WALKS ON ORDINALS AND THEIR CHARACTERISTICS

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Fields Institute, Sept. 6, 2012

Outline

A basis problem

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of L-structures on ω such that for every $\mathfrak{A} \in \mathcal{K}_L$ there is $1 \leq i \leq n(L)$ and an **infinite set** *$M \subseteq \omega$ such that*

$$\mathfrak{A} \restriction M = \mathfrak{B}_i \restriction M.$$

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Question

Can there be a similar result for other index-sets Γ in place of ω? What about the set ω_1 of all countable ordinals?

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Theorem (Ramsey 1930)

For every positive integer k *the class of* **equivalence relations** $\mathcal{E}$ *on*

$$[\omega]^k = \{x \subseteq \omega : |x| = k\}$$

with **finite quotients** $[\omega]^k/\mathcal{E}$ *has the* **1-element Ramsey basis**

$$\mathcal{E}_k = \{(a, b) \in [\omega]^k \times [\omega]^k : a = a\},$$

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Theorem (Erdős-Rado 1950)

For every positive integer k the class of **all** *equivalence relations on $[\omega]^k$ has the 2^k-element Ramsey basis*

$$E_I \ (I \in \mathcal{P}(k)),$$

where for $I \subseteq \{0, 1, ..., k-1\}$ and $a, b \in [\omega]^k$ we set $a \ E_I \ b$ iff $a \restriction I = b \restriction I$.

Accessible cardinals

Remark

1. No other **accessible** index set Γ can have such a strong property, a 1-element Ramsey basis for even equivalence relations on $[\Gamma]^2$.
2. For example, the class of equivalence relations on $[\mathbb{R}]^2$ has no finite Ramsey basis (Galvin-Shelah 1973).

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Question

Are there **accessible** index sets Γ for which the class of equivalence relations on $[\Gamma]^2$ admits a **finite** Ramsey basis? What about the set ω_1 of all countable ordinals?

Theorem (Erdős-Hajnal-Rado, 1965)

If Γ *is, for example, equal to*

$$\beth_\omega = \sup\{2^\omega, 2^{2^\omega}, \ldots\}$$

then for every positive integer k *there is an equivalence relation* $\mathcal{E}_k$ *on* $[\Gamma]^k$ *with* 2^{k-1} *classes such that for every other equivalence relation* $\mathcal{E}$ *on* $[\Gamma]^k$ *with* **finite quotient space** *there is* $X \subseteq \Gamma$ *of cardinality* Γ *such that*

$$\mathcal{E} \restriction [X]^k \subseteq \mathcal{E}_k \restriction [X]^k.$$

Moreover $\mathcal{E}_k$ *is* **irreducible** *in the sense that*

$$|[X]^k/\mathcal{E}_k| = 2^{k-1}$$

for every $X \subseteq \Gamma$ *of cardinality* Γ.

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Let $\mathcal{GS}_2$ be the equivalence relation on $[\omega_1]^2$ defined by letting $\{\alpha, \beta\}$ be equivalent to $\{\gamma, \delta\}$ iff

$$(\forall R \in \{<, <_S, <_A\})[\alpha R \beta \Leftrightarrow \gamma R \delta].$$

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Theorem (Sierpinski 1933; Galvin-Shelah 1973)

The equivalence relation $\mathcal{GS}_2$ is **irreducible***, i.e.,*

$$|[X]^2/\mathcal{GS}_2| = 4$$

for all uncountable $X \subseteq \omega_1$.

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4. The class of uncountable **Hausdorff spaces** have a finite basis. In particular, the class of uncountable regular spaces has a 3-element basis.
5. If a graph $\mathcal{G}$ on the vertex-set ω_1 has an uncountable complete or discrete subgraph iff $\mathcal{G}$ has such a subgraph in a **forcing extension** which preserves ω_1

Von Neumann's ordinals and Cantor's normal form

Von Neumann's ordinals:

$$\beta = \{\alpha : \alpha < \beta\}$$

$$0 = \emptyset, \quad 1 = \{0\}, \quad 2 = \{0,1\}, \quad 3 = \{0,1,2\}, ...,$$

$$\omega = \{0,1,2,....\}, \quad \omega + 1 = \omega \cup \{\omega\}, \quad \omega + 2 = \omega \cup \{\omega, \omega + 1\}, ...$$

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Cantor's normal form:

$$\alpha = n_1\omega^{\alpha_1} + n_2\omega^{\alpha_2} + \cdots + n_k\omega^{\alpha_k}$$

where $\alpha_1 > \alpha_2 > \cdots > \alpha_k \geq 0$ are ordinals and $n_1, n_2, ..., n_k$ natural numbers.

