Potential Enstrophy in Stratified Turbulence

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Introduction

- Potential enstrophy
  - integrated squared potential vorticity: $V = \frac{1}{2}\langle q^2 \rangle$
  - neglecting forcing & dissipation: $\mathrm{D}q/\mathrm{D}t = 0$, V is conserved
  - V-conservation important in QG turbulence (enstrophy cascade, inverse energy cascade)
  - what happens at larger *Ro* – atmospheric mesoscale & oceanic sub-mesoscale?
- Stratified turbulence
  - homogeneous turbulence in stratified fluid with weak or no rotation
  - model for geophysical turbulence at small-scale end of atmos meso and ocean sub-meso
  - connects large-scale QG turbulence with small-scale isotropic turbulence
  - waves, vortical modes, thin shear layers, K-H (reviews: Riley & Lelong 2000; Riley & Lindborg 2013)

Potential vorticity & enstrophy

- Ertel PV for Boussinesq fluid: $q = (f\hat{\mathbf{z}} + \boldsymbol{\omega}) \cdot (N^2\hat{\mathbf{z}} + \nabla b) = q_0 + q_1 + q_2$, where

$$q_0 = fN^2, \quad q_1 = N^2\omega_z + f\partial_z b, \quad q_2 = \boldsymbol{\omega} \cdot \nabla b$$

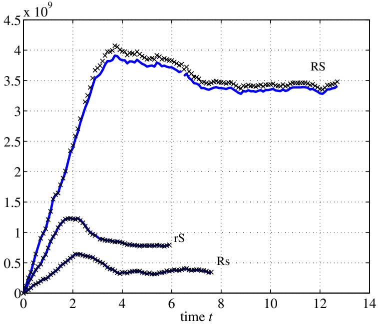
  - $f =$ Coriolis, $N =$ Brunt–Väisälä freq, $b =$buoyancy

- q is quadratic in $\boldsymbol{\omega}$ and b, so $V = V_2 + V_3 + V_4$ is a quartic invariant.
  - no detailed conservation of V by wavenumber triads
  - weird: viscosity & diffusion are not strictly dissipative (Herring, Kerr, Rotunno 1994)

$$\frac{\mathrm{D}q}{\mathrm{D}t} = \left(N^2\hat{\mathbf{z}} + \nabla b\right) \cdot \left(\nu\nabla^2\boldsymbol{\omega}\right) + \kappa\left(f\hat{\mathbf{z}} + \boldsymbol{\omega}\right) \cdot \nabla\left(\nabla^2 b\right)$$

Potential vorticity & enstrophy

- But under certain conditions, is V approximately quadratic? (i.e. $q \approx$ linear?)
  - yes, for QG turbulence
  - what about for large Ro?
  - Kurien, Smith & Wingate (2006), Aluie & Kurien (2011): V is $\approx$ quadratic for stratified turbulence
  - how generic is this result?



Aluie & Kurien, EPL 96, 44006, 2011

So what?

- Cascade theories:
  - quadratic $V \Rightarrow$ triad-by-triad conservation, like kinetic energy
  - relationship between energy and p. enstrophy: e.g. for $f = 0$ have $V(\boldsymbol{k}) = N^2 k_h^2 E_R(\boldsymbol{k})$
  - joint conservation constrains cascade as in 2D, QG: inverse cascade? (Lilly 1983)
- Decomposition into waves and vortices:
  - linear decomposition into vortical modes (with q_1) and gravity waves (no q_1)
  - e.g. stratified turbulence (Lelong & Riley 1991), rotating-stratified turbulence (Bartello 1995)
  - motivates decomp of KE spectra into horizontally rotational ($\approx$ vortical) and divergent ($\approx$ wave)
  - popular/easy decomposition, but meaningless if higher-order V terms important

Scale analysis of potential vorticity

- Usual scaling of terms (Lilly 1983, Riley & Lelong 2000) gives:

$$q_1 = N^2\omega_z + f\partial_z b \sim N^2 \frac{U}{L_h} \max\left(1, Fr_v^2/Ro\right) \quad \left(\text{assuming } \; b \sim U^2/L_v\right),$$

$$q_2 = \boldsymbol{\omega} \cdot \nabla b \sim \frac{U^3}{L_h L_v^2},$$

$$\Rightarrow q_2/q_1 \sim \min\left(Fr_v^2, Ro\right), \quad \text{where}$$

$$Fr_v = U/NL_v, \quad Ro = U/fL_h$$

- For strong rotation, $q_2/q_1 \sim Ro \ll 1$, so $V \approx V_2$ is quadratic
- For weak rotation $q_2/q_1 \sim Fr_v^2$ (W & Bartello 2006). How big is Fr_v?

