical

Lemma 4

The conditioning set of any independence statement implied by $\boldsymbol{G}^{\boldsymbol{N}}_{\mathbf{reg}}$ for the endpoints of any of its Vs, includes the inner node if it is a transmitting V and excludes the inner node if it is collision V

To make Vs dependence-inducing, we take an edge-minimal regression graph for $\boldsymbol{f_N}$, assume the three equivalent decompositions of a joint dependence and require in addition singleton transitivity

Singleton transitivity. For $\boldsymbol{i, h, k}$ distinct nodes and $\boldsymbol{d \subseteq N \setminus \{i, h, k\}}$

$$\boldsymbol{(i \perp\!\!\!\perp k | d \text{ and } i \perp\!\!\!\perp k | hd) \implies (i \perp\!\!\!\perp h | d \text{ or } k \perp\!\!\!\perp h | d)}$$

Thus, for a conditional independence of $\boldsymbol{Y_i, Y_k}$ given $\boldsymbol{Y_d}$ and given $\boldsymbol{Y_h, Y_d}$ to hold both, there has to be at least one additional independence given $\boldsymbol{Y_c}$ involving $\boldsymbol{Y_h}$

An edge-minimal $\boldsymbol{G^N_{\mathrm{reg}}}$ forms a **dependence base** for $\boldsymbol{f_N}$, generated over it, if singleton transitivity holds (always for $\boldsymbol{f_{g_j|g_{>j}}, f_{g_{>j}}}$ a cut for all $\boldsymbol{j}$)

Proposition 1

Dependence inducing V**s**. For $(\boldsymbol{i}, \mathrm{o}, \boldsymbol{k})$ any V of a dependence base $G^{\boldsymbol{N}}_{\mathbf{reg}}$ and each $\boldsymbol{c} \subseteq \boldsymbol{N} \setminus \{\boldsymbol{i}, \boldsymbol{k}, \mathrm{o}\}$ such that this regression graph implies one of $\boldsymbol{i} \perp\!\!\!\perp \boldsymbol{k}|\boldsymbol{c}$ or $\boldsymbol{i} \perp\!\!\!\perp \boldsymbol{k}|\mathrm{o}\boldsymbol{c}$, the following two equivalent statements hold:

– $(\boldsymbol{i}, \mathrm{o}, \boldsymbol{k})$ forms a collision V $\iff$ $(\boldsymbol{i} \perp\!\!\!\perp \boldsymbol{k}|\boldsymbol{c} \implies \boldsymbol{i} \pitchfork \boldsymbol{k}|\mathrm{o}\boldsymbol{c})$

– $(\boldsymbol{i}, \mathrm{o}, \boldsymbol{k})$ forms a transmitting V $\iff$ $(\boldsymbol{i} \perp\!\!\!\perp \boldsymbol{k}|\mathrm{o}\boldsymbol{c} \implies \boldsymbol{i} \pitchfork \boldsymbol{k}|\boldsymbol{c})$

Thus, in a dependence base $G^{\boldsymbol{N}}_{\mathbf{reg}}$, conditioning on the inner node of a collision **V** and marginalizing over the inner node of transmitting **V** is dependence-inducing for the endpoints of the **V** given any appropriate $\boldsymbol{c}$

Definition 2

Traceable regressions. For $\{\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}\}$ partitioning $\boldsymbol{N}$, we say $\boldsymbol{f_N}$ results from traceable regressions if

1. it could have been generated over a dependence base regression graph, $G^{\boldsymbol{N}}_{\mathbf{reg}}$,
2. it has the three equivalent decompositions of the joint independence $\boldsymbol{b} \perp\!\!\!\perp \boldsymbol{ac}|\boldsymbol{d}$
3. dependence-inducing V's of $G^{\boldsymbol{N}}_{\mathbf{reg}}$ are also dependence-inducing for $\boldsymbol{f_N}$

Thus, traceable regression behave like regular Gaussian families generated over an edge-minimal $G^{\boldsymbol{N}}_{\mathbf{reg}}$

Next goal:

