Minutes from
Open Problem Session

21^{st} Ontario Combinatorics Workshop
Nipissing University

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Problem 1. **[Tzvetalin Vassilev]** This problem is related to the Middle Level Conjecture. Given a binary hypercube of dimension n, we can partition vertices in levels by the number of zeros. For example, if $n = 5$:

level 0	11111
level 1	11110, 11101, 11011, 10111, 01111
level 2	11100, 11010, 11001, ...
level 3	10100, 01100, 10001, 00101,...
level 4	10000, 01000, 00100, 00010, 00001
level 5	00000

Note that the edges of the hypercube are only between the levels.

Conjecture If n is odd, there is always a Hamilton cycle between middle levels.

It is known that:

n	number of Hamilton cycles between middle layers
3	1
5	27 or 28
7	$\sim 7 \cdot 10^5$

Problem 1a. How close to the Hamilton property can one get by imposing that the adjacent vertices satisfy an additional property: first or last bit replaced, shifts for number of bits, prioritize shifts...

Note, a k-shift of a vertex $a_1a_2...a_ka_{k+1}....a_n$ is $a_{k+1}....a_nb_1b_2...b_k$. It is known that when

n=5 1,2-shifts do not work
n=7 1,2,3-shifts do not work

A cycle is antipodal if its diametrically opposite vertices in are complements of each other. The following is an example of an antipodal cycle of length 6:

- 110 - 010 - 011 - 001 - 101 - 100 -

Problem 1b. What are conditions on n so that there is an antipodal Hamilton cycle between middle levels?

Brett Stevens' comment related to this problem: Example of the universal cycle for 2-subsets of 3 elements is: position elements 1,2,3 around a circle. Take a 2-window on 1 and 2 and rotate it to obtain all 2-subsets. In particular, we get $\{1,2\}, \{2,3\}, \{1,3\}$.

This can be generalized to k-subsets of n-element set. If it was possible to sweep the $k-1$-level in the same way, the middle level problem would be its corollary.

Universal cycle was proved to be impossible for $n-2$. What can be said about $n-3$?

Bijection between the 2-subsets of $\{1,2,3\}$ and binary words of length 3:

000	$\emptyset$
001	$\{3\}$
010	$\{2\}$
100	$\{1\}$
011	$\{2,3\}$
101	$\{1,3\}$
110	$\{1,2\}$
111	$\{1,2,3\}$

Problem 2. [Tzvetalin Vassilev] Assume we are given n students who need to referee each other papers. A matching algorithm for stable marriage can be used to find one such assignment.

Assume we have additional constraints:

- Every student receives the same number of evaluations.
- Partition students in groups of incompetent, average and critical students. Then one can impose the following two types of constraints:
  - No more than a evaluations from a student in group i.
  - No less than b evaluations from a student in group j.
- Assume that everyone gets to write equal number of evaluations (might be relaxed to more than or equal to or less than or equal to some number of evaluations).

Is there a universal solution? Can a solution for a particular case be used to obtain a solution of another case?

Problem 3. [Frank Ruskey] Consider the following $2 \times 2^{3-1}$ array and its labeling by: A, B, C.

A	A, B	A, C	
C	B, C	A, B, C	B

Notice that:

- For any given label, the subarray of cells where this label is present gives a simple connected curve.
- Every label is equally present in the array.
- The cells of the array give the power set of $\{A, B, C\}$.

Analogous labeling can be done for a $2 \times 2^{n-1}$ array with n labels when $n \in \{4, 5\}$.
Conjecture This cannot be done for $n = 6$.

Problem 4. [Brett Stevens] Consider a circular sequence over $\mathbb{Z}_v$ of length v^2 such that:

1. every of v^2 ordered pairs $(v_1, v_2) \in \mathbb{Z}_v \times \mathbb{Z}_v$ occurs once in the sequence
2. every v elements represent the set $\mathbb{Z}_v$ (more formally, there exists an i such that $\{s_{i+j}, s_{i+j+1}, \ldots, s_{i+j+v-1}\} = \mathbb{Z}_v$ for all j).

2'. other than $(x, x) \in \mathbb{Z}_v \times \mathbb{Z}_v$, x's are maximally separated (ordered version of Davis and Simmons' pair designs).

Sequences having only property 1 can be obtained by finding an Euler tour in looped ${K_v}^*$ (double every edge of K_v, orient them in opposite directions and add a loop on every vertex).

Construct sequences satisfying properties (1. and 2.) or (1. and 2'.)

Problem 5. [**Brett Stevens**] Latin square of order n is:

- **row complete** if every order 2-set appears once consecutively in rows;

- **column complete** if transpose of the array is row complete;

- **complete** if it is both row and column complete.

Theorem For every even positive integer n there is a complete Latin square.
Proof: First row and first column are:

$$1 \quad 2 \quad 0 \quad 3 \quad (n-1) \quad 4 \quad \ldots \quad \tfrac{n}{2} \quad (\tfrac{n}{2}+2) \quad (\tfrac{n}{2}+1)$$

Remaining entries are:

$$L_{ij} = L_{i1} + L_{1j} - 1 \pmod{n}. \qquad\qquad \text{QED}$$

Wang, Cheng, in "Complete Latin squares of order p^n exist for odd primes p and $n > 2$." Discrete Math. 252 (2002), no. 1-3, 189 – 201., gives solution for odd prime powers.

The problem of the existence of complete Latin squares for odd primes is otherwise open.