Equations of motion

- Incompressible, Boussinesq, constant N
- Non-dimensionalize (e.g. Riley *et al.* 1981, Lilly 1983):

$$Fr_h \equiv \frac{U}{NL_h}, \quad Fr_v \equiv \frac{U}{NL_v}, \quad Re \equiv \frac{UL_h}{\nu}, \quad \alpha \equiv \frac{L_v}{L_h} \equiv \frac{Fr_h}{Fr_v}.$$

$$\begin{aligned}
\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} \boldsymbol{u} + Fr_v^2 w \frac{\partial \boldsymbol{u}}{\partial z} + \frac{1}{Ro} \hat{\boldsymbol{z}} \times \boldsymbol{u} &= -\boldsymbol{\nabla} p + \frac{1}{Re} \left(\nabla^2 + \frac{1}{\alpha^2} \frac{\partial^2}{\partial z^2} \right) \boldsymbol{u}, \\
Fr_h^2 \left(\frac{\partial w}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} w + Fr_v^2 w \frac{\partial w}{\partial z} \right) &= -\frac{\partial p}{\partial z} + b + \frac{Fr_h^2}{Re} \left(\nabla^2 + \frac{1}{\alpha^2} \frac{\partial^2}{\partial z^2} \right) w, \\
\boldsymbol{\nabla} \cdot \boldsymbol{u} + Fr_v^2 \frac{\partial w}{\partial z} &= 0, \\
\frac{\partial b}{\partial t} + \boldsymbol{u} \cdot \boldsymbol{\nabla} b + Fr_v^2 w \frac{\partial b}{\partial z} + w &= \frac{1}{Re} \left(\nabla^2 + \frac{1}{\alpha^2} \frac{\partial^2}{\partial z^2} \right) b.
\end{aligned}$$

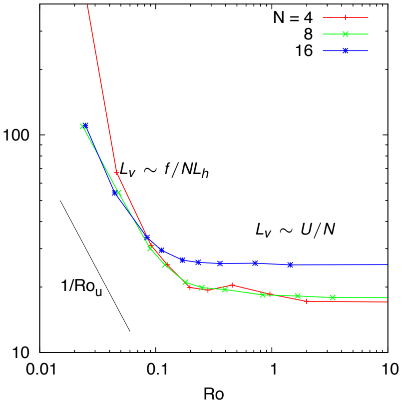
- stratified turbulence means $Fr_h \ll 1$, $Re \gg 1$. What about Fr_v?
- $Fr_v \ll 1 \Rightarrow$ quasi-2D, $Fr_v \sim O(1) \Rightarrow$ anisotropic 3D

Vertical scales in geophysical turbulence

- Size of Fr_v depends on L_v:
  - in QG turbulence, $L_v/L_h \sim f/N \Rightarrow Fr_v \sim Ro \ll 1$
  - In stratified turbulence $L_v \sim U/N \Rightarrow Fr_v \sim 1$ (e.g. Billant & Chomaz 2001)
  - $U/N = L_b$ buoyancy scale, “pancake” thickness (W & Bartello 04) at which $Ri \sim O(1)$.
  - but, need to be careful: assumes large Reynolds number $Re = UL_h/\nu$
- If *Re* not large enough, L_v is set by viscosity:
  - depends on buoyancy Reynolds number $Re_b = ReFr_h^2$ (e.g. Smyth & Moum 2000)
  - turbulence requires large Re_b (Riley & de Bruyn Kops 2003, Brethouwer et al 2007)
  - for $Re_b \gg 1$, viscous effects small and $Fr_v \sim 1$
  - for $Re_b \lesssim 1$, viscous effects important and $Fr_v \sim Re_b^{1/2}$
- Suggests that quadratic potential enstrophy may only be realized for $Re_b \ll 1$

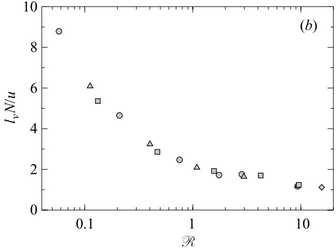
Vertical scales in geophysical turbulence

Rotating–Stratified: $L_v N/U$ vs Ro



W & Bartello, *J. Fluid Mech.* 568, 89-108, 2006

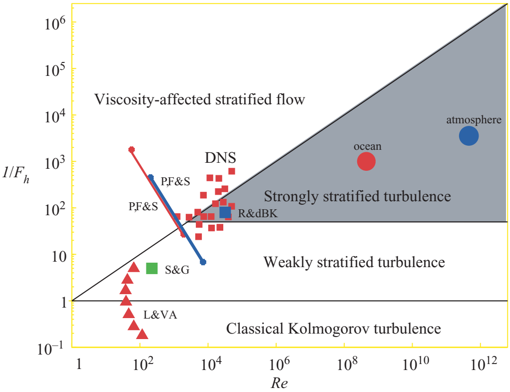
Stratified only: $L_v N/U$ vs Re_b



Brethouwer et al., *J. Fluid Mech.* 585, 343-368, 2007

Geophysical vs. lab/DNS regimes

- Typical values for atmospheric mesoscale: $Fr_h = 10^{-3}$ $Re = 10^{10}$, $Re_b = 10^4$
- Lