The Russo-Dye Theorem in Nest Subalgebras of factors

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COSy, May, 2013

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Define the immediate successor of $N \in \mathcal{N}$

$$N_+ = inf\{M \in \mathcal{N} : M > N\}$$

and the immediate predecessor of N

$$N_- = sup\{M \in \mathcal{N} : M < N\}.$$

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Let $\mathcal{H} = \mathbf{C}^n$, $\{e_1, e_2, \cdots, e_n\}$ the orthogonal basis for $\mathcal{H}$. Let $N_k = \text{span}\{e_1, e_2, \cdots, e_k\}$, then the nest algebra $\mathcal{T}_n$ is all the $n \times n$ upper triangular matrices.

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Russo-Dye Theorem is not true for this case. Every unitary element in $\mathcal{T}_n$ must be diagonal, and so the closure of the convex hull of all the unitary elements is smaller than the whole unit ball of $\mathcal{T}_n$.

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A nest is said to be admissible in $B(\mathcal{H})$, if both 0_+ and $I - I_-$ are either zero or infinite rank.

If $\mathcal{N}$ is an admissible nest in $B(\mathcal{H})$, then there is a projection $P = \sum_{k\in\mathbf{Z}} E_k$ in the nest algebra, which is called an admissible projection for $\mathcal{N}$.

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The one dimensional projections are equivalent in the Murry-von Neumann sense.

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Let $\mathcal{M}$ be a von Neumann algebra and $\mathcal{N} \subset \mathcal{M}$ be a nest, then the nest subalgebra of the von Neumann algebra is the set of all the operators T in $\mathcal{M}$ such that $TN \subseteq N$ for every N in $\mathcal{N}$, which is denoted by $\mathcal{M} \cap Alg\mathcal{N}$.

Definition

Let $\mathcal{N}$ be a nest in a factor $\mathcal{R}$. $\mathcal{N}$ is said to be an admissible nest in $\mathcal{R}$, if for any proper projection $N \in \mathcal{N}$, i.e., $N \neq 0$ or I, both N and $I - N$ are infinite projections in $\mathcal{R}$.

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Lemma

Let $\mathcal{N}$ *be an admissible nest in a factor* $\mathcal{R}$ *with* $0_+ > 0$ *and* $I_- = I$*. Then there exist two projection sequences:* $\{N_k \in \mathcal{N} : \ k = 0, 1, 2, \cdots\}$ *increasing to* I *with*

$$0 = N_0 < 0_+ = N_1 < N_2 < N_3 < \cdots < I$$

and $\{E_k \in \mathcal{R} \cap Alg\mathcal{N} : \ k \in \mathbf{Z}\}$ *with* $E_i \sim E_j$*, for* $i, \ j \in \mathbf{Z}$*, such that*

$$\begin{cases} E_k < 0_+, & k = 0, -1, -2, \cdots . \\ E_k \leq N_{k+1} - N_k, & k = 1, 2, \cdots . \end{cases}$$

Lemma

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and $\{E_k \in \mathcal{R} \cap Alg\mathcal{N} : \ k \in \mathbf{Z}\}$ with $E_i \sim E_j$, for $i, \ j \in \mathbf{Z}$, such that

$$\begin{cases} E_k < 0_+, & k = 0, -1, -2, \cdots . \\ E_k \leq N_{k+1} - N_k, & k = 1, 2, \cdots . \end{cases}$$

We call $\hat{\mathcal{N}} = \{I, N_k : k = 0, 1, 2, \cdots\}$ and $\{E_k : \ k \in \mathbf{Z}\}$ the basic sub-nest of $\mathcal{N}$ and the basic equivalent projection sequence for $\hat{\mathcal{N}}$, respectively.

Theorem

Let $\mathcal{N}$ be an admissible nest in a factor $\mathcal{R}$, then each element $A \in \mathcal{R} \cap Alg\mathcal{N}$ with $\| A \| < 1 - \frac{1}{n}$ is the average of $16n^2$ unitary elements in $\mathcal{R} \cap Alg\mathcal{N}$. Thus the convex hull of all the unitary elements contains the open unit ball of $\mathcal{R} \cap Alg\mathcal{N}$.
If $\mathcal{N}$ is not admissible, then the weak operator topology closure of the convex hull of all the unitary elements in $\mathcal{R} \cap Alg\mathcal{N}$ is not the unit ball.

