The Choquet boundary of an operator system

Matthew Kennedy
(joint work with Ken Davidson)

Carleton University

May 28, 2013

Operator systems and completely positive maps

Definition

An **operator system** is a unital self-adjoint subspace of a unital C^*-algebra.

Definition

An **operator system** is a unital self-adjoint subspace of a unital C^*-algebra.

For a non-self-adjoint subalgebra (or subspace) $\mathcal{M}$ contained in a unital C^*-algebra, can consider corresponding operator system $\mathcal{S} = \mathcal{M} + \mathcal{M}^* + \mathbb{C}1$.

Definition

For operator systems $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{S}$, a map $\phi : \mathcal{S}_1 \to \mathcal{S}_2$ induces maps $\phi_n : \mathcal{M}_n(\mathcal{S}_1) \to \mathcal{M}_n(\mathcal{S}_2)$ by

$$\phi_n([s_{ij}]) = [\phi(s_{ij})].$$

We say ϕ is **completely positive** if each ϕ_n is positive.

Definition

For operator systems $\mathcal{S}_1, \mathcal{S}_2 \in \mathfrak{S}$, a map $\phi : \mathcal{S}_1 \to \mathcal{S}_2$ induces maps $\phi_n : \mathcal{M}_n(\mathcal{S}_1) \to \mathcal{M}_n(\mathcal{S}_2)$ by

$$\phi_n([s_{ij}]) = [\phi(s_{ij})].$$

We say ϕ is **completely positive** if each ϕ_n is positive.

The collection of operator systems forms a category, **the category of operator systems** $\mathfrak{S}$. The morphisms between operator systems are the completely positive maps. The isomorphisms are the unital completely positive maps with unital completely positive inverse.

Stinespring (1955) introduces the notion of a completely positive map.

> *W.F. Stinespring, Positive functions on C*-algebras, Proceedings of the AMS 6 (1955), No 6, 211–216.*

Stinespring (1955) introduces the notion of a completely positive map.

W.F. Stinespring, Positive functions on C-algebras, Proceedings of the AMS 6 (1955), No 6, 211–216.*

Referenced by Nakamura, Takesaki and Umegaki in 1955, C. Davis in 1958, E. Størmer in 1963, B. Russo and H.A. Dye in 1966.

Stinespring (1955) introduces the notion of a completely positive map.

W.F. Stinespring, Positive functions on C-algebras, Proceedings of the AMS 6 (1955), No 6, 211–216.*

Referenced by Nakamura, Takesaki and Umegaki in 1955, C. Davis in 1958, E. Størmer in 1963, B. Russo and H.A. Dye in 1966.

Arveson (1969/1972) uses completely positive maps as the basis of his work on non-commutative dilation theory and non-self-adjoint operator algebras.

W.B. Arveson, Subalgebras of C-algebras, Acta Math. 123 (1969), 141–224.*

W.B. Arveson, Subalgebras of C-algebras II, Acta Math. 128 (1972), 271–308.*

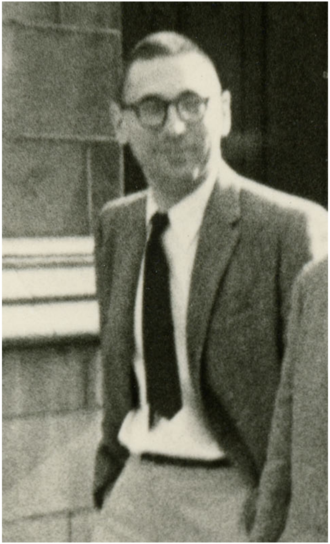
 

Figure: Stinespring's paper and Arveson's series of papers each now have over 1,000 citations. (To put this in perspective, Einstein's paper on Brownian motion has about 800.)

A **dilation** of a UCP (unital completely positive) map $\phi : \mathcal{S} \to \mathcal{B}(H)$ is a UCP map $\psi : \mathcal{S} \to \mathcal{B}(K)$, where $K = H \oplus K'$ and

$$\psi(s) = \begin{pmatrix} \phi(s) & * \\ * & * \end{pmatrix}, \quad \forall s \in \mathcal{S}.$$

A **dilation** of a UCP (unital completely positive) map $\phi : \mathcal{S} \to \mathcal{B}(H)$ is a UCP map $\psi : \mathcal{S} \to \mathcal{B}(K)$, where $K = H \oplus K'$ and

$$\psi(s) = \begin{pmatrix} \phi(s) & * \\ * & * \end{pmatrix}, \quad \forall s \in \mathcal{S}.$$

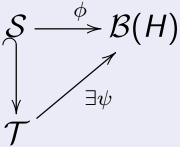
Theorem (Stinespring's dilation theorem)

*Every UCP map $\phi : \mathcal{S} \to \mathcal{B}(H)$ dilates to a *-representation of $\mathrm{C}^*(\mathcal{S})$.*

Arveson's extension theorem is the operator system analogue of the Hahn-Banach theorem.

