Computing equilibrium states for self-similar actions

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Self-similar group actions

Continuing from Marcelo Laca's talk...

- Suppose X is a finite set of cardinality $|X|$;
  - let X^n denote the set of words of length n in X,
  - let $X^* = \bigcup_{n \in \mathbb{N}} X^n$.
- A faithful action of a group G on X^* is *self-similar* if, for all $g \in G$ and $x \in X$, there exist unique $g|_x \in G$ such that

$$g \cdot (xw) = (g \cdot x)(g|_x \cdot w) \quad \text{for all finite words } w \in X^*.$$

  The pair (G, X) is referred to as a *self-similar action* and the group element $g|_x$ is called the *restriction* of g to x.

Contracting self-similar actions and Moore diagrams

- A self-similar action (G, X) is *contracting* if there is a finite $S \subset G$ such that for every $g \in G$ there exists $n \in \mathbb{N}$ with

$$\{g|_v : v \in X^*, |v| \geq n\} \subset S.$$

- The *nucleus* of a contracting (G, X) is the smallest such S:

$$\mathcal{N} := \bigcup_{g \in G} \bigcap_{n=0}^{\infty} \{g|_v : v \in X^*, |v| \geq n\}.$$

- Let $\mathcal{N}$ be the nucleus of (G, X). The *Moore diagram* of $\mathcal{N}$ is the labelled directed graph with vertices in $\mathcal{N}$ and edges labelled:

$$g \xrightarrow{(x,y)} g|_x$$

for each self similarity relation $g \cdot (xw) = y(g|_x \cdot w)$.

Theorem (Laca-Raeburn-Ramagge-W '13)

1. *If* $\beta \in [0, \log |X|)$, *there are no* KMS_β *states for* σ;
2. *if* $\beta \in (\log |X|, \infty]$, *for each normalized trace* τ *on* $C^*(G)$ *define* $\psi_{\beta,\tau}(S_v U_g S_w^*) = 0$ *if* $v \neq w$, *and*

$$\psi_{\beta,\tau}(S_v U_g S_v^*) = (1 - |X|e^{-\beta}) \sum_{k=0}^{\infty} e^{-\beta(k+|v|)} \Big(\sum_{\substack{y \in X^k \\ g \cdot y = y}} \tau(\delta_{g|_y}) \Big)$$

   the map $\tau \mapsto \psi_{\beta,\tau}$ *is an affine homeomorphism of Choquet simplices onto the* KMS_β *states of* $\mathcal{T}(G, X)$.

3. *the* $KMS_{\log |X|}$ *states of* $\mathcal{T}(G, X)$ *arise from KMS states of* $\mathcal{O}(G, X)$; *and there is at least this one:*

$$\psi_{\log |X|}(S_v U_g S_w^*) = \begin{cases} |X|^{-|v|} c_g & \text{if } v = w \\ 0 & \text{otherwise.} \end{cases}$$

   If (G, X) *is contractible, this is the only one.*

The asymptotic proportion of points fixed by $g \in G$

Let τ be the usual trace on $C^*(G)$, i.e. $\tau(\delta_g) = 0$ unless $g = e$; then let $\beta \searrow \log|X|$,

$$\psi_{\beta,\tau}(U_g) = (1 - |X|e^{-\beta}) \sum_{k=0}^{\infty} e^{-\beta k} \Big(\sum_{\substack{y \in X^k \\ g \cdot y = y}} \tau(\delta_{g|_y}) \Big) \longrightarrow \ ??$$

For each $n \in \mathbb{N}$ and $g \in G$ define

$$F_g^n := \{v \in X^n : g \cdot v = v \text{ and } g|_v = e\}.$$

Then $\quad |F_g^k||X|^{-k} \nearrow c_g \in [0,1)$ and since

$$\sum_{\substack{y \in X^k \\ g \cdot y = y}} \tau(\delta_{g|_y}) = |F_g^k|,$$

the above limit is also c_g. How do we actually compute c_g ?