Fundamental sequences below $\varepsilon_0 = \min\{\alpha : \alpha = \omega^\alpha\}$

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$$C_\alpha = \{c_\alpha(0), c_\alpha(1), c_\alpha(2), ...\} \nearrow \alpha :$$

$$c_{\alpha+1}(n) = \alpha,$$

$$c_\omega(n) = n,$$

$$c_{\beta+\omega^{\alpha+1}}(n) = \beta + n\omega^\alpha,$$

$$c_{\beta+\omega^\alpha}(n) = \beta + \omega^{c_\alpha(n)},$$

$$c_{\varepsilon_0}(n+1) = \omega^{c_{\varepsilon_0}(n)}.$$

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For a given integer n, the length of the classical walk from α to 0 starting with $\alpha \curvearrowright c_\alpha(n)$ is equal to $H_\alpha(n)$.

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Definition (G.H. Hardy, 1904)

$$H_0(n) = n,$$

$$H_{\alpha+1}(n) = H_\alpha(n+1),$$

$$H_\alpha(n) = H_{c_\alpha(n)}(n).$$

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such that for all $i < k$, the step $\beta_i \curvearrowright \beta_{i+1}$ is the minimal step from β_i towards α, i.e.

$$\beta_{i+1} = c_{\beta_i}(n(\alpha, \beta_i)).$$

The full code of the walk

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The full code of the minimal walk is given by the formula

$$\rho_0(\alpha, \beta) = n(\alpha, \beta)^\frown \rho_0(\alpha, c_\beta(n(\alpha, \beta))),$$

with the boundary value

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Note that this is simply the sequence of integers

$$\rho_0(\alpha,\beta) = (n(\alpha,\beta_i) : i < k)$$

that code the steps of the minimal walk

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$$\rho_1(\alpha, \beta) = \max \begin{cases} n(\alpha, \beta), \\ \rho_1(\alpha, c_\beta(n(\alpha, \beta))) \end{cases}$$

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The length of the walk is given by

$$\rho_2(\alpha, \beta) = \rho_2(\alpha, c_\beta(n(\alpha, \beta))) + 1$$

with the boundary value

$$\rho_2(\alpha, \alpha) = 0.$$

The fundamental property of $\rho_0(\alpha, \beta)$

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is identified with the rational number

$$\frac{1}{n_0 + \frac{1}{n_1 + \frac{1}{n_2 + \cdots \frac{1}{n_k}}}}$$

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then the Von Neumann equality

$$\beta = \{\alpha : \alpha < \beta\}$$

becomes the identification

$$\beta \cong \{\rho_0(\alpha, \beta) : \alpha < \beta\} \subseteq \mathbb{Q}.$$

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(**Coherence:**)
For all $\alpha < \beta$,

$$\{\xi < \alpha : \rho_1(\xi, \alpha) \neq \rho_1(\xi, \beta)\}$$

is a finite set.

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(**Unboundedness:**)
For every pair A and B of uncountable subsets of ω_1,

$$\sup\{\rho_2(\alpha, \beta) : \alpha \in A, \beta \in B, \quad \alpha < \beta\} = \infty.$$

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(ℓ_∞-**Coherence:**)
For every $\alpha < \beta < \omega_1$,

$$\sup_{\xi < \alpha} |\rho_1(\xi, \alpha) - \rho_2(\xi, \beta)| < \infty.$$

The metric characteristic

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$$\rho(\alpha,\beta) = \max \begin{cases} n(\alpha,\beta) \\ \rho(\alpha, c_\beta(n(\alpha,\beta))) \\ \rho(c_\beta(n),\alpha) & n < n(\alpha,\beta). \end{cases}$$

with the boundary value $\rho(\alpha,\alpha) = 0$.