Obtaining a matrix criterion to decide whether a dependence base $\boldsymbol{G}^{\boldsymbol{N}}_{\mathbf{reg}}$ implies $\boldsymbol{\alpha} \perp\!\!\!\perp \boldsymbol{\beta}|\boldsymbol{c}$ or $\boldsymbol{\alpha} \pitchfork \boldsymbol{\beta}|\boldsymbol{c}$ for partitioning

We use **edge matrix representation of $\boldsymbol{G}^{\boldsymbol{N}}_{\mathbf{reg}}$** : adjacency matrices with ones added along the diagonal so that sums of products of submatrices become well-defined

First task:

Given $\boldsymbol{N} = (\boldsymbol{u}, \boldsymbol{v})$ and the edge matrices of $\boldsymbol{G}^{\boldsymbol{N}}_{\mathbf{reg}}$ for $\boldsymbol{f_N} = \boldsymbol{f_{u|v} f_v}$ find the implied edge-matrices for another split $\boldsymbol{N} = (\boldsymbol{a}, \boldsymbol{b})$ with $\boldsymbol{a} = \boldsymbol{\alpha} \cup \boldsymbol{m}$, $\boldsymbol{b} = \boldsymbol{\beta} \cup \boldsymbol{c}$ and $\boldsymbol{G}^{\boldsymbol{N}-\boldsymbol{a}|\boldsymbol{b}}_{\mathbf{reg}}$ for $\boldsymbol{f_N} = \boldsymbol{f_{a|b} f_b}$ having multivariate regression of $\boldsymbol{Y_a}$ on $\boldsymbol{Y_b}$ and a concentration graph for $\boldsymbol{Y_b}$

Regression graphs have three types of edge sets, $\boldsymbol{E}_{\prec\!\!-}$, $\boldsymbol{E}_{---}$, and $\boldsymbol{E}_{\text{—}}$

The edge matrix components of $\boldsymbol{G^N_{\mathbf{reg}}}$ are a $\boldsymbol{d_N \times d_N}$ upper block-triangular matrix $\boldsymbol{\mathcal{H}_{NN} = (\mathcal{H}_{ik})}$ such that

$$\boldsymbol{\mathcal{H}_{ik}} = \begin{cases} \mathbf{1} & \text{if and only if } \boldsymbol{i \prec\!\!-\!\!-\, k} \text{ or } \boldsymbol{i \text{ —— } k} \text{ in } \boldsymbol{G^N_{\mathbf{reg}}} \text{ or } \boldsymbol{i = k,} \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

and a $\boldsymbol{d_u \times d_u}$ symmetric matrix $\boldsymbol{\mathcal{W}_{uu} = (\mathcal{W}_{ik})}$ such that

$$\boldsymbol{\mathcal{W}_{ik}} = \begin{cases} \mathbf{1} & \text{if and only if } \boldsymbol{i\text{---}k} \text{ in } \boldsymbol{G^N_{\mathbf{reg}}} \text{ or } \boldsymbol{i = k,} \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

where, $\boldsymbol{E}_{---}$ corresponds to $\boldsymbol{\mathcal{W}_{uu}}$, $\boldsymbol{E}_{\text{—}}$ to $\boldsymbol{\mathcal{H}_{vv}}$, and $\boldsymbol{E}_{\prec\!\!-}$ to $\boldsymbol{\mathcal{H}_{uN}}$ ($\boldsymbol{\mathcal{W}_{uv} = 0}$, $\boldsymbol{\mathcal{W}_{vu} = 0}$, $\boldsymbol{\mathcal{W}_{vv} = \mathcal{H}_{vv}}$)