Calculating c_g using the Moore diagram

- To calculate values of the KMS states explicitly, we need to evaluate the limit

$$c_g = \lim_{k\to\infty} |F_g^k||X|^{-k}$$

- Each $v \in F_g^k$ corresponds to a path μ_v in the Moore diagram:

$$\mu_v := g \xrightarrow{(v_1,v_1)} g|_{v_1} \xrightarrow{(v_2,v_2)} g|_{v_1 v_2} \xrightarrow{(v_3,v_3)} \cdots \xrightarrow{(v_k,v_k)} g|_v = e$$

- Notice that all the labels have the form (x, x).
- Every path with labels (x, x) arises this way.

The odometer

- Let $X = \{0, 1\}$ and $G = \mathbb{Z}$
- $(\mathbb{Z}, X)$ is a self-similar action described by:

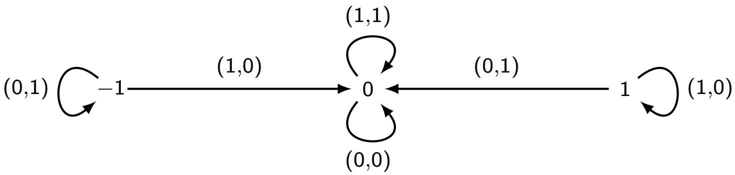
$$1 \cdot 0w = 1w \qquad\qquad 1 \cdot 1w = 0(1 \cdot w)$$

  for every finite word $w \in X^*$
- For example, $3 \in \mathbb{Z}$ acts on the word 01100 by

$$3 \cdot 01100 = 2 \cdot 11100 = 1 \cdot 00010 = 10010.$$

The odometer

- The nucleus of the odometer action is $\mathcal{N} = \{0, 1, -1\}$.
- The Moore diagram is:



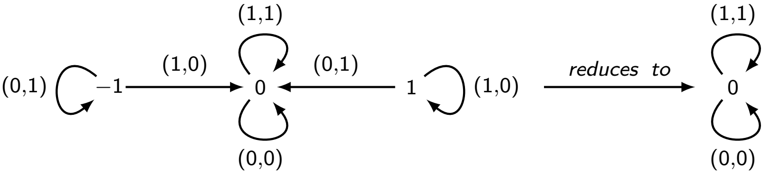
The odometer

Proposition

The C^*-algebra $\mathcal{O}(\mathbb{Z}, X)$ has a unique $KMS_{\log 2}$ state, which is given on the nucleus $\mathcal{N} = \{0, 1, -1\}$ by

$$\psi(U_n) = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{for } n = \pm 1 \end{cases}$$

- *Sketch of proof.*



- $F_g^k = F_{g^{-1}}^k = \varnothing$ so we have $c_1 = c_{-1} = \lim_{k\to\infty} 0 \cdot 2^{-k} = 0$.

The basilica group [Grigorchuk and Żuk 2003]

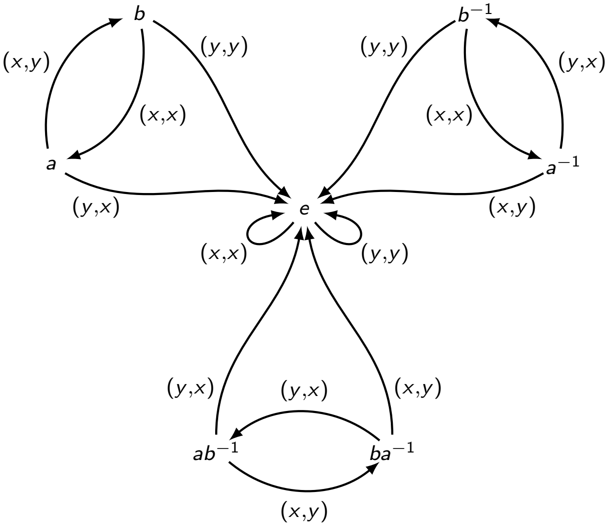
- Let $X = \{x, y\}$
- Generators a and b have (faithful) self-similar action defined by

$$a \cdot (xw) = y(b \cdot w) \qquad a \cdot (yw) = xw$$
$$b \cdot (xw) = x(a \cdot w) \qquad b \cdot (yw) = yw$$

  for $w \in X^*$.
- The *basilica group* B is the group generated by $\{a, b\}$. The pair (B, X) is then a self-similar action.
- The nucleus is $\mathcal{N} = \{e, a, b, a^{-1}, b^{-1}, ba^{-1}, ab^{-1}\}$.
- The basilica group is torsion free, has exponential growth, and is amenable but not elementary amenable.