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(**Triangle inequalities:**)
For all $\alpha < \beta < \gamma$,

$$\rho(\alpha,\gamma) \leq \max\{\rho(\alpha,\beta), \rho(\beta,\gamma)\},$$

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Some applications of the ρ-structure

Recall that a normalized sequence (x_n) in some normed space $(X, \|\cdot\|)$ is **unconditional** whenever there is a constant $C \geq 1$ such that

$$\Big\| \sum_{i \in I} a_i x_i \Big\| \leq C \Big\| \sum_{j \in J} a_j x_j \Big\|$$

for any pair $I \subseteq J$ of finite subsets of ω and for every sequence $(a_j : j \in J)$ of scalars.

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Theorem (Argyros-LopezAbad-T., 2005)

There is a reflexive space of density $\aleph_1$ with no infinite unconditional basic sequence.

Theorem (LopezAbad-T., 2011)

For every $k < \omega$ there is a weakly null sequence of length ω_k with no infinite unconditional basic subsequence.

The characteristic tree

The characteristic tree

To any characteristic $a : [\omega_1]^2 \to \omega$, we associate the corresponding **tree**

$$T(a) = \{a(\,\cdot, \beta) \restriction \alpha : \alpha \leq \beta < \omega_1\}$$

and the corresponding **distance function**

$$\Delta_a : [\omega_1]^2 \to \omega_1 \cup \{\infty\}$$

defined by

$$\Delta_a(\alpha, \beta) = \min\{\xi < \alpha : a(\xi, \alpha) \neq a(\xi, \beta)\}.$$

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Definition

A characteristics $a : [\omega_1]^2 \to \omega$ is **Lipschitz** *if for every map $f : A \to \omega_1$ on an uncountable subset A of ω_1 such that $f(\alpha) > \alpha$ for all $\alpha \in A$ there is uncountable $B \subseteq A$ such that*

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$$\Delta_a(\alpha, \beta) = \Delta(f(\alpha), f(\beta)) \neq \infty \text{ for all } \alpha, \beta \in B, \alpha < \beta.$$

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Two characteristics $a : [\omega_1]^2 \to \omega$ and $b : [\omega_1]^2 \to \omega$ are **metrically equivalent** if there is an uncountable $X \subseteq \omega_1$ such that

(i) $\Delta_a(\alpha, \beta) \neq \infty$ and $\Delta_b(\alpha, \beta) \neq \infty$ for all $\alpha, \beta \in X$ with $\alpha < \beta$,

(ii) for every quadruple $\alpha, \beta, \gamma, \delta \in X$ such that $\alpha < \beta$ and $\gamma < \delta$,

$$\Delta_a(\alpha, \beta) > \Delta_a(\gamma, \delta) \text{ if and only if } \Delta_b(\alpha, \beta) > \Delta_b(\gamma, \delta).$$

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Theorem (T., 2007)

Assuming $\mathfrak{mm} > \omega_1$, every pair of Lipschitz characteristics $a : [\omega_1]^2 \to \omega$ and $b : [\omega_1]^2 \to \omega$ are metrically equivalent.

Theorem (T., 2000)

1. *The characteristics* $\rho, \rho_0, \rho_1, \rho_2$ *of the minimal walk are all Lipschitz.*
2. *Assuming* $\mathfrak{mm} > \omega_1$, *all Lipschitz trees are* **shift equivalent** *in the sense that for every pair* $a : [\omega_1]^2 \to \omega$ *and* $b : [\omega_1]^2 \to \omega$ *of Lipschitz characteristics there is a strictly increasing partial map* $\sigma : \omega_1 \to \omega_1$ *such that*

$$T(a) \equiv T(b)^{(\sigma)} \text{ or } T(b) \equiv T(a)^{(\sigma)}.$$

3. *Assuming* $\mathfrak{mm} > \omega_1$, *the class* $[T(\rho_1)]$ *of Lipschitz trees is* Σ_1*-definable in* $(H(\omega_1), \in)$ *and it is cofinal and coinitial in the class of all counterexamples to König's lemma at the level* ω_1.

Corollary

Assuming $\mathfrak{mm} > \omega_1$, *up to the metric equivalence, the characteristics* $\rho, \rho_0, \rho_1, \rho_2$ *of the minimal walk do not depend on the choice of the fundamental sequence* C_α $(\alpha < \omega_1)$.

