Equilibrium states for self-similar actions

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Equilibrium states on the Cuntz-Pimsner algebras of self-similar actions

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Self similar group actions: G = a group, X = a finite set

X^n set of words of length n, $\quad X^0 = \{\varnothing\}$, $\quad X^* := \bigcup_{n=0}^{\infty} X^n$.

A *self similar action* (G, X) is an action $G \circlearrowright X^*$ such that,

$$g \cdot (xw) = (g \cdot x)(g|_x \cdot w) \quad \text{for all } w \in X^*.$$

for unique $g \cdot x \in X$ and $g|_x \in G$ (the *restriction* of g to x).

We may replace the letter x by an initial word v: for $g \in G$ and $v \in X^k$ there exists a unique $g|v \in G$ such that

$$g \cdot (vw) = (g \cdot v)(g|_v \cdot w) \quad \text{for all } w \in X^*.$$

with $\quad g \cdot v = (g \cdot v_1)(g|_{v_1} \cdot v_2) \cdots (g|_{v_1}|_{v_2 \cdots}|_{v_{k-1}} \cdot v_k)$

and $\quad g|_v = (g|_{v_1})|_{v_2} \cdots |_{v_k}$

Example: The Grigorchuk group G

(Finitely generated by elements of order 2, intermediate growth, amenable but not elementary-amenable).

$X = \{x, y\}$; $G \circlearrowright X^*$ has generators a, b, c, d defined recursively:

$$\begin{aligned} a \cdot (xw) &= yw & a \cdot (yw) &= xw \\ b \cdot (xw) &= x(a \cdot w) & b \cdot (yw) &= y(c \cdot w) \\ c \cdot (xw) &= x(a \cdot w) & c \cdot (yw) &= y(d \cdot w) \\ d \cdot (xw) &= xw & d \cdot (yw) &= y(b \cdot w) \end{aligned}$$

Proposition

The generators a, b, c, d of G all have order two, and satisfy $cd = b = dc$, $db = c = bd$ and $bc = d = cb$. The self-similar action (G, X) is contracting with nucleus $\mathcal{N} = \{e, a, b, c, d\}$.

Contracting SSAs, nucleus and Moore diagrams

- (G, X) is *contracting* if there is a finite $S \subset G$ such that for every $g \in G$ there exists $n \in \mathbb{N}$ with

$$\{g|_v : v \in X^*, |v| \geqslant n\} \subset S.$$

- The *nucleus* of a contracting (G, X) is the smallest such S:

$$\mathcal{N} := \bigcup_{g \in G} \bigcap_{n=0}^{\infty} \{g|_v : v \in X^*, |v| \geqslant n\}.$$

- For $g \in S$ ($S \subset G$ closed under restriction), the *Moore diagram* with vertex set S has a directed edge $g \xrightarrow{(x,y)} h = g|_x$ for each self similarity relation $g \cdot (xw) = y(h \cdot w)$.

Moore diagram for the nucleus of the Grigorchuk group

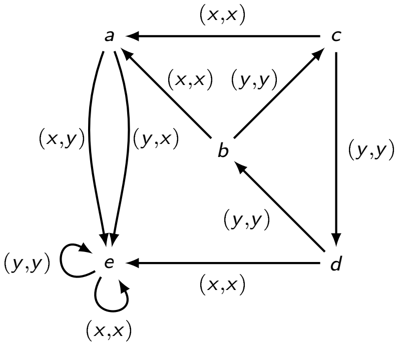

Figure: put an edge from g to $h = g|_x$ with label $(x, g \cdot x)$

$$\begin{aligned} a \cdot (xw) &= yw & a \cdot (yw) &= xw \\ b \cdot (xw) &= x(a \cdot w) & b \cdot (yw) &= y(c \cdot w) \\ c \cdot (xw) &= x(a \cdot w) & c \cdot (yw) &= y(d \cdot w) \\ d \cdot (xw) &= xw & d \cdot (yw) &= y(b \cdot w) \end{aligned}$$

SSAs from odometers, integer matrices, basilica group, ...