Sketch of the proof: Suppose $\{E_k\}$ is the basic equivalent projection sequence. Let $P = \sum_{k=-\infty}^{+\infty} E_{2k}$ and $Q = \sum_{k=-\infty}^{+\infty} E_{2k+1}$. Split P and Q into $2n$ orthogonal projections $P_1, P_2, \cdots, P_{2n}$ and $Q_1, Q_2, \cdots, Q_{2n}$, respectively. i.e., $P = \sum_{k=1}^{2n} P_k$ and $Q = \sum_{k=1}^{2n} Q_k$. It follows that $X_i = Q_i^{\perp} - \frac{1}{2n} Q^{\perp}$ and $Y_j = P_j^{\perp} - \frac{1}{2n} P^{\perp}$ are contractions such that

$$\sum_{i=1}^{2n} X_i = (2n-1)I = \sum_{j=1}^{2n} Y_j.$$

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Moreover, $(1-\frac{1}{2n})^2 > 1 - \frac{1}{n}$. Thus

$$A_{ij} = (\frac{2n}{2n-1})^2 X_i A Y_j = Q_i^\perp A_{ij} P_j^\perp, \text{ for } 1 \leq i, j \leq n$$

are strict contractions in $\mathcal{R} \cap Alg\mathcal{N}$ and $A = \frac{1}{4n^2} \sum_{i=1}^{2n} \sum_{j=1}^{2n} A_{ij}$.

Dilate $A_{ij} = Q_i^{\perp} A_{ij} P_j^{\perp}$ to be a unitary element U_{ij} in $\mathcal{R} \cap Alg\mathcal{N}$ such that $A_{ij} = Q_i^{\perp} U_{ij} P_j^{\perp}$.

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That is:

$$\begin{aligned} \frac{1}{16n^2} \sum_{i=1}^{2n} \sum_{j=1}^{2n} \sum_{k=1}^{4} U_{ijk} &= \frac{1}{4n^2} \sum_{i=1}^{2n} \sum_{j=1}^{2n} A_{ij} \\ &= A. \end{aligned}$$

Definition

Let $\mathcal{N}$ be a nest in a von Neumann algebra $\mathcal{M}$. $\mathcal{N}$ is said to be an admissible nest in $\mathcal{M}$, if for each projection $N \in \mathcal{N}$, both $N - N_c$ and $I - N - (I - N)_c$ are either 0 or proper infinite in $\mathcal{M}$.

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Here N_c is the union of all the central projections P_a of $\mathcal{M}$ such that $P_a \leq N$. N_c is the maximal central projection such that $N_c \leq P$.

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In factors, all the central projections are trivial and a projection is infinite if and only if it is proper infinite.

For every von Neumann algebra $\mathcal{M}$, there are orthogonal center projections P_{I_n}, $1 \leq n < \infty$, P_{I_∞}, P_{II_1}, P_{II_∞} and P_{III} with sum I and $P_{I_\infty}\mathcal{M}$, $P_{II_1}\mathcal{M}$, $P_{II_\infty}\mathcal{M}$ and $P_{III}\mathcal{M}$ are type I_n, type I_∞, type II_1, type II_∞ and type III von Neumann algebras, respectively.

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if $\mathcal{N}$ is an admissible nest in $\mathcal{M}$, then each of the summands is decomposable

$$P_A\mathcal{M} = \sum P_A^j\mathcal{M}, \qquad A = I_n, I_\infty, II_1, II_\infty, III$$

P_A^j's are orthogonal central projections and each $P_A^j\mathcal{N}$ in $P_A^j\mathcal{M}$ is either a trivial nest or

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The nests $P_A^j\mathcal{N}$, $A = I_n, II_1$ are trivial in $P_A^j\mathcal{M}$.

Type III case, use central carrier, Type II_∞ and Type I_∞ case, use tracial weight on each factor, we can obtain the basic sub-nest $\{N_{t_n}\}$ and basic equivalent projection sequence $\{E_n\}$ for each summand of $P_A^j \chi_{Y_n} \mathcal{M}$, $A = I_\infty, II_\infty, III$.

Theorem

Let $\mathcal{N}$ be an admissible nest in a von Neumann $\mathcal{M}$ acting on a separable infinite dimensional Hilbert space $\mathcal{H}$, then each element $A \in \mathcal{M} \cap Alg\mathcal{N}$ with $\|A\| < 1 - \frac{1}{n}$ is the average of $16n^2$ unitary elements in $\mathcal{M} \cap Alg\mathcal{N}$. Thus the convex hull of all the unitary elements contains the whole open unit ball.

If $\mathcal{N}$ is not admissible, the weak operator topology closure of the convex hull of all the unitary elements in $\mathcal{M} \cap Alg\mathcal{N}$ is not the unit ball of $\mathcal{M} \cap Alg\mathcal{N}$.

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If $N - N_c$ is infinite but not proper infinite, then there is a central projection Q in $\mathcal{M}$ such that $QN - QN_c \neq 0$ is finite in $\mathcal{M}$. For any unitary element $U \in \mathcal{M} \cap Alg\mathcal{N}$, we have $(QN - QN_c)U(QN - QN_c)^{\perp} = 0$.

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This means

$$C_{(QN-QN_c)}C_{(QN-QN_c)^{\perp}} = 0.$$

We have $C_{QN-QN_c} = QN - QN_c$ and $C_{(QN-QN_c)^{\perp}} = (QN - QN_c)^{\perp}$. Therefore $QN - QN_c \neq 0$ is a central projections. But $QN_c = (QN)_c$ is the maximal central projection such that $QN_c \leq QN$, a contradiction.

Thank You!