The upper trace and its oscillations

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The upper trace of the walk

$$\beta = \beta_0 \curvearrowright \beta_1 \curvearrowright \cdots \curvearrowright \beta_k = \alpha$$

from β towards $\alpha < \beta$ is the set

$$\mathrm{Tr}(\alpha, \beta) = \{\beta_i : i \leq k\}.$$

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The oscillation mapping is given by

$$\mathrm{o}_0(\alpha, \beta) = \mathrm{osc}(\mathrm{Tr}(\Delta(\alpha, \beta) - 1, \alpha), \mathrm{Tr}(\Delta(\alpha, \beta) - 1, \beta)),$$

where

$$\Delta(\alpha, \beta) = \min\{\xi : \rho_0(\xi, \alpha) \neq \rho_0(\xi, \beta)\}.$$

The fundamental property of the oscillation mapping

Theorem (T., 1987)

For every uncountable $\Gamma \subseteq \omega_1$ *and every integer* $n \geq 2$ *there exist* $\alpha < \beta$ *in* Γ *such that* $\mathrm{o}_0(\alpha, \beta) = n$.

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Corollary

The class of equivalence relations on $[\omega_1]^2$ *does not have a finite Ramsey basis.*

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Corollary

The class of graphs on ω_1 *does not have a finite basis.*

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Theorem (T., 1987)

For every uncountable $\Gamma \subseteq \omega_1$ *and every integer* $n \geq 2$ *there exist* $\alpha < \beta$ *in* Γ *such that* $\mathrm{o}_0(\alpha, \beta) = n$*.*

Corollary

The class of equivalence relations on $[\omega_1]^2$ *does not have a finite Ramsey basis.*

Corollary

The class of graphs on ω_1 *does not have a finite basis.*

Question

Can similar results be proved for other basis problems mentioned above?

The canonical ordering on ω_1

For $\alpha \neq \beta$ in ω_1, set

$$\alpha <_{\rho_0} \beta \text{ iff } \rho_0(\Delta(\alpha,\beta),\alpha) < \rho_0(\Delta(\alpha,\beta),\beta).$$

Let

$$C(\rho_0) = (\omega_1, <_{\rho_0}).$$

Theorem (T., 1987)

1. *$C(\rho_0)$ is a linearly ordered set whose cartesian square can be decomposed into countably many chains.*
2. *Assuming $\mathfrak{m} > \omega_1$, the ordering $C(\rho_0)$ is a minimal uncountable linear ordering and its equivalence class*

$$[C(\rho_0)] = \{K \in \mathcal{LO} : K \leq C(\rho_0) \text{ and } C(\rho_0) \leq K\}$$

   does not depend on the choice of the sequence C_α $(\alpha < \omega_1)$.

3. *Assuming $\mathfrak{m} > \omega_1$, the class $[C(\rho_0)]$ is Σ_1-definable in $(H(\omega_2), \in)$.*

Theorem (Moore, 2005)

Assuming $\mathfrak{mm} > \omega_1$,

$$C(\rho_0) \leq L \text{ or } C(\rho_0)^* \leq L$$

for every **non-separable** *linear ordering* L *such that*

$$\omega_1 \not\leq L \text{ and } \omega_1^* \not\leq L.$$

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Theorem (Baumgartner, 1973)

Assume $\mathfrak{mm} > \omega_1$ *and let* B *be* **any set of reals** *of cardinality* $\aleph_1$ *with its usual ordering. Then*

$$B \leq L$$

for every **separable** *linear ordering* L.

Let
$$\mathcal{A} = \{L \in \mathcal{LO} : B \not\leq L,\ \omega_1 \not\leq L \text{ and } \omega_1^* \not\leq L\}.$$

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Theorem (Martinez-Ranero, 2010)

Assuming $\mathfrak{mm} > \omega_1$, the class $\mathcal{A}$ is **well-quasi-ordered,** *i.e., for every sequence*

$$(L_i : i < \omega) \subseteq \mathcal{A}$$

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Note that this includes to the following classical result which verifies an old conjecture of Fraïssé.

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Remark

Note that this includes to the following classical result which verifies an old conjecture of Fraïssé.

