The unilateral shift as a Hilbert module over the disc algebra

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In 1989, Douglas and Paulsen reformulated several interesting operator theoretic problems using the language of module theory. This suggested the use of cohomological methods such as extension groups to further the study of problems such as commutant lifting.

A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is said to be *polynomially bounded* if there exists a constant $C > 0$ such that for every polynomial φ, we have

$$\|\varphi(T)\| \leq C\|\varphi\|_\infty$$

where

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The map

$$A(\mathbb{D}) \times \mathcal{H} \to \mathcal{H}$$

$$(\varphi, h) \mapsto \varphi(T)h$$

gives rise to a structure of an $A(\mathbb{D})$-module on $\mathcal{H}$, and we say that $(\mathcal{H}, T)$ is a *Hilbert module*.

Extension groups

Given two Hilbert modules $(\mathcal{H}_1, T_1)$ and $(\mathcal{H}_2, T_2)$, we can consider the extension group $\mathrm{Ext}^1_{A(\mathbb{D})}(T_2, T_1)$, which consists of equivalence classes of exact sequences

$$0 \to \mathcal{H}_1 \to \mathcal{K} \to \mathcal{H}_2 \to 0$$

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Theorem (Carlson-Clark 1995)

Let $(\mathcal{H}_1, T_1)$ and $(\mathcal{H}_2, T_2)$ be Hilbert modules. Then, the group

$$\mathrm{Ext}^1_{A(\mathbb{D})}(T_2, T_1)$$

is isomorphic to $\mathcal{A}/\mathcal{J}$, where $\mathcal{A}$ is the space of operators $X : \mathcal{H}_2 \to \mathcal{H}_1$ for which the operator

$$\begin{pmatrix} T_1 & X \\ 0 & T_2 \end{pmatrix}$$

is polynomially bounded, and $\mathcal{J}$ is the space of operators of the form $T_1 L - L T_2$ for some bounded operator $L : \mathcal{H}_2 \to \mathcal{H}_1$.

An important question in the study of extension groups is that of determining which Hilbert modules $(\mathcal{H}_2, T_2)$ have the property that

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Projective Hilbert modules

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A characterization of projective Hilbert modules has long been sought.

Results of Carlson and Clark

The unilateral shift operator

$$S_{\mathcal{E}} : H^2(\mathcal{E}) \to H^2(\mathcal{E})$$

is defined as

$$(S_{\mathcal{E}} f)(z) = zf(z)$$

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Let $(\mathcal{H}, T)$ be a Hilbert module. Then, an operator $X : \mathcal{H} \to \mathcal{E}$ gives rise to an element $[X] \in \mathrm{Ext}^1_{A(\mathbb{D})}(T, S_{\mathcal{E}})$ if and only if there exists a constant $c > 0$ such that

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We bring the reader's attention to the fact that the group $\operatorname{Ext}^1_{A(\mathbb{D})}(T, S_{\mathcal{E}})$ is really of a "scalar" nature: it consists of elements $[X]$ where the operator $X : \mathcal{H} \to H^2(\mathcal{E})$ has range contained in the constant functions $\mathcal{E}$.

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(iii) *the operator* T *is similar to an isometry.*

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Question Are unilateral shifts projective?

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Indeed, recall that he constructed an operator-valued Hankel matrix $X : H^2(\mathcal{E}) \to H^2(\mathcal{E})$ with the property that

$$R(X) = \begin{pmatrix} S_{\mathcal{E}}^* & X \\ 0 & S_{\mathcal{E}} \end{pmatrix}$$

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Whether or not things are different for finite multiplicities is still an open problem, and is the driving force behind our results.
The difficulty in attacking the main question is two-fold: first we need to exhibit an element $[X] \in \text{Ext}^1_{A(\mathbb{D})}(T, S_{\mathcal{E}}^*)$, and then we need to check whether it is trivial or not.

How to find elements of the extension group

We introduce a special class of operators $X : \mathcal{H}_2 \to \mathcal{H}_1$ on which we will focus.