Theorem (Arveson's Extension Theorem)

If $\phi : \mathcal{S} \to \mathcal{B}(H)$ is CP (completely positive) and $\mathcal{S} \subseteq \mathcal{T}$, then there is a CP map $\psi : \mathcal{T} \to \mathcal{B}(H)$ extending ϕ, i.e.



Boundary representations and the C*-envelope

Arveson's Philosophy

1. View an operator system as a subspace of a canonically determined C*-algebra, but
2. Decouple the structure of the operator system from any particular representation as operators.

Arveson's Philosophy

1. View an operator system as a subspace of a canonically determined C*-algebra, but
2. Decouple the structure of the operator system from any particular representation as operators.

Somewhat analogous to the theory of concrete vs abstract C*-algebras, and concrete von Neumann algebras vs W*-algebras.

If $\phi : \mathcal{S} \to \mathcal{B}$ is an operator system isomorphism on $\mathcal{S}$, then $\phi(\mathcal{S})$ is an isomorphic copy of $\mathcal{S}$. The C*-envelope of $\mathcal{S}$ is the "smallest" C*-algebra generated by an isomorphic copy of $\mathcal{S}$.

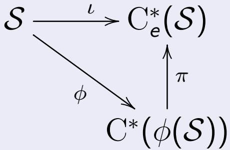
If $\phi : \mathcal{S} \to \mathcal{B}$ is an operator system isomorphism on $\mathcal{S}$, then $\phi(\mathcal{S})$ is an isomorphic copy of $\mathcal{S}$. The C*-envelope of $\mathcal{S}$ is the "smallest" C*-algebra generated by an isomorphic copy of $\mathcal{S}$.

Definition

The **C*-envelope** $\mathrm{C}^*_e(\mathcal{S})$ is the C^*-algebra generated by an isomorphic copy $\iota(\mathcal{S})$ of $\mathcal{S}$ with the following universal property: For every isomorphic copy $\phi(\mathcal{S})$ of $\mathcal{S}$, there is a surjective *-homomorphism

$$\pi : \mathrm{C}^*(\phi(\mathcal{S})) \to \mathrm{C}^*_e(\mathcal{S})$$

such that $\pi \circ j = \iota$, i.e.



Example

Let $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$. The disk algebra is $A(\mathbb{D}) = H^\infty(\mathbb{D}) \cap C(\overline{\mathbb{D}})$. By the maximum modulus principle, the norm on $A(\mathbb{D})$ is completely determined on $\partial\mathbb{D}$. So the restriction map $A(\mathbb{D}) \to C(\partial\mathbb{D})$ is completely isometric. But no smaller space suffices to norm $A(\mathbb{D})$. Hence $\mathrm{C}^*_e(A(\mathbb{D})) = C(\partial\mathbb{D})$.

We need to be able to construct the C*-envelope $\mathrm{C}^*_e(\mathcal{S})$ using only knowledge of $\mathcal{S}$.

We need to be able to construct the C*-envelope $\mathrm{C}^*_e(\mathcal{S})$ using only knowledge of $\mathcal{S}$.

Definition

An irreducible representation $\sigma : \mathrm{C}^*(\mathcal{S}) \to \mathcal{B}(H)$ is a **boundary representation** for $\mathcal{S}$ if the restriction $\sigma\mid_{\mathcal{S}}$ of σ to $\mathcal{S}$ has a *unique* UCP extension.

We need to be able to construct the C*-envelope $\mathrm{C}^*_e(\mathcal{S})$ using only knowledge of $\mathcal{S}$.

Definition

An irreducible representation $\sigma : \mathrm{C}^*(\mathcal{S}) \to \mathcal{B}(H)$ is a **boundary representation** for $\mathcal{S}$ if the restriction $\sigma\,|_{\mathcal{S}}$ of σ to $\mathcal{S}$ has a *unique* UCP extension.

Boundary representations give irreducible representations of $\mathrm{C}^*_e(\mathcal{S})$.

