Keen model with Erlang distributed delay

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Keen Model

With capital assets being driven by

$$\dot{K} = \kappa(\pi_n)Y - \delta K \tag{1}$$

we get the following dynamical system

$$\dot{\omega} = \omega(\Phi(\lambda) - \alpha) \tag{2}$$

$$\dot{\lambda} = \lambda\left(\frac{\kappa(\pi_n)}{\nu} - \alpha - \beta - \delta\right) \tag{3}$$

$$\dot{d} = \kappa(\pi_n) - \pi_n - d\left(\frac{\kappa(\pi_n)}{\nu} - \delta\right) \tag{4}$$

Introducing the delay

Capital assets should be delayed from the moment of investment:

$$\dot{K}(t) = \kappa(\pi_n(t-\tau))Y(t-\tau) - \delta K(t) \tag{5}$$

To avoid complications related to Delayed-Differential Equations, we introduce investment stages:

$$\begin{aligned}
\dot{\Theta}_1 &= \kappa(\pi_n)Y - \frac{n}{\tau}\Theta_1 \\
\dot{\Theta}_2 &= \frac{n}{\tau}(\Theta_1 - \Theta_2) \\
&\vdots \\
\dot{\Theta}_n &= \frac{n}{\tau}(\Theta_{n-1} - \Theta_n) \\
\dot{K} &= \frac{n}{\tau}\Theta_n - \delta K \\
\dot{D} &= (\kappa(\pi_n) - \pi_n)\,Y
\end{aligned} \tag{6}$$

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\end{aligned} \tag{6}$$

- Dollar travels between the investment substages taking an exponential (with mean $\frac{\tau}{n}$) time in each of them.
- To see this, suppose that during investment stage k, the only process occurring was transference to stage $k+1$. $\dot{\Theta}_k$ would then be

$$d\Theta_k/dt = -\frac{n}{\tau}\Theta_k \tag{7}$$

that is, if we start with \$$M$ dollars at time 0, \$$Me^{-\frac{n}{\tau}t}$ will remain there at time t.

- Still confused? If we start with \$$M$ at time 0, and dollars (or cents!) leave at an exponentially distributed time, at time t we can expect to still have

$$\$M.\mathbb{P}[exp.dist.r.v. > t] = \$M(1 - F(t)^1) = \$Me^{-\frac{n}{\tau}t} \tag{8}$$

[1] F(t) is the CDF for the exponentially distributed random variable describing the waiting time.

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- The total time it takes for each dollar invested then follows an Erlang distribution with shape parameter n and rate $\frac{n}{\tau}$, which has mean τ and variance τ^2/n.

$$\left(X_i \sim \text{Exponential}(n/\tau) \implies \sum_{i=1}^{n} X_i \sim \text{Erlang}(n, n/\tau)\right) \tag{9}$$

- In the limit $n \to \infty$, the distribution converges to a deterministic time delay of τ, which represents the Delayed-Differential Equation we tried to avoid.

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Dividing the Θ_k variables by Y, $\theta_k = \Theta_k / Y$, we can derive the $n+3$-dimensional system

$$
\begin{aligned}
\dot{\omega} &= \omega(\Phi(\lambda) - \alpha) \\
\dot{\lambda} &= \lambda \left(\frac{n}{\tau\nu}\theta_n - (\alpha + \beta + \delta) \right) \\
\dot{d} &= \kappa(\pi_n) - \pi_n - d \left(\frac{n}{\tau\nu}\theta_n - \delta \right) \\
\dot{\theta}_1 &= \kappa(\pi_n) - \theta_1 \left[\frac{n}{\tau} \left(1 + \frac{1}{\nu}\theta_n \right) - \delta \right] \\
\dot{\theta}_2 &= \frac{n}{\tau}(\theta_1 - \theta_2) - \theta_2 \left(\frac{n}{\tau\nu}\theta_n - \delta \right) \\
&\vdots \\
\dot{\theta}_k &= \frac{n}{\tau}(\theta_{k-1} - \theta_k) - \theta_k \left(\frac{n}{\tau\nu}\theta_n - \delta \right) \\
&\vdots \\
\dot{\theta}_n &= \frac{n}{\tau}(\theta_{n-1} - \theta_n) - \theta_n \left(\frac{n}{\tau\nu}\theta_n - \delta \right)
\end{aligned}
\tag{10}
$$

The "good" equilibrium has become

$$
\begin{aligned}
\hat{\lambda}_1 &= \Phi^{-1}(\alpha) \\
\hat{\theta}_{n,1} &= \frac{\tau\nu}{n}(\alpha+\beta+\delta) \\
&\vdots \\
\hat{\theta}_{n-k,1} &= \hat{\theta}_{n,1}\left[\frac{\tau}{n}(\alpha+\beta+n/\tau)\right]^{k} \\
&\vdots \\
\hat{\theta}_{1,1} &= \hat{\theta}_{n,1}\left[\frac{\tau}{n}(\alpha+\beta+n/\tau)\right]^{n-1} \\
\hat{\pi}_{n,1} &= \kappa^{-1}\left[\hat{\theta}_{1,1}(\alpha+\beta+n/\tau)\right] \\
\hat{d}_1 &= \frac{\kappa(\hat{\pi}_{n,1})-\hat{\pi}_{n,1}}{\alpha+\beta} \\
\hat{\omega}_1 &= 1-\hat{\pi}_{n,1}-r\hat{d}_1
\end{aligned}
\tag{11}
$$

