The Constraint Satisfaction Problem

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Sudoku

	6	3		4	5		2	9
5				9	2		1	8
			6					
	8			7	9	1		
							8	2
		2				4	6	
2								1
3		6			8	5		
			1		3		7	

Sudoku

x_{11}	6	3	x_{14}	4	5	x_{17}	2	9
5	x_{22}	x_{23}	x_{24}	9	2	x_{27}	1	8
x_{31}	x_{32}	x_{33}	6	x_{35}	x_{36}	x_{37}	x_{38}	x_{39}
x_{41}	8	x_{43}	x_{44}	7	9	1	x_{48}	x_{49}
x_{51}	x_{52}	x_{53}	x_{54}	x_{55}	x_{56}	x_{57}	8	2
x_{61}	x_{62}	2	x_{64}	x_{65}	x_{66}	4	6	x_{69}
2	x_{72}	x_{73}	x_{74}	x_{75}	x_{76}	x_{77}	x_{78}	1
3	x_{82}	6	x_{84}	x_{85}	8	5	x_{88}	x_{89}
x_{91}	x_{92}	x_{93}	1	x_{95}	3	x_{97}	7	x_{99}

Sudoku

An alternate formulation as a decision problem

Is there a way to assign elements from $\{1,2,3,4,5,6,7,8,9\}$ to each variable x_{ij} so that for each i:

$$\begin{aligned}
(x_{i1}, x_{i2}, \ldots, x_{i9}) &\in Sym\{1,2,3,4,5,6,7,8,9\} \\
(x_{1i}, x_{2i}, \ldots, x_{9i}) &\in Sym\{1,2,3,4,5,6,7,8,9\} \\
(x_{11}, x_{12}, \ldots, x_{33}) &\in Sym\{1,2,3,4,5,6,7,8,9\} \\
(x_{14}, x_{15}, \ldots, x_{36}) &\in Sym\{1,2,3,4,5,6,7,8,9\} \\
&\vdots \\
(x_{77}, x_{78}, \ldots, x_{99}) &\in Sym\{1,2,3,4,5,6,7,8,9\} \\
x_{12} &\in \{6\} \\
x_{13} &\in \{3\} \\
&\vdots
\end{aligned}$$

Boolean Satisfiability

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$$\Phi = x_1 \wedge (x_3 \vee x_4) \wedge \neg x_4 \wedge (\neg x_1 \vee x_2) \wedge (\neg x_1 \vee \neg x_3 \vee x_4)$$

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An Equivalent Formulation

Is there a way to assign values from $\{0, 1\}$ to the variables x_1, x_2, x_3, x_4 so that:

$$\begin{aligned} x_1 &\in \{1\} \qquad x_4 \in \{0\} \\ (x_3, x_4) &\in \{(0,1), (1,0), (1,1)\} \\ (x_1, x_2) &\in \{(0,0), (0,1), (1,1)\} \\ (x_1, x_3, x_4) &\in \{(0,0,0), (0,0,1), (0,1,0), (0,1,1) \\ &\quad (1,0,0), (1,0,1), (1,1,1)\} \end{aligned}$$

The Constraint Satisfaction Problem

Instance

A triple $P = (V, A, \mathcal{C})$ with

- V a nonempty, finite set of variables,
- A a nonempty, finite domain,
- $\mathcal{C}$ a set of constraints $\{C_1, \ldots, C_q\}$ where each C_i is a pair $(\vec{s}_i, R_i)$ with
  - $\vec{s}_i$ a tuple of variables of length m_i, called the scope of C_i, and
  - R_i a subset of A^{m_i}, called the constraint relation of C_i.

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Question

Is there a solution to P, i.e., is there a function $f : V \to A$ such that for each $i \leq q$, the m_i-tuple $f(\vec{s}_i) \in R_i$?

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Problem

Identify natural subclasses of the CSP for which there are efficient algorithms for solving them.

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Problem

Identify algebraic properties of sets of constraint relations that can be used to construct good algorithms for solving CSPS over those relations.

Some Constraint Relations

Fact

There is a fast algorithm for solving instances of the CSP whose constraint relations come from the following set:

$$\{(0,5,2),(0,8,2),(0,9,5),(1,8,2),(1,8,5),(4,7,6)\}$$

$$\{(0,2,6),(0,3,2),(0,8,1),(0,9,0),(1,2,2),(1,3,1),(1,3,2),(1,3,5),(4,2,4)\}$$

$$\{(0,0,1),(0,1,3),(0,1,8),(1,1,3),(1,2,1),(1,2,3),(1,5,3),(1,6,1),(4,2,3),(4,4,3),(4,5,3)\}$$

$$\{(5,2,5),(5,2,6),(7,2,0),(7,2,1),(7,2,4),(8,2,2),(9,5,2),(9,6,2)\}$$

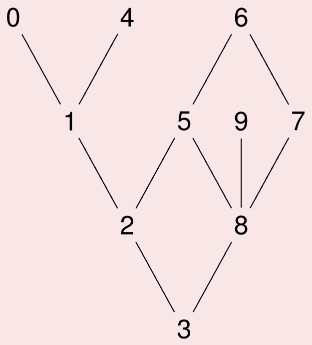
$$\{(5,5,9,1),(7,6,8,8),(8,2,3,3),(8,5,3,2),(8,5,3,3),(8,5,8,3),(9,5,2,2)\}$$

Some Constraint Relations, continued

Hidden Structure

The constraint relations on the previous slide are compatible with the following order and associated binary operation. It follows that the constraint language is tractable.

An Ordering of the Domain



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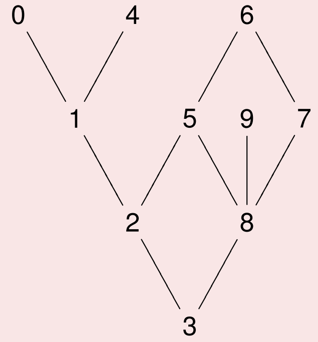

A Compatible Operation

- For $x, y \in \{0, 1, 2, \ldots, 9\}$, define $x \wedge y = \text{g.l.b.}(x, y)$.
- Then for all x, y, z,
  - $x \wedge x = x$,
  - $x \wedge y = y \wedge x$,
  - $x \wedge (y \wedge z) = (x \wedge y) \wedge z$.