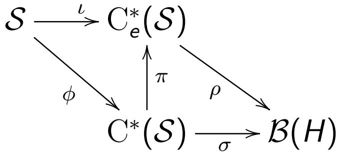
Let $\sigma : \mathrm{C}^*(\mathcal{S}) \to \mathcal{B}(H)$ be a boundary representation. By the universal property of $\mathrm{C}^*_e(\mathcal{S})$ there is an operator system isomorphism $\iota : \mathcal{S} \to \mathrm{C}^*_e(\mathcal{S})$ and a surjective *-homomorphism $\pi : \mathrm{C}^*(\mathcal{S}) \to \mathrm{C}^*_e(\mathcal{S})$.

Let $\sigma : \mathrm{C}^*(\mathcal{S}) \to \mathcal{B}(H)$ be a boundary representation. By the universal property of $\mathrm{C}^*_e(\mathcal{S})$ there is an operator system isomorphism $\iota : \mathcal{S} \to \mathrm{C}^*_e(\mathcal{S})$ and a surjective *-homomorphism $\pi : \mathrm{C}^*(\mathcal{S}) \to \mathrm{C}^*_e(\mathcal{S})$.

We can extend $\sigma \circ \iota \mid_{\mathcal{S}}$ to a UCP map $\rho : \mathrm{C}^*_e(\mathcal{S}) \to \mathcal{B}(H)$. Then $\rho \circ \pi = \sigma$ on $\mathcal{S}$. By the unique extension property, $\rho \circ \pi = \sigma$ on all of $\mathrm{C}^*(\mathcal{S})$. Hence ρ is an irreducible *-representation of $\mathrm{C}^*_e(\mathcal{S})$.

Let $\sigma : \mathrm{C}^*(\mathcal{S}) \to \mathcal{B}(H)$ be a boundary representation. By the universal property of $\mathrm{C}^*_e(\mathcal{S})$ there is an operator system isomorphism $\iota : \mathcal{S} \to \mathrm{C}^*_e(\mathcal{S})$ and a surjective *-homomorphism $\pi : \mathrm{C}^*(\mathcal{S}) \to \mathrm{C}^*_e(\mathcal{S})$.

We can extend $\sigma \circ \iota \mid_{\mathcal{S}}$ to a UCP map $\rho : \mathrm{C}^*_e(\mathcal{S}) \to \mathcal{B}(H)$. Then $\rho \circ \pi = \sigma$ on $\mathcal{S}$. By the unique extension property, $\rho \circ \pi = \sigma$ on all of $\mathrm{C}^*(\mathcal{S})$. Hence ρ is an irreducible *-representation of $\mathrm{C}^*_e(\mathcal{S})$.



If there are enough boundary representations, then we can use them to construct $\mathrm{C}^*_e(\mathcal{S})$ from $\mathcal{S}$.

Theorem (Arveson)

If there are sufficiently many boundary representations $\{\sigma_\lambda\}$ to completely norm $\mathcal{S}$, then letting $\sigma = \oplus \sigma_\lambda$,

$$\mathrm{C}^*_e(\mathcal{S}) = \mathrm{C}^*(\sigma(\mathcal{S})).$$

Example

Let $\mathcal{A} \subseteq C(X)$ be a function system. The irreducible representations of $C(X)$ are the point evaluations δ_x for $x \in X$, which are given by representing measures μ on $\mathcal{A}$,

$$f(x) = \int_X f \, \mathrm{d}\mu, \quad \forall f \in \mathcal{A}.$$

Thus δ_x is a boundary representation for $\mathcal{A}$ if and only if x has a unique representing measure on $\mathcal{A}$. The set of such points is precisely the classical **Choquet boundary** of X with respect to $\mathcal{A}$.

Example

Let $\mathcal{A} \subseteq C(X)$ be a function system. The irreducible representations of $C(X)$ are the point evaluations δ_x for $x \in X$, which are given by representing measures μ on $\mathcal{A}$,

$$f(x) = \int_X f \, \mathrm{d}\mu, \quad \forall f \in \mathcal{A}.$$

Thus δ_x is a boundary representation for $\mathcal{A}$ if and only if x has a unique representing measure on $\mathcal{A}$. The set of such points is precisely the classical **Choquet boundary** of X with respect to $\mathcal{A}$.

Arveson calls the set of boundary representations of an operator system $\mathcal{S}$ the **(non-commutative) Choquet boundary.**

Two big problems

Although Arveson was able to construct boundary representations, and hence the C*-envelope, in some special cases, he was unable to do so in general. The following questions were left unanswered.