Example

For a Gaussian family in a mean-centered $\boldsymbol{Y_N}$ generated over $\boldsymbol{G^N_{\mathrm{reg}}}$ with just two concurrent response sets $\boldsymbol{a, b}$, the parameter matrices are for

$$\boldsymbol{H_{NN}Y_N = \varepsilon_N, \quad \mathrm{cov}(\varepsilon_N) = W_{NN}}$$

$$\boldsymbol{H_{NN}} = \begin{pmatrix} \boldsymbol{I_{aa}} & -\boldsymbol{\Pi_{a|b.v}} & -\boldsymbol{\Pi_{a|v.b}} \\ \boldsymbol{0_{ba}} & \boldsymbol{I_{bb}} & -\boldsymbol{\Pi_{b|v}} \\ \boldsymbol{0_{va}} & \boldsymbol{0_{vb}} & \boldsymbol{\Sigma^{vv.ab}} \end{pmatrix} \quad \boldsymbol{W_{NN}} = \begin{pmatrix} \boldsymbol{\Sigma_{aa|bv}} & \boldsymbol{0_{ab}} & \boldsymbol{0_{av}} \\ \boldsymbol{0_{ba}} & \boldsymbol{\Sigma_{bb|v}} & \boldsymbol{0_{bv}} \\ \boldsymbol{0_{va}} & \boldsymbol{0_{vb}} & \boldsymbol{\Sigma^{vv.ab}} \end{pmatrix}$$

where the Yule-Cochran notation is used: $\boldsymbol{\Pi_{a|bv} = (\Pi_{a|b.v}\ \Pi_{a|v.b})}$;
edge matrices $\boldsymbol{\mathcal{H}_{NN}}$, $\boldsymbol{\mathcal{W}_{NN}}$ implicitly define such Gaussian families

Partial closure

The edge matrix calculus of Wermuth, Wiedenbeck and Cox (2006) uses partial closure, denoted by $\mathbf{zer}_{\boldsymbol{a}}(\boldsymbol{\mathcal{F}})$, which operates on all nodes $\boldsymbol{i}$ in $\boldsymbol{a} \subseteq \boldsymbol{N}$ of a symmetric edge matrix $\boldsymbol{\mathcal{F}}$

After a reordering to have node $\boldsymbol{i}$ in position (1,1) of $\tilde{\boldsymbol{\mathcal{F}}}$ and $\boldsymbol{b} = \boldsymbol{N} \setminus \boldsymbol{i}$

$$\mathbf{zer}_{i}\, \tilde{\boldsymbol{\mathcal{F}}} = \mathbf{In}\left[\begin{pmatrix} 1 & \boldsymbol{\mathcal{F}}_{ib} \\ \boldsymbol{\mathcal{F}}_{bi} & \boldsymbol{\mathcal{F}}_{bb} + \boldsymbol{\mathcal{F}}_{bi}\boldsymbol{\mathcal{F}}_{ib} \end{pmatrix}\right]$$

is seen to **close, by an edge, every V with inner node $\boldsymbol{i}$**

Basic properties of partial closure

Partial closure is

(i) commutative

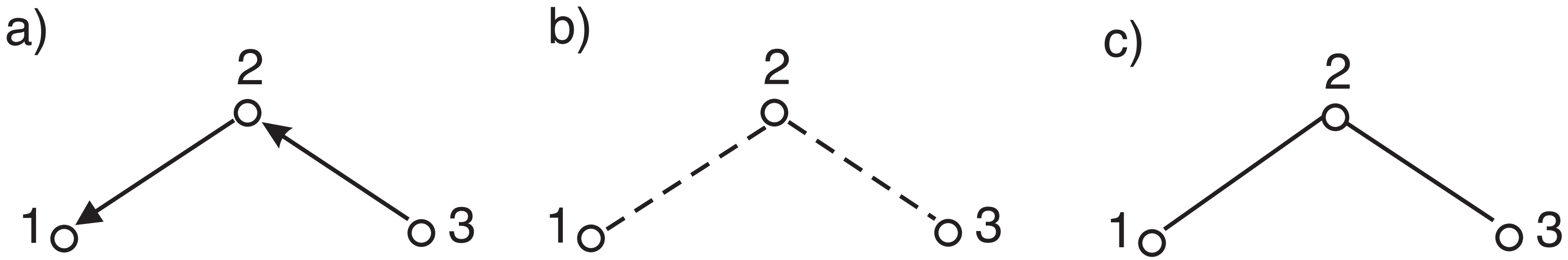
(ii) cannot be undone and

(iii) is exchangeable with selecting a submatrix

Lemma 5

Partial closure applied to $G^N_{\mathbf{reg}}$. For $N = (a, b)$, the transformation $\mathcal{K}_{NN} = \mathrm{zer}_a(\mathcal{H}_{NN})$ closes each a-line anterior path and $\mathcal{Q}_{uu} = \mathrm{zer}_b(\mathcal{W}_{uu})$ each dashed, b-line collision path

Examples of three dependence base, 3-node graphs



Active path (1,2,3) induces in a) $\mathbf{1} \pitchfork \mathbf{3}$, in b) $\mathbf{1} \pitchfork \mathbf{3}|\mathbf{2}$, and in c) $\mathbf{1} \pitchfork \mathbf{3}$

By letting the edge induced by the three V ’s **‘remember the type of edge at the path endpoints’** the induced edges become in

a) $\mathbf{1} \prec\!\!\!-\!\!\!- \mathbf{3},$ b) $\mathbf{1} - - - \mathbf{3},$ c) $\mathbf{1} \relbar\!\!\!\relbar\!\!\!\relbar \mathbf{3}$

For $N=(a,b)$, o_a nodes in a, o_b nodes in b and i, k the endpoints of paths that are active for $G^{N-a|b}_{\mathbf{reg}}$, there remain three types of active ik-path given b in the graph having edge matrices $\mathcal{K}_{NN}$ and $\mathcal{Q}_{uu}$:

$$i \prec\!\!\!-\, \mathrm{o}_a \text{---} \mathrm{o}_b \prec\!\!\!-\, k, \quad i \prec\!\!\!-\, \mathrm{o}_a \text{---} \mathrm{o}_a -\!\!\!\succ k, \quad i -\!\!\!\succ \mathrm{o}_b \text{---} \mathrm{o}_b \prec\!\!\!-\, k$$

Proposition 2

The active path remaining in $\mathcal{K}_{NN} = \mathrm{zer}_a(\mathcal{H}_{NN})$, $\mathcal{Q}_{uu} = \mathrm{zer}_b(\mathcal{W}_{uu})$ for $G^{N-a|b}_{\mathbf{reg}}$ are closed with the induced edge matrices $\mathcal{P}_{a|b}$, $\mathcal{S}_{aa|b}$, $\mathcal{S}^{bb}$

$$\mathcal{P}_{a|b} = \mathrm{In}[\mathcal{K}_{ab} + \mathcal{K}_{aa}\mathcal{Q}_{ab}\mathcal{K}_{bb}]$$

$$\mathcal{S}_{aa|b} = \mathrm{In}[\mathcal{K}_{aa}\mathcal{Q}_{aa}\mathcal{K}^{\mathrm{T}}_{aa}], \quad \mathcal{S}^{bb.a} = \mathrm{In}[\mathcal{H}^{\mathrm{T}}_{bb}\mathcal{Q}_{bb}\mathcal{H}_{bb}]$$

After remembering the types of edge at the path endpoints, we have with

$\boldsymbol{\mathcal{P}_{a|b}}$ an induced bipartite graph of arrows pointing from $\boldsymbol{b}$ to $\boldsymbol{a}$

$\boldsymbol{\mathcal{S}_{aa|b}}$ an induced covariance graph

$\boldsymbol{\mathcal{S}^{bb.a}}$ an induced concentration graph

Lemma 6

Edge matrices induced by $\boldsymbol{G^N_{\mathrm{reg}}}$ for $\boldsymbol{f_{\alpha\beta|c}}$. The subgraph induced by nodes $\boldsymbol{\alpha} \cup \boldsymbol{\beta}$ in $\boldsymbol{G^{N-a|b}_{\mathrm{reg}}}$ captures the independence implications of $\boldsymbol{G^N_{\mathrm{reg}}}$ for $\boldsymbol{f_{\alpha|\beta c} f_{\beta|c}}$ with multivariate regression of $\boldsymbol{Y_\alpha}$ on $\boldsymbol{Y_\beta, Y_c}$ and conditional concentration graph for $\boldsymbol{Y_\beta}$ given $\boldsymbol{Y_c}$