The Jacobian matrix for the linearized system at this equilibrium is

$$\begin{bmatrix}
0 & \bar{\omega}\Phi'(\bar{\lambda}) & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & \frac{n}{\tau\nu}\bar{\lambda} \\
1-\kappa'(\bar{\pi}_n) & 0 & \begin{matrix} r - r\kappa'(\bar{\pi}_n) \\ -(\alpha+\beta) \end{matrix} & 0 & 0 & \dots & 0 & 0 & -\frac{n}{\tau\nu}\bar{d} \\
-\kappa'(\bar{\pi}_n) & 0 & -r\kappa'(\bar{\pi}_n) & \begin{matrix} -\alpha-\beta \\ -\frac{n}{\tau} \end{matrix} & 0 & \dots & 0 & 0 & -\frac{n}{\tau\nu}\bar{\theta}_1 \\
0 & 0 & 0 & \frac{n}{\tau} & \begin{matrix} -\alpha-\beta \\ -\frac{n}{\tau} \end{matrix} & \dots & 0 & 0 & -\frac{n}{\tau\nu}\bar{\theta}_2 \\
\vdots & \vdots & \vdots & & & \ddots & & & \vdots \\
0 & 0 & 0 & 0 & 0 & \dots & \frac{n}{\tau} & \begin{matrix} -\alpha-\beta \\ -\frac{n}{\tau} \end{matrix} & -\frac{n}{\tau\nu}\bar{\theta}_{n-1} \\
0 & 0 & 0 & 0 & 0 & \dots & 0 & \frac{n}{\tau} & \begin{matrix} -\frac{n}{\tau}-2\alpha \\ -2\beta-\delta \end{matrix}
\end{bmatrix} \tag{12}$$

Armed with the Jacobian, we can investigate when stability is lost for each n, in terms of τ.

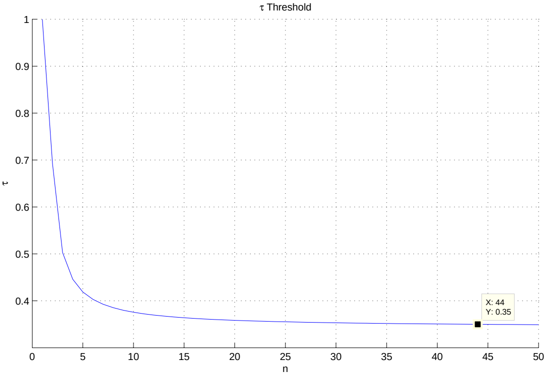

Figure 1: Stability threshold value for τ as a function of n

Using XPPAUT, we verified that the there is a **supercritical** Hopf bifurcation for τ larger than the threshold seen on Figure1. The stable equilibrium point unfolds in a stable cycle, while the equilibrium point loses its local stability, Figure 2.

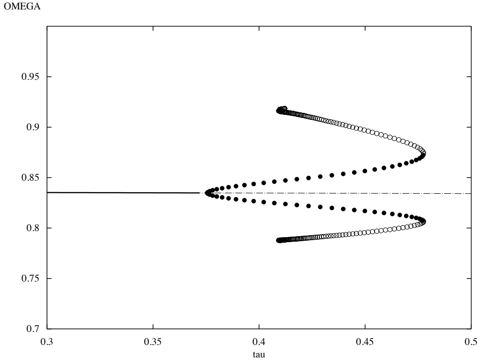

Figure 2: Supercritical Hopf bifurcation for $n = 10$

Simulations

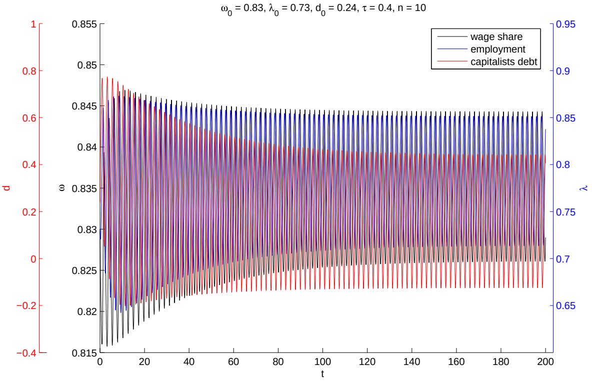

Figure 3: Solution converging to the stable cycle, $n = 10$

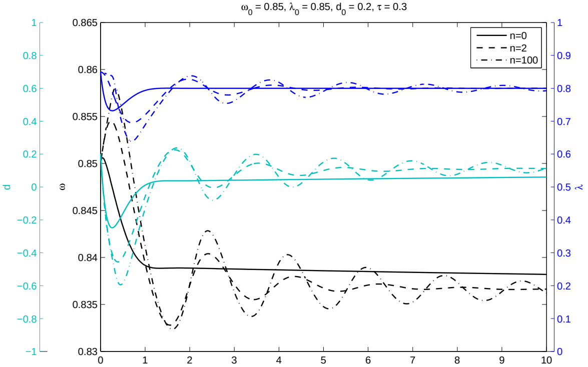

Figure 4: Solutions for different values of n

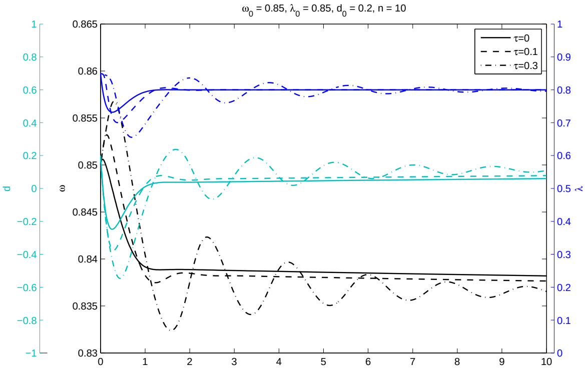

Figure 5: Solutions for different values of τ