The basilica group

The Moore diagram of the nucleus:



The basilica group

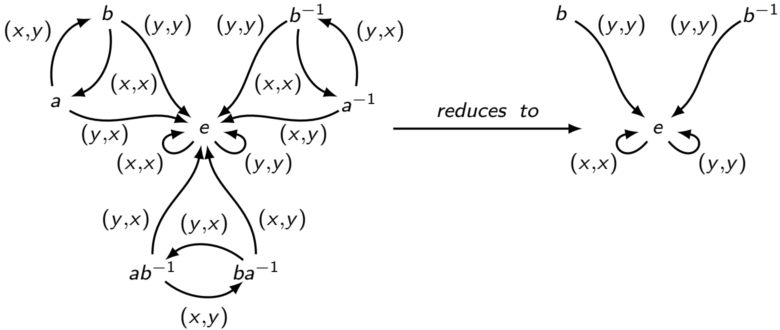
Proposition

The C^*-algebra $\mathcal{O}(B, X)$ has a unique $KMS_{\log 2}$ state, which is given on the nucleus $\mathcal{N} = \{e, a, b, a^{-1}, b^{-1}, ab^{-1}, ba^{-1}\}$ by

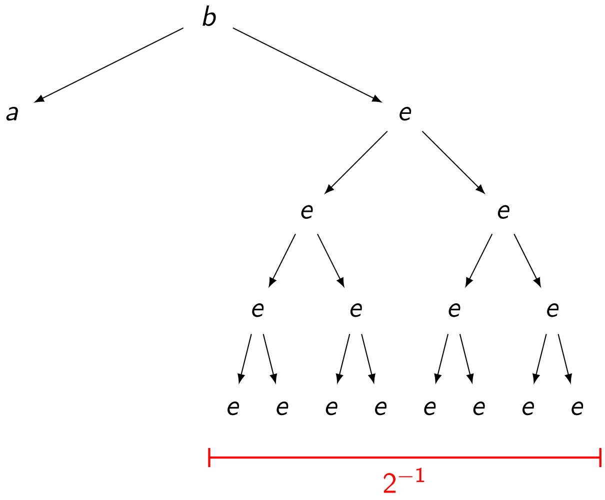
$$\psi(u_g) = \begin{cases} 1 & \text{for } g = e \\ \frac{1}{2} & \text{for } g = b, b^{-1} \\ 0 & \text{for } g = a, a^{-1}, ab^{-1}, ba^{-1}. \end{cases}$$

The basilica group

Sketch of proof.



Computation of c_b for the basilica group



$$c_b = \frac{1}{2}$$

The Grigorchuk group [Grigorchuk 1980]

- Let $X = \{x, y\}$
- Generators a, b, c, and d have (faithful) self-similar action defined by

$$\begin{aligned} a \cdot xw &= yw & a \cdot yw &= xw \\ b \cdot xw &= x(a \cdot w) & b \cdot yw &= y(c \cdot w) \\ c \cdot xw &= x(a \cdot w) & c \cdot yw &= y(d \cdot w) \\ d \cdot xw &= xw & d \cdot yw &= y(b \cdot w). \end{aligned}$$

