

# The mysterious nature of nested recurrence relations

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## Abstract

In the 1979 Pulitzer Prize winning book “Gödel, Escher, Bach: an Eternal Golden Braid”, Douglas R. Hofstadter introduced the recurrence relation

$$Q(n) = Q(n - Q(n - 1)) + Q(n - Q(n - 2))$$

with  $Q(1) = Q(2) = 1$ . We call this a nested recurrence relation because it has a sub-expression of the form

$$\dots Q(\dots Q(\dots) \dots) \dots$$

Other than composition, the only operations that are used are addition and subtraction. Some nested recurrence relations have a solution and combinatorial interpretations, while others seemingly do not. For example it is still unknown whether  $Q(n)$  is defined for all  $n$ . On the other hand, for the closely related recurrence  $T(n) = T(n - 1 - T(n - 1)) + T(n - 2 - T(n - 2))$  with  $T(1) = T(2) = 1$ , the number  $T(n)$  counts the maximum number of leaves at the lowest level in a binary tree with  $n$  nodes. Some nested recurrence relations are undecidable in the sense that there is no algorithm to decide whether, given a set of initial conditions, they are well-defined for all  $n > 0$ . The purpose of this talk is to introduce nested recurrence relations, discuss some of the known results, and present tantalizing open problems. This is joint work with Marcel Celaya, Steve Tanny, Brad Jackson, Alejandro Erickson and Bram Isgur.