A dichotomy for expansions of the real field

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Thank you!

- The Fields Institute for hosting the thematic program on *O-minimal Structures and Real Analytic Geometry*,
- the Deutscher Akademischer Austausch Dienst for funding my stay, and
- above all, the organizers for running such a fantastic program.

I am very fortunate that this excellent program came at such a great time for me.

Setting

Let $\mathcal{X}$ be a collection of subsets of $\mathbb{R}^n$. Let $\mathcal{R} = (\mathbb{R}, +, \cdot, (X)_{X\in\mathcal{X}})$ be an expansion of $(\mathbb{R}, +, \cdot)$. We want to study the definable sets in $\mathcal{R}$.

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Question

What can be said about $\mathcal{R}$ in general if $\mathcal{R}$ does not define $\mathbb{Z}$? In particular, is there anything (geometrically) that can be said about the sets definable in $\mathcal{R}$ (without further assumptions on $\mathcal{R}$)?

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A result from Fields

Dichotomy - H.

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An application - Spirals

Consider a logarithmic spiral

$$\mathbb{S}_\omega := \{(e^t \cos \omega t, e^t \sin \omega t) : t \in \mathbb{R}\}.$$

The expansion $(\mathbb{R}, +, \cdot, \mathbb{S}_\omega)$ is tame. But it defines an infinite discrete set

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An application - More spirals

Let $U \subseteq \mathbb{R}^2$ be open and connected and let $F : U \to \mathbb{R}^2$ be a vector field with an isolated singularity at the origin. Let Γ be a nontrivial trajectory of F; that is the image of a map $\gamma : (0,1] \to \mathbb{R}^2$ such that

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If the eigenvalues of the Jacobian at the origin are imaginary, then

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Minkowski dimension

Given $E \subseteq \mathbb{R}^n$ bounded and $r > 0$, let $N(E, r)$ be the number of closed balls of radius r needed to cover E. Put

$$\overline{\dim}_{\mathrm{M}} E = \varlimsup_{r \downarrow 0} \log N(E, r) / \log(1/r),$$

(with $\log 0 := -\infty$), the **upper Minkowski dimension** of E. We say that E is **M-null** if $\overline{\dim}_{\mathrm{M}} E \leq 0$.

There are many equivalent formulations and different names, in particular, $\overline{\dim}_{\mathrm{M}}$ is also known as upper box-counting dimension.

Two examples

Minkowski dimension distinguishes between countable sets:

$$\overline{\dim}_{\mathrm{M}}\{\frac{1}{n+1} : n \in \mathbb{N}\} = \frac{1}{2},$$

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Dichotomy - Fornasiero-H.-Miller - (PAMS, to appear)

Let $\mathcal{R}$ be an expansion of $(\mathbb{R}, +, \cdot)$ such that $\mathcal{R}$ does not define $\mathbb{Z}$. Then every bounded nowhere dense definable subset of $\mathbb{R}$ is M-null.

Strategy

Let $\mathcal{R}$ be an expansion of $(\mathbb{R}, +, \cdot)$ such that $\mathcal{R}$ defines a set $E \subseteq \mathbb{R}$ such that E is nowhere dense, but *not* M-null.

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There is a discrete set $D \subseteq \mathbb{R}$ such that $\mathcal{R}$ defines a map $f : D \longrightarrow E$ such that $f(D) = E$.

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Conclude that $\mathbb{Z}$ is definable, since there is a function $D^m \to \mathbb{R}$ whose image is dense in $\mathbb{R}$.