Odometer: Let $X = \{0, 1, \cdots, N-1\}$, $G = \{g^k : k \in \mathbb{Z}\}$ with $g :=$ "add 1 modulo N with carry-over to the right" then $g|_i = e$ for $i < N-1$ and $g|_{N-1} = g$.

Integer Matrix A**:** Let $X := \mathbb{Z}^n/(A^t)\mathbb{Z}^n$ for $A \in \mathrm{Mat}_n(\mathbb{Z})$, with $|\det A| > 1$. $G = \mathbb{Z}^d$ acting by 'addition modulo $(A^t)\mathbb{Z}^n$ with carry over to the right' (uses fixed set of representatives for $\mathbb{Z}^n/(A^t)\mathbb{Z}^n$). (G, X) is contracting if $|\lambda| > 1$ for all eigenvalues of A.

Basilica group: Let $X = \{x, y\}$ and recursively define a and b by

$$a \cdot (xw) = y(b \cdot w) \qquad a \cdot (yw) = xw$$
$$b \cdot (xw) = x(a \cdot w) \qquad b \cdot (yw) = yw$$

The *basilica group* B is the group generated by $\{a, b\}$, it gives a contracting self similar action.

C*(G) bimodule for (G, X) (after V. Nekrashevych)

Take the usual right-Hilbert $C^*(G)$-module on X,

$$M = \bigoplus_{x \in X} C^*(G)$$

$M = \{m = (m_x) : m_x \in C^*(G)\}$, with module action $(m_x) \cdot a = (m_x a)$ and inner product

$$\langle m, n \rangle = \sum_{x \in X} m_x^* n_x.$$

Then $(e_x)_y = 1_{C^*(G)} \delta_{y,x}$ gives orthonormal basis elements for M, and there is a left action of $C^*(G)$ on M arising from:

$$U_g(e_x \cdot a) = e_{g \cdot x} \cdot (\delta_{g|_x} a)$$

The C*-algebras $\mathcal{T}(G, X)$ and $\mathcal{O}(G, X)$

The bimodule C*-algebras have natural presentations:

$\mathcal{T}(G, X)$:= universal C^*-algebra with generators $\{S_x : x \in X\}$ and $\{U_g : g \in G\}$ such that

(T1) $S_y^* S_x = \begin{cases} 1 & \text{if x =y} \\ 0 & \text{if } x \neq y \end{cases}$ $\quad S \rightsquigarrow \mathcal{T}_{|X|}$

(T2) $U_g U_h = U_{gh}; \quad U_g^* = U_{g^{-1}}; \quad U_e = 1$ $\quad U \rightsquigarrow C^*(G)$

(T3) $U_g S_x = S_{g \cdot x} U_{g|_x}$ $\quad$ self-similarity comm. rels. $g \cdot (xw) = (g \cdot x)(g|_x \cdot w)$

$\mathcal{O}(G, X)$:= quotient of $\mathcal{T}(G, X)$ by the extra relation

(O) $\sum_{x \in X} \tilde{S}_x \tilde{S}_x^* = 1$ $\quad \tilde{S} \rightsquigarrow \mathcal{O}_{|X|}$

Spanning set and dynamics

For a word $v = x_1 x_2 \cdots x_n$, we let $S_v := S_{x_1} S_{x_2} \cdots S_{x_n}$.

- $$\mathcal{T}(G, X) = \overline{\text{span}}\{S_v U_g S_w^* : v, w \in X^*,\ g \in G\}.$$
- If (G, X) is contracting,
$$\mathcal{O}(G, X) = \overline{\text{span}}\{\tilde{S}_v U_g \tilde{S}_w^* : v, w \in X^*,\ g \in \mathcal{N}\}$$
- The dynamics on $\mathcal{T}(G, X)$, and on $\mathcal{O}(G, X)$ are defined by
$$\sigma_t(S_v U_g S_w^*) = e^{t(|v|-|w|)} S_v U_g S_w^*$$
- Interested in (KMS) equilibrium states of $(\mathcal{T}(G, X), \sigma)$ and of $(\mathcal{O}(G, X), \sigma)$.

KMS states

- Given a continuous action $\sigma : \mathbb{R} \to \mathrm{Aut}(A)$, there is a dense *-subalgebra of σ-analytic elements $a \in A$ such that $t \mapsto \sigma_t(a)$ extends to an entire function $z \mapsto \sigma_z(a)$.