Questions

1. Does every operator system have sufficiently many boundary representations?
2. Does every operator system have a C*-envelope?

Choi-Effros (1977) prove an injective operator system is (completely order isomorphic to) a C*-algebra.

Choi-Effros (1977) prove an injective operator system is (completely order isomorphic to) a C*-algebra.

Theorem (Hamana (1979))

Every operator system is contained in a unique minimal injective operator system.

Choi-Effros (1977) prove an injective operator system is (completely order isomorphic to) a C*-algebra.

Theorem (Hamana (1979))

Every operator system is contained in a unique minimal injective operator system.

Corollary

Every operator system has a C-envelope.*

Choi-Effros (1977) prove an injective operator system is (completely order isomorphic to) a C*-algebra.

Theorem (Hamana (1979))

Every operator system is contained in a unique minimal injective operator system.

Corollary

Every operator system has a C-envelope.*

Very difficult to "get your hands on" this construction. Does not give boundary representations.

Muhly-Solel (1998) give a homological characterization of boundary representations (using the notions of orthoprojectivity and orthoinjectivity).

Muhly-Solel (1998) give a homological characterization of boundary representations (using the notions of orthoprojectivity and orthoinjectivity).

Dritschel-McCullough (2005) characterize the unique extension property, clarifying Muhly and Solel's work, and give a new proof of the existence of the C*-envelope. Uses ideas of Agler.

Muhly-Solel (1998) give a homological characterization of boundary representations (using the notions of orthoprojectivity and orthoinjectivity).

Dritschel-McCullough (2005) characterize the unique extension property, clarifying Muhly and Solel's work, and give a new proof of the existence of the C*-envelope. Uses ideas of Agler.

Say a UCP map $\phi : \mathcal{S} \to \mathcal{B}(H)$ is **maximal** if, whenever ψ is a UCP dilation of ϕ, $\psi = \phi \oplus \psi'$. A UCP map is maximal if and only if it has the unique extension property.

Muhly-Solel (1998) give a homological characterization of boundary representations (using the notions of orthoprojectivity and orthoinjectivity).

Dritschel-McCullough (2005) characterize the unique extension property, clarifying Muhly and Solel's work, and give a new proof of the existence of the C*-envelope. Uses ideas of Agler.

Say a UCP map $\phi : \mathcal{S} \to \mathcal{B}(H)$ is **maximal** if, whenever ψ is a UCP dilation of ϕ, $\psi = \phi \oplus \psi'$. A UCP map is maximal if and only if it has the unique extension property.

Theorem (Dritschel-McCullough (2005))

There are maximal representations $\{\sigma_\lambda\}$ such that letting $\sigma = \oplus\sigma_\lambda$,

$$\mathrm{C}^*_e(\mathcal{S}) = \mathrm{C}^*(\sigma(\mathcal{S})).$$

Arveson (1998) publishes the third paper in his series.

W.B. Arveson, Subalgebras of C-algebras III: Multivariable operator theory, Acta Math. 181 (1998), 159–228.*

Arveson (1998) publishes the third paper in his series.

W.B. Arveson, Subalgebras of C-algebras III: Multivariable operator theory, Acta Math. 181 (1998), 159–228.*

Arveson (2008) returns to the questions he raised in 1969.

W.B. Arveson, The noncommutative Choquet boundary, Journal of the AMS 21 (2008), No. 4, 1065–1084.

Arveson (1998) publishes the third paper in his series.

W.B. Arveson, Subalgebras of C-algebras III: Multivariable operator theory, Acta Math. 181 (1998), 159–228.*

Arveson (2008) returns to the questions he raised in 1969.

W.B. Arveson, The noncommutative Choquet boundary, Journal of the AMS 21 (2008), No. 4, 1065–1084.

Gives a new proof of Dritschel-McCullough's results using ideas of Ozawa. Using an intricate direct integral argument, shows that when $\mathcal{S}$ is separable, a maximal representation is a.e. an integral of boundary representations.

Arveson (1998) publishes the third paper in his series.

W.B. Arveson, Subalgebras of C-algebras III: Multivariable operator theory, Acta Math. 181 (1998), 159–228.*

Arveson (2008) returns to the questions he raised in 1969.

W.B. Arveson, The noncommutative Choquet boundary, Journal of the AMS 21 (2008), No. 4, 1065–1084.

Gives a new proof of Dritschel-McCullough's results using ideas of Ozawa. Using an intricate direct integral argument, shows that when $\mathcal{S}$ is separable, a maximal representation is a.e. an integral of boundary representations.