This subgraph has induced edge matrices

$$\boldsymbol{\mathcal{P}_{\alpha|\beta.c} = [\mathcal{P}_{a|b}]_{\alpha,\beta} \;\; \mathcal{S}_{\alpha\alpha|b} = [\mathcal{S}_{aa|b}]_{\alpha,\alpha} \;\; \mathcal{S}^{\beta\beta.a} = [\mathcal{S}^{bb.a}]_{\beta\beta}}$$

Proposition 3

Edge criteria for implied independences and dependences

A dependence base $G^N_{\mathbf{reg}}$ implies $\boldsymbol{\alpha} \perp\!\!\!\perp \boldsymbol{\beta}|\boldsymbol{c}$ if $\boldsymbol{\mathcal{P}}_{\alpha|\beta.c} = \mathbf{0}$ and it implies $\boldsymbol{\alpha} \pitchfork \boldsymbol{\beta}|\boldsymbol{c}$ if $\boldsymbol{\mathcal{P}}_{\alpha|\beta.c} \neq \mathbf{0}$

Corollary

The transformations of $G^N_{\mathbf{reg}}$ to get $\boldsymbol{\mathcal{P}}_{\alpha|\beta.c}$ use implicitly set transitivity since edges may be introduced but never removed

For $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d}$ disjoint subsets of index set $\boldsymbol{N}$, **set transitivity** means

$$(\boldsymbol{a} \perp\!\!\!\perp \boldsymbol{b}|\boldsymbol{d} \text{ and } \boldsymbol{a} \perp\!\!\!\perp \boldsymbol{b}|\boldsymbol{cd}) \implies (\boldsymbol{a} \perp\!\!\!\perp \boldsymbol{c}|\boldsymbol{d} \text{ or } \boldsymbol{b} \perp\!\!\!\perp \boldsymbol{c}|\boldsymbol{d})$$

Thus, the implications of the graph for a generated family ignores path cancellations, that are possible for a member

Most recent relevant work

Sadeghi and Lauritzen (2012), submitted and http://arxiv.org/abs/1109.5909

Wermuth (2011) Bernoulli

Wermuth (2012) submitted and http://arxiv.org/abs/1110.1986

Wermuth and Sadeghi (2012), to appear as invited discussion paper in TEST

A regular Gaussian family violating set transitivity. For $\boldsymbol{N = (u, v)}$, let $\boldsymbol{Y_u}$ and $\boldsymbol{Y_v}$ be mean-centered vector variables with a joint Gaussian distribution. Let them have equal dimension, $\boldsymbol{d_u}$, the components of $\boldsymbol{Y_u}$ and of $\boldsymbol{Y_u}$ be mutually independent and all elements in the covariance matrix $\mathbf{cov}(\boldsymbol{Y_u, Y_v}) = \boldsymbol{\Sigma_{uv}}$ be nonzero, then

$$\mathbf{cov}(\boldsymbol{Y_u}) = \boldsymbol{\Sigma_{uu}} \text{ diagonal}, \quad \mathbf{cov}(\boldsymbol{Y_v}) = \boldsymbol{\Sigma_{vv}} \text{ diagonal}$$

Let further the components of $\boldsymbol{Y_v}$ have equal variances $\boldsymbol{\omega > 1}$ and the equal variances of the components $\boldsymbol{Y_u}$ be $\boldsymbol{\kappa > \omega + 1}$. Whenever in the described situation $\boldsymbol{\Sigma_{uv}}$ is orthogonal, then also

$$\mathbf{cov}(\boldsymbol{Y_u | Y_v}) = \boldsymbol{\Sigma_{uu|v}} \text{ diagonal}, \quad \mathbf{cov}(\boldsymbol{Y_v | Y_u}) = \boldsymbol{\Sigma_{vv|u}} \text{ diagonal}$$