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Lemma

Let $(\mathcal{H}_1, T_1)$ and $(\mathcal{H}_2, T_2)$ be Hilbert modules. Let $X : \mathcal{H}_2 \to \mathcal{H}_1$ be a bounded operator such that $T_1^N X T_2^N = 0$ for some integer $N \geq 0$. Then, the operator $R : \mathcal{H}_1 \oplus \mathcal{H}_2 \to \mathcal{H}_1 \oplus \mathcal{H}_2$ defined as

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is polynomially bounded.

Note that if $T_1 = S_{\mathcal{E}}^*$, then the condition $S_{\mathcal{E}}^{*N} X = 0$ really says that the range of the operator $X : \mathcal{H} \to H^2(\mathcal{E})$ is contained in the polynomials of degree at most $N - 1$.

We can now define the object appearing in our main result.

Definition

Let $\mathcal{E}$ be a separable Hilbert space. Given two Hilbert modules $(H^2(\mathcal{E}), T_1)$ and $(\mathcal{H}, T_2)$, we define the *polynomial subgroup* $\mathrm{Ext}^1_{\mathrm{poly}}(T_2, T_1)$ of $\mathrm{Ext}^1_{A(\mathbb{D})}(T_2, T_1)$ to be the subgroup of elements $[X]$ such that $S_{\mathcal{E}}^{*N} X T_2^N = 0$ for some integer $N \geq 0$.

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Now, how can we tell when such operators satisfy $[X] = 0$ in $\mathrm{Ext}^1_{A(\mathbb{D})}(T, S_{\mathcal{E}}^*)$?

Characterization

Definition

Let $(\mathcal{H}, T)$ be a Hilbert module and let $\mathcal{E}$ be a separable Hilbert space. We denote by $Z_{\mathcal{E}}(T)$ the subspace of $B(\mathcal{H}, \mathcal{E})$ consisting of the operators $X \in B(\mathcal{H}, \mathcal{E})$ with the property that there exists a constant $c_X > 0$ such that

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By the results of Carlson and Clark, we see that the set $Z_{\mathcal{E}}(T)$ consists exactly of those operators $X : \mathcal{H} \to \mathcal{E}$ which give rise to an element $[X] \in \operatorname{Ext}^1_{A(\mathbb{D})}(T, S_{\mathcal{E}})$.

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Theorem (C., 2013)

Let $S_{\mathcal{E}} : H^2(\mathcal{E}) \to H^2(\mathcal{E})$ be the unilateral shift and let $(\mathcal{H}, T)$ be a Hilbert module. Then

$$B(\mathcal{H}, \mathcal{E})T + Z_{\mathcal{E}}(T) = B(\mathcal{H}, \mathcal{E})$$

if and only if

$$\mathrm{Ext}^1_{\mathrm{poly}}(T, S_{\mathcal{E}}^*) = 0.$$

Functional model

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Let $T \in B(\mathcal{H})$ be a completely non-unitary contraction. Then, there exists a contractive holomorphic function $\Theta \in H^\infty(B(\mathcal{E}, \mathcal{E}_*))$ with the property that T is unitarily equivalent to $S(\Theta)$.

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The ability to work on a function space allows us to obtain the following crucial fact.

Theorem (C., 2013)

Let $\mathcal{F}, \mathcal{F}_*, \mathcal{E}$ be separable Hilbert spaces. Let $\Theta \in H^\infty(B(\mathcal{F}, \mathcal{F}_*))$ be a contractive holomorphic function. Then,

$$B(H(\Theta), \mathcal{E}) = B(H(\Theta), \mathcal{E}) S(\Theta)^* + Z_{\mathcal{E}}(S(\Theta)^*).$$

Idea behind the proof

Focusing on the simple case where $\mathcal{E} = \mathbb{C}$ and $S(\Theta) = S_{\mathbb{C}}$, we need to establish

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The general case is based on this idea.

Main result

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This result might be seen as supporting the idea that the shift is projective. However, notice that it holds regardless of multiplicity.

Remarks on the main result

The theorem illustrates a clear difference between $S_{\mathcal{E}}$ and $S_{\mathcal{E}}^*$ on the level of extension groups:

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We do not know whether equality holds if we require that the shift be of finite multiplicity.

Concluding remarks

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Thank you!