A Krein-Milman type theorem for C*-algebras

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Extreme points for the set of positive operators

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- K convex and compact implies there exists at least one ext. point (Zorn's lemma)

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- study the **extreme points** of S and the **closure of the convex hull** of these extreme points

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- an extreme point of such a set is a *-homomorphisms (E. Stormer 1963)

State space vs Pure states

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- $S(C(X)) = \bar{co}(P(C(X))$

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- $||\xi(f) - \frac{1}{N} \sum \phi_i(f)|| < \epsilon$ for all $f \in F$.

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- note $1(x) \equiv 1 \notin C[0,1]_a$ and $\psi(1(x)) = 1(x)$
  $C[0,1]_a$ is not an algebra but some features can still be used.

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- **Thank You!**