Theorem (Laver, 1970)

The class $\mathcal{LO}_\omega$ *of* **countable** *linear orderings is well-quasi-ordered.*

Oscillation on lower trace

Oscillation on lower trace

The lower trace of the minimal walk

$$\beta = \beta_0 \curvearrowright \beta_1 \curvearrowright \cdots \curvearrowright \beta_k = \alpha$$

is the set

$$L(\alpha, \beta) = \{\max\{\max(C_{\beta_i} \cap \alpha) : i \leq j\} : j < k\}.$$

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The corresponding **oscillation function** is defined as follows

$$\mathrm{o}_1(\alpha, \beta) = |\{\xi \in L(\alpha, \beta) : \rho_1(\xi, \alpha) \leq \rho_1(\xi, \beta) \wedge \rho_1(\xi^+, \alpha) > \rho_1(\xi^+, \beta)\}|,$$

where for $\xi \in L(\alpha, \beta)$,

$$\xi^+ = \min(L(\alpha, \beta) \setminus \xi + 1).$$

Theorem (Moore, 2005)

1. *For every pair A, B of uncountable subsets of ω_1, the set*

$$\{\mathrm{o}_1(\alpha, \beta) : \alpha \in A, \beta \in B, \alpha < \beta\}$$

is a **syndetic** *set of integers.*

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Assuming $\mathfrak{mm} > \omega_1$, every regular hereditarily separable space is Lindelöf.

Question ($\mathfrak{mm} > \omega_1$)

Does the class of uncountable (regular) **first countable** spaces have finite basis?

Conjecture (Fremlin, 1988)

Assume $\mathfrak{mm} > \omega_1$. Show that every compact space K either

1. K contains an uncountable discrete subspace, or

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Example

The split interval is the product $[0,1] \times \{0,1\}$ ordered lexicographically. It has no uncountable discrete subspace and is a 2-to-1 preimage of the unit interval.

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Let K be a compact subset of a Tychonoff cube $[0,1]^X$ consisting of Baire-class-1 functions on some Polish space X. Then either

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1. *K contains an uncountable discrete subspace, or*
2. *K is an at most 2-to-1 preimage of a compact metric space.*

The canonical ultrafilter on ω_1

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For a characteristic $a : [\omega_1]^2 \to \omega$ and $X \subseteq \omega_1$, we set

$$\Delta_a[X] = \{\Delta_a(\alpha, \beta) : \alpha, \beta \in X, \alpha < \beta \text{ and } \Delta_a(\alpha, \beta) \neq \infty\}.$$

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Proposition

If a characteristic $a : [\omega_1]^2 \to \omega$ is Lipschitz then for every pair X and Y of uncountable subsets of ω_1 there is an uncountable subset Z of X such that $\Delta_a[Z] \subseteq \Delta_a[X] \cap \Delta_a[Y]$.

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Corollary

If a characteristic $a : [\omega_1]^2 \to \omega$ is Lipschitz then the family

$$\{\Delta_a[X] : X \subseteq \omega_1 \textit{ and } X \textit{ is uncountable}\}$$

generates a **uniform filter** *$\mathcal{U}_a$ on ω_1.*

Theorem (T., 2000)

1. *Assuming* $\mathfrak{m} > \omega_1$, *for every Lipschitz characteristic* $a : [\omega_1]^2 \to \omega$, *the filter* $\mathcal{U}_a$ *is in fact an* **ultrafilter**.
2. *Assuming* $\mathfrak{mm} > \omega_1$, *for Lipschitz characteristics* $a : [\omega_1]^2 \to \omega$ *and* $b : [\omega_1]^2 \to \omega$,

$$T(a) \equiv T(b) \text{ iff } \mathcal{U}_a = \mathcal{U}_b.$$

3. *Assuming* $\mathfrak{mm} > \omega_1$, *for every pair of Lipschitz characteristics* $a : [\omega_1]^2 \to \omega$ *and* $b : [\omega_1]^2 \to \omega$,

$$\mathcal{U}_a \equiv_{\mathrm{RK}} \mathcal{U}_b.$$

Corollary

Assuming $\mathfrak{mm} > \omega_1$, *the filter* $\mathcal{U}_{\rho_0}$ *is a* Σ_1*-definable, in* $(H(\omega_1), \in)$, *uniform ultrafilter on* ω_1 *whose Rudin-Keisler class does not depend on the choice of the fundamental sequence* C_α $(\alpha < \omega_1)$ *that defines the characteristic* ρ_0 *of the minimal walk.*