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Lemma

Let $E \subseteq \mathbb{R}$ be bounded. If $\overline{\dim}_{\mathrm{M}} E > 0$, then there exist $n \in \mathbb{N}$ and linear $T\colon \mathbb{R}^n \to \mathbb{R}$ such that $Q(T(E^n))$ is dense in $\mathbb{R}$, where

$$Q(X) := \{\frac{x_1 - x_2}{x_3 - x_4} : x_1, x_2, x_3, x_4 \in X, x_3 \neq x_4\}$$

Proof of Step 2

Since $\overline{\dim}_{\mathrm{M}} E^n = n\,\overline{\dim}_{\mathrm{M}} E$, $\lim_{n\to\infty} \overline{\dim}_{\mathrm{M}} E^n = \infty$. By Falconer and Howroyd, there exist $n \in \mathbb{N}$ and a linear $T : \mathbb{R}^n \to \mathbb{R}$ such that $\overline{\dim}_{\mathrm{M}} T(E^n) > 1/2$.

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Proof of Step 2

Since $\overline{\dim}_{\mathrm{M}} E^n = n\,\overline{\dim}_{\mathrm{M}} E$, $\lim_{n\to\infty} \overline{\dim}_{\mathrm{M}} E^n = \infty$. By Falconer and Howroyd, there exist $n \in \mathbb{N}$ and a linear $T\colon \mathbb{R}^n \to \mathbb{R}$ such that $\overline{\dim}_{\mathrm{M}} T(E^n) > 1/2$. By replacing E with $T(E^n)$, it suffices to consider the case that $\overline{\dim}_{\mathrm{M}} E > 1/2$ and show that $Q(E)$ is dense in $\mathbb{R}$. Suppose not.Observe that $Q(E)$ is the set of slopes of nonvertical lines connecting pairs of points in E^2. Thus, the difference set $\{\, u - v : u, v \in E^2 \,\}$ of E^2 is disjoint from some open double cone $C \subseteq \mathbb{R}^2$ centered at the origin. Let ℓ be the line through the origin perpendicular to the axis of C. Then the restriction to E^2 of the projection of $\mathbb{R}^2$ onto ℓ is injective, and the compositional inverse is Lipshitz. Hence, E^2 is contained in a rotation of the graph of a Lipshitz function from some bounded subinterval of $\mathbb{R}$ into $\mathbb{R}$. It follows that $\overline{\dim}_{\mathrm{M}} E^2 \le 1$. But then $\overline{\dim}_{\mathrm{M}} E = (\overline{\dim}_{\mathrm{M}} E^2)/2 \le 1/2$, a contradiction.

Conjecture

Let $\mathcal{R}$ be an expansion of $(\mathbb{R},+,\cdot)$ that does not define $\mathbb{Z}$. Let $X \subseteq \mathbb{R}^n$ be definable in $\mathcal{R}$. Then $\overline{\dim}_{\mathrm{M}} \overline{X} = \dim \overline{X}$.

Conjecture

Let $\mathcal{R}$ be an expansion of $(\mathbb{R},+,\cdot)$ by a spiralling trajectory Γ of an o-minimal vector field and $\mathcal{R}$ does not define $\mathbb{Z}$. Then $\overline{\dim}_{\mathrm{M}} \Gamma = 1$ and the length of Γ is finite.

Optimality

Dichotomy - Fornasiero-H.-Miller

Let $\mathcal{R}$ be an expansion of $(\mathbb{R}, +, \cdot)$ such that $\mathcal{R}$ does not define $\mathbb{Z}$. Then every bounded definable subset of $\mathbb{R}$ is either somewhere dense or M-null.

M-null

There are Cantor sets $K \subseteq \mathbb{R}$ such that $(\mathbb{R}, +, \cdot, K)$ defines sets in every projective level, yet every subset of $\mathbb{R}$ definable in $(\mathbb{R}, +, \cdot, K)$ either has interior or is nowhere dense (Friedman, Miller, Kurdyka, Speissegger).

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Somewhere dense

$\mathbb{Z}$ is not definable in the expansion of $(\mathbb{R}, +, \cdot)$ by $\{(2^j, 2^k 3^l) : j, k, l \in \mathbb{Z}\}$ (Günaydın), yet it evidently defines both an infinite discrete set and a dense subset of $\mathbb{R}^{>0}$ that has empty interior.

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