Theorem (Arveson)

Every separable operator system has sufficiently many boundary representations.

Our results

Theorem (Davidson-K (2013))

Every operator system has sufficiently many boundary representations.

Theorem (Davidson-K (2013))

Every operator system has sufficiently many boundary representations.

Proof is dilation-theoretic and works in complete generality. Very much in the style of Arveson's original work.

A completely positive map ϕ is **pure** if whenever $0 \leq \psi \leq \phi$ implies $\psi = \lambda\phi$.

Lemma (Arveson (1969))

If $\phi : \mathcal{S} \to \mathcal{B}(H)$ is pure and maximal, then it extends to a boundary representation.

A completely positive map ϕ is **pure** if whenever $0 \leq \psi \leq \phi$ implies $\psi = \lambda\phi$.

Lemma (Arveson (1969))

If $\phi : \mathcal{S} \to \mathcal{B}(H)$ is pure and maximal, then it extends to a boundary representation.

Our strategy is to extend a pure UCP map in small steps, taking care to preserve purity, until we attain maximality.

Say a UCP map $\phi : \mathcal{S} \to \mathcal{B}(H)$ is **maximal** at $(s, x) \in \mathcal{S} \times H$ if, whenever $\psi : \mathcal{S} \to \mathcal{B}(K)$ dilates ϕ, $\|\psi(s)x\| = \|\phi(s)x\|$.

Key Lemma

If $\phi : \mathcal{S} \to \mathcal{B}(H)$ is a pure UCP map and $(s, x) \in \mathcal{S} \times H$, then there is a *pure* UCP map $\psi : \mathcal{S} \to \mathcal{B}(H \oplus \mathbb{C})$ dilating ϕ that is maximal at (s, x).

Say a UCP map $\phi : \mathcal{S} \to \mathcal{B}(H)$ is **maximal** at $(s, x) \in \mathcal{S} \times H$ if, whenever $\psi : \mathcal{S} \to \mathcal{B}(K)$ dilates ϕ, $\|\psi(s)x\| = \|\phi(s)x\|$.

Key Lemma

If $\phi : \mathcal{S} \to \mathcal{B}(H)$ is a pure UCP map and $(s, x) \in \mathcal{S} \times H$, then there is a *pure* UCP map $\psi : \mathcal{S} \to \mathcal{B}(H \oplus \mathbb{C})$ dilating ϕ that is maximal at (s, x).

- For a UCP map $\psi : \mathcal{S} \to \mathcal{B}(H \oplus K)$, the compression to $\operatorname{span}\{H, \psi(s)x\}$ has the same norm at (s, x).

Say a UCP map $\phi : \mathcal{S} \to \mathcal{B}(H)$ is **maximal** at $(s, x) \in \mathcal{S} \times H$ if, whenever $\psi : \mathcal{S} \to \mathcal{B}(K)$ dilates ϕ, $\|\psi(s)x\| = \|\phi(s)x\|$.

Key Lemma

If $\phi : \mathcal{S} \to \mathcal{B}(H)$ is a pure UCP map and $(s, x) \in \mathcal{S} \times H$, then there is a *pure* UCP map $\psi : \mathcal{S} \to \mathcal{B}(H \oplus \mathbb{C})$ dilating ϕ that is maximal at (s, x).

- For a UCP map $\psi : \mathcal{S} \to \mathcal{B}(H \oplus K)$, the compression to span$\{H, \psi(s)x\}$ has the same norm at (s, x).
- The set $\{\psi : \mathcal{S} \to \mathcal{B}(K) \mid \psi \text{ dilates } \phi\}$ is point-weak* compact, so can find at least one dilation $\psi : \mathcal{S} \to \mathcal{B}(H \oplus K)$ that is maximal at (s, x), say $\psi(s)x = \phi(x) \oplus \eta$.

Say a UCP map $\phi : \mathcal{S} \to \mathcal{B}(H)$ is **maximal** at $(s, x) \in \mathcal{S} \times H$ if, whenever $\psi : \mathcal{S} \to \mathcal{B}(K)$ dilates ϕ, $\|\psi(s)x\| = \|\phi(s)x\|$.

Key Lemma

If $\phi : \mathcal{S} \to \mathcal{B}(H)$ is a pure UCP map and $(s, x) \in \mathcal{S} \times H$, then there is a *pure* UCP map $\psi : \mathcal{S} \to \mathcal{B}(H \oplus \mathbb{C})$ dilating ϕ that is maximal at (s, x).