The canonical selective ultrafilter on ω

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Recall that to any characteristic $a : [\omega_1]^2 \to \omega$ we associate the corresponding filter on ω_1,

$$\mathcal{U}_a = \{Y \subseteq \omega_1 : (\exists X \subseteq \omega_1)\ [X \text{ is uncountable and } \Delta_a[X] \subseteq Y]\}.$$

It is also natural to consider its Rudin-Keisler images to ω via maps $f : \omega_1 \to \omega$,

$$f[\mathcal{U}_a] = \{X \subseteq \omega : f^{-1}(X) \in \mathcal{U}_a\}.$$

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Assuming $\mathfrak{m} > \omega_1$, for every Lipschitz characteristic $a : [\omega_1]^2 \to \omega$ and every $f : \omega_1 \to \omega$, the filter

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is an ultrafilter on ω that is Σ_1-definable in $(H(\omega_1), \in)$.

Question

Which kind of ultrafilter is $\mathcal{V}_a^f$? How canonical is it?

Recall that an ultrafilter $\mathcal{W}$ on ω is **selective** if for every $h : \omega \to \omega$ there is $M \in \mathcal{W}$ such that

$$h \restriction M \text{ is } \textbf{one-to-one} \text{ or } \textbf{constant}.$$

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Proposition (T., 1990)

Assume that every set of reals in $L(\mathbb{R})$ is $2^{\mathfrak{c}}$-universally Baire. Then every selective ultrafilter on ω is $L(\mathbb{R})$-generic filter for the forcing notion $\mathcal{P}(\omega)/\mathrm{Fin}$.

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This assumption is fulfilled if, for example, there exist some large cardinals in the universe.

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Remark

This assumption is fulfilled if, for example, there exist some large cardinals in the universe.

Theorem (T., 2007)

Assuming $\mathfrak{m} > \omega_1$, for every Lipschitz characteristic $a : [\omega_1]^2 \to \omega$ and every mapping $f : \omega_1 \to \omega$, the filter $\mathcal{V}_a^f = f[\mathcal{U}_a]$ is a selective ultrafilter on ω.

Theorem (T., 2007)

Suppose that $a : [\omega_1]^2 \to \omega$ *and* $b : [\omega_1]^2 \to \omega$ *are two metrically equivalent Lipschitz characteristics.*

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Suppose that $f : \omega_1 \to \omega$ *and* $g : \omega_1 \to \omega$ *map* $\mathcal{U}_a$ *and* $\mathcal{U}_b$ *to two non-principal filters* $f[\mathcal{U}_a]$ *and* $g[\mathcal{U}_b]$ *on* ω.

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Question

Suppose that $a : [\omega_1]^2 \to \omega$ is equal to one of the standard characteristics ρ, ρ_0, ρ_1, or ρ_2 of the minimal walk.

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Question

Suppose that $a : [\omega_1]^2 \to \omega$ is equal to one of the standard characteristics ρ, ρ_0, ρ_1, or ρ_2 of the minimal walk.
Is there a **canonical map** $f : \omega_1 \to \omega$ so that the corresponding filter $f[\mathcal{U}_a]$ on ω is non-principal?

Let Λ be the set of countable limit ordinals. Let

$$d_\lambda : \omega_1 \to \omega$$

be the distance function to Λ, i.e., $d(\lambda + n) = n$ for $\lambda \in \Lambda$.

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Theorem (T., 2007)

Assuming $\mathfrak{mm} > \omega_1$, the selective ultrafilter $\mathcal{V}_{\rho_0}$ has its Rudin-Keisler class

$$[\mathcal{V}_{\rho_0}]_{\mathrm{RK}} = \{h[\mathcal{V}_{\rho_0}] : h \textit{ a permutation of } \omega\}$$

independent on the choice of the fundamental sequence C_α ($\alpha < \omega_1$) and Σ_1-definable in the structure $(H(\omega_1), \in)$.