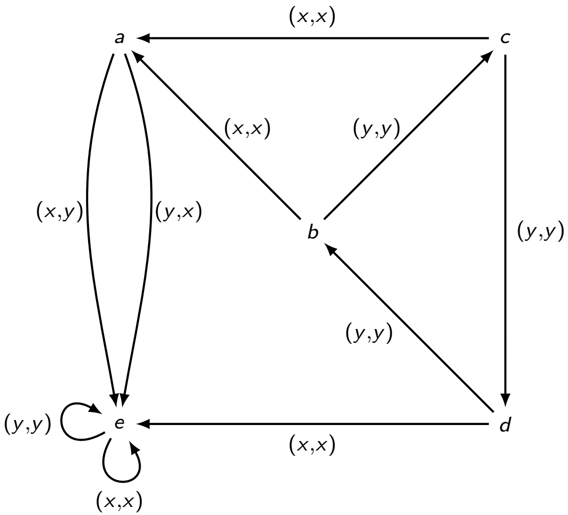
- The nucleus of the Grigorchuk group is

$$\mathcal{N} = \{e, a, b, c, d\}.$$

- The Grigorchuk group has intermediate growth and is a finitely generated infinite torsion group.

The Grigorchuk group

The Moore diagram of the nucleus:



The Grigorchuk group

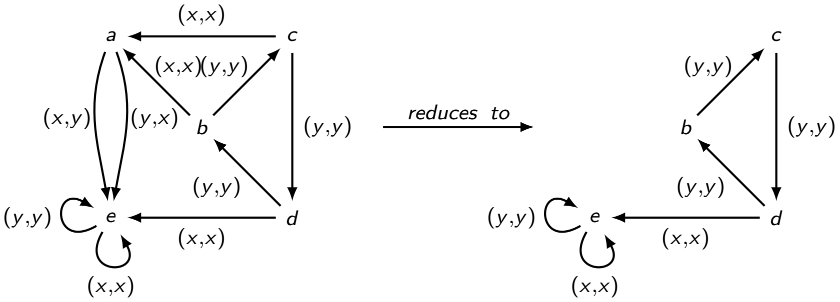
Proposition

Let (G, X) *be the self-similar action of the Grigorchuk group. Then* $(\mathcal{O}(G, X), \sigma)$ *has a unique* $KMS_{\log 2}$ *state* ψ *which is given on the nucleus* $\mathcal{N} = \{e, a, b, c, d\}$ *by*

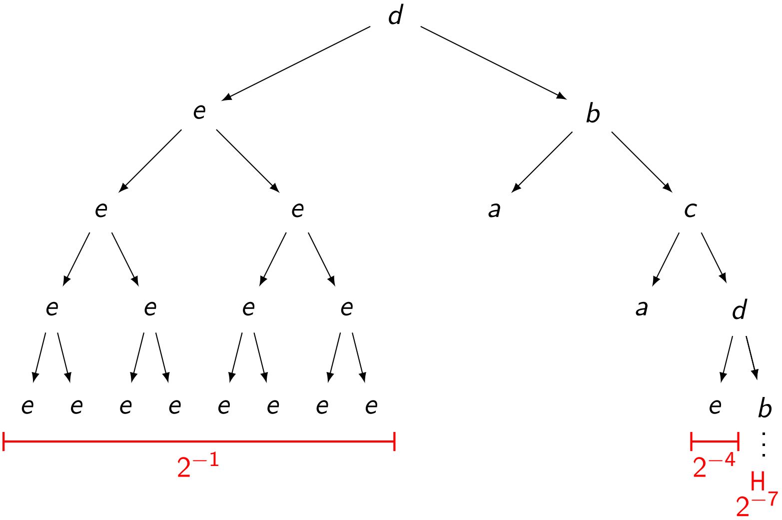
$$\psi(U_g) = \begin{cases} 1 & \text{for } g = e \\ 0 & \text{for } g = a \\ 1/7 & \text{for } g = b \\ 2/7 & \text{for } g = c \\ 4/7 & \text{for } g = d. \end{cases}$$

The Grigorchuk group

Sketch of proof.



Computation of c_d for the Grigorchuk group



$$c_d = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{3n} = \frac{1}{2}\left(\frac{1}{1-\frac{1}{8}}\right) = \frac{4}{7}.$$

References:

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4. V. Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs **vol. 117**,. Amer. Math. Soc., Providence, RI, 2005.