- For a UCP map $\psi : \mathcal{S} \to \mathcal{B}(H \oplus K)$, the compression to $\text{span}\{H, \psi(s)x\}$ has the same norm at (s, x).
- The set $\{\psi : \mathcal{S} \to \mathcal{B}(K) \mid \psi \text{ dilates } \phi\}$ is point-weak* compact, so can find at least one dilation $\psi : \mathcal{S} \to \mathcal{B}(H \oplus K)$ that is maximal at (s, x), say $\psi(s)x = \phi(x) \oplus \eta$.
- Take an extreme point of the set $\{\psi : \mathcal{S} \to \mathcal{B}(H \oplus \mathbb{C}) \mid \psi \text{ dilates } \phi,\ \psi(s)x = \phi(s)x \oplus \eta\}$. Delicate argument proves purity.

Theorem

Every pure UCP map $\phi : \mathcal{S} \to \mathcal{B}(H)$ *dilates to a maximal pure UCP map, which extends to a boundary representation.*

Theorem

Every pure UCP map $\phi : \mathcal{S} \to \mathcal{B}(H)$ dilates to a maximal pure UCP map, which extends to a boundary representation.

Easy transfinite induction argument on the key lemma obtains dilation that is maximal at each pair $(s, x) \in \mathcal{S} \times H$.

Theorem

Every pure UCP map $\phi : \mathcal{S} \to \mathcal{B}(H)$ dilates to a maximal pure UCP map, which extends to a boundary representation.

Easy transfinite induction argument on the key lemma obtains dilation that is maximal at each pair $(s, x) \in \mathcal{S} \times H$.
If $\mathcal{S}$ is separable and $\dim H < \infty$, then can work entirely with finite rank maps.

Theorem

There are sufficiently many boundary representations to completely norm $\mathcal{S}$.

Theorem

There are sufficiently many boundary representations to completely norm $\mathcal{S}$.

First proof uses C*-convexity of matrix states, and the Krein-Milman type theorem of Webster-Winkler (1999) for C*-convex sets. A result of Farenick (2000) shows the C*-extreme points of the matrix states coincide with the pure matrix states. (More recently, Farenick gave a very nice direct proof of this result that avoids the Webster-Winkler theorem.)

Theorem

There are sufficiently many boundary representations to completely norm $\mathcal{S}$.

First proof uses C*-convexity of matrix states, and the Krein-Milman type theorem of Webster-Winkler (1999) for C*-convex sets. A result of Farenick (2000) shows the C*-extreme points of the matrix states coincide with the pure matrix states. (More recently, Farenick gave a very nice direct proof of this result that avoids the Webster-Winkler theorem.)

Shorter second proof suggested by Kleski. Easy to obtain that the boundary representations of $\mathcal{M}_n(\mathcal{S})$ norm $\mathcal{M}_n(\mathcal{S})$. A result of Hopenwasser implies boundary representations of $\mathcal{M}_n(\mathcal{S})$ correspond to boundary representations of $\mathcal{S}$.

The future

Over 40 years of work (too many names to mention) has led to the development of many techniques and applications.

Over 40 years of work (too many names to mention) has led to the development of many techniques and applications.

In recent years, a great deal of evidence has been compiled showing that noncommutative techniques are needed even in the classical commutative setting. For example, the Drury-Arveson multiplier algebra H_d^∞ has been much more tractable than $H^\infty(\mathbb{B}_d)$. One explanation is that $\mathrm{C}_e^*(H_d^\infty)$ is noncommutative, while $\mathrm{C}_e^*(H^\infty(\mathbb{B}_d))$ is commutative. Classical notions of measure and boundary may not suffice for $d \geq 2$ variables.

Over 40 years of work (too many names to mention) has led to the development of many techniques and applications.

In recent years, a great deal of evidence has been compiled showing that noncommutative techniques are needed even in the classical commutative setting. For example, the Drury-Arveson multiplier algebra H_d^∞ has been much more tractable than $H^\infty(\mathbb{B}_d)$. One explanation is that $\mathrm{C}_e^*(H_d^\infty)$ is noncommutative, while $\mathrm{C}_e^*(H^\infty(\mathbb{B}_d))$ is commutative. Classical notions of measure and boundary may not suffice for $d \geq 2$ variables.

All restrictions have now been removed on the use of Arveson's ideas from 1969. Perhaps we can now realize his vision.

Thanks!