Problem 6. [**Andrea Burgess**] A bipartite graph has a monogamous 4-cycle decomposition if given two 4-cycles (a, b, c, d) and (a, b', c, d'), we must have that $\{b, d\} = \{b', c'\}$. In "Monogamous decompositions of complete bipartite graphs, symmetric H-squares, and self-orthogonal 1-factorizations. ", by Lindner, C. C. and Rosa, A. Australas. J. Combin. 20 (1999), 251–256., it is proved that there exists a monogamous 4-cycle decomposition of $K_{s,t}$ if and only if $s, t \geq 6$ and $t \leq 2s - 2$.

Study of monogamous 4-cycle packings of $K_{s,t}$ regards the two main questions:

- the maximal size of the packing, D;
- constructions of the optimal packing.

These designs are equivalent to generalized packings with $\vec{k} = (2, 2)$.
It is known that

$$D \leq \min\left\{ \binom{s}{t}, \left\lfloor \frac{s}{2} \left\lfloor \frac{t}{2} \right\rfloor \right\rfloor, \left\lfloor \frac{t}{2} \left\lfloor \frac{s}{2} \right\rfloor \right\rfloor \right\}.$$

The bound is met when:

- s and t are both even;
- if one of s, t is odd;
- if both s and t are odd and $t > 2s$ (with finitely many possible exceptions).

When both s and t are odd and $s \leq t \leq 2s-1$, this problem is open. It is known that $D \geq \frac{(t-1)(s-1)}{4}$, with some possible exceptions. The idea of the proof is as follows: add a vertex to each bipartite set so that both sides are even. Then, there exists an optimal packing. Delete all cycles containing the added vertices. The construction is not optimal when $s = t = 5$ since it give a packing with four cycles, and an optimal solution has five.

Problem 7. [Ada Chan] This problem regards type II matrices.

Let $W = [w_{ij}]$ be an $n \times n$ complex matrix such that $w_{ij} \neq 0$ for any i, j. Then $W^{(-)} = [\frac{1}{w_{ij}}]$. Then W is a type II matrix if $WW^{(-)T} = nI$.

Examples: if $|w_{ij}| = 1$ for all i, j, these are (complex) Hadamard matrices:

$$n = 2 \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$n = 3 \quad \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{bmatrix}, \omega^3 = 1.$$

When:

- $n \leq 5$, complex Hadamard matrices are classified
- $n = 6$, solution is conjectured
- $n > 6$ is open problem

Relation to quantum walks: Let $A(X)$ be the adjacency matrix of X. Let t be time. Then continuous time quantum walk satisfies that

$$|e_{uv}^{-itA(X)}|^2 = \text{ probability of starting at vertex } u \text{ and ending at vertex } v \text{ after time } t$$

Graph X on n vertices has instantaneous uniform mixing (IUM) at time τ if

$$|e_{uv}^{-i\tau A(X)}|^2 = \frac{1}{n}, \forall u, v$$

Conjecture: Only C_3 and C_4 have IUM.

It is known that if n is even or an odd prime $n \geq 5$, C_n does not have IUM.

Question: What about n odd, not a prime?

Problem 8. [Robert Bailey] Consider the following generalization of the adjacency matrix of a graph. Let G be a graph on n vertices. Then

$$S(G) = \left\{ A = [A_{ij}] \; : \; A_{ij} = \begin{cases} 0, & \text{if } i \neq j \text{ and } i \nsim j \\ \neq 0, & \text{if } i \neq j \text{ and } i \sim j \end{cases} \quad \text{and } A = A^T \right\}.$$

That is, $S(G)$ is a collection of matrices for which G is the graph, and we do not care about diagonal elements.

General type of questions: what are algebraic properties satisfied by this set of matrices for a fixed graph?

Let $g(G)$ be the least number of eigenvalues for any $A \in S(G)$.

If $g(G) = 1$, then $A = nI$, so G is an empty graph.

When is $g(G) = 2$? Real symmetric matrices matrices with two eigenvalues are orthogonal, i.e. $AA^T = I$.

If G is a bipartite graph, $A \in S(G)$ is a block matrix:

$$A = \left[\begin{array}{c|c} D & B \\ \hline B^T & D' \end{array} \right],$$

where D and D' are diagonal matrices. Hence, B defines A. Therefore, the analog of this question for bipartite graphs is to determine orthogonal B's.

Example: When $G = K_{n,n}$, $2 \leq q(G) \leq 3$. Moreover,

$$B = I - \frac{1}{n} J$$

is an orthogonal, nowhere zero matrix, so $q(G) = 2$. On the other hand, $q(K_{m,n}) = 3$ when $m \neq n$.
Question: What about $G = K_{n,n} - I$? From the adjacency matrix of G, we know that $q(G) \leq 4$. Deletion of an edge is not a monotonic operation for $q(G)$; $q(G)$ might go up or down.
Question: Does there exist an $n \times n$ real matrix with zero on diagonal, non-zero elsewhere which is orthogonal? Yes, if n is even (conference matrix when $n \equiv 0 \pmod 4$; generally, it can be done for $n \equiv 0 \pmod 2$). Is there a solution when n is odd?

There are examples for $n = 5, 7$. The following is one such matrix for $n = 5$. Let $a, b = \frac{-1 \pm \sqrt{3}}{2}$.

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & a & 1 & b \\ 1 & b & 0 & a & 1 \\ 1 & 1 & b & 0 & a \\ 1 & a & 1 & b & 0 \end{bmatrix}$$