- Definition
  The state φ of A satisfies the KMS condition at inverse temperature $\beta \in [0, \infty)$ if whenever a and b are analytic for σ,

$$\varphi(ab) = \varphi(b\ \sigma_{i\beta}(a)).$$

- Note: it suffices to verify the above for elements that span a dense subalgebra, e.g, in our case, the spanning set $\{S_v U_g S_w^*\}$

Theorem (L. Raeburn Ramagge Whittaker '13)

1. *If* $\beta \in [0, \log|X|)$*, there are no* KMS_β *states for* σ*;*
2. *if* $\beta \in (\log|X|, \infty]$*, for each normalized trace* τ *on* $C^*(G)$ *define* $\psi_{\beta,\tau}(S_v U_g S_w^*) = 0$ *if* $v \neq w$*, and*

$$\psi_{\beta,\tau}(S_v U_g S_v^*) = (1 - |X|e^{-\beta}) \sum_{k=0}^{\infty} e^{-\beta(k+|v|)} \Big(\sum_{\substack{y \in X^k \\ g \cdot y = y}} \tau(\delta_{g|_y}) \Big)$$

the map $\tau \mapsto \psi_{\beta,\tau}$ *is an affine homeomorphism of Choquet simplices onto the* KMS_β *states of* $\mathcal{T}(G, X)$*.*

3. *the* $KMS_{\log|X|}$ *states of* $\mathcal{T}(G, X)$ *arise from KMS states of* $\mathcal{O}(G, X)$*; and there is at least this one:*

$$\psi_{\log|X|}(S_v U_g S_w^*) = \begin{cases} |X|^{-|v|} c_g & \text{if } v = w \\ 0 & \text{otherwise.} \end{cases}$$

If (G, X) *is contractible, this is the only one.*

There is a $\mathrm{KMS}_{\log|X|}$ state of $\mathcal{O}(G,X)$ given by

$$\psi_{\log|X|}(\tilde{S}_v U_g \tilde{S}_w^*) = \begin{cases} |X|^{-|v|} c_g & \text{if } v = w \\ 0 & \text{otherwise.} \end{cases}$$

If (G,X) is contractible, this is the only one.

Hence $\lim_{\beta \searrow \log|X|} \psi_{\beta,\tau} = \psi_{\log|X|}$ for every τ.

What is c_g?

The asymptotic proportion of points fixed by $g \in G$

Let τ = usual trace on $C^*(G)$, i.e. $\tau(\delta_g) = 0$ unless $g = e$; then the following limit exists as $\beta \searrow \log|X|$,

$$\psi_{\beta,\tau}(U_g) = (1 - |X|e^{-\beta}) \sum_{k=0}^{\infty} e^{-\beta k} \Big(\sum_{\substack{y \in X^k \\ g \cdot y = y}} \tau(\delta_{g|_y}) \Big) \longrightarrow c_g$$

For each $n \in \mathbb{N}$ and $g \in G$ define

$$F_g^n := \{v \in X^n : g \cdot v = v \text{ and } g|v = e\}.$$

Clearly $\sum_{\substack{y \in X^k \\ g \cdot y = y}} \tau(\delta_{g|_y}) = |F_g^k|$ and it turns out that

$$\frac{|F_g^k|}{|X^k|} \nearrow c_g \in [0, 1).$$

The asymptotic proportion of g-invariant sets.

In the contractive case the same limit is obtained starting from any normalized trace on $C^*(G)$.

For instance, if we use the trace τ_1 defined as the integrated version of the trivial representation, $\tau_1(U_g) = 1$ for every $g \in G$, we are led to use the measure of g-invariant sets at level k. So instead of $|F_g^k|$ we need to compute the cardinality of the set

$$G_g^k := \{w \in X^k : g \cdot w = w\}.$$

This yields the same limit: $\lim_{k\to\infty} \frac{|G_g^k|}{|X^k|} = c_g$, and again, it suffices to compute it for $g \in \mathcal{N}$.

Next, in Mike Whittaker's talk, we'll see how to compute $|F_g^k|$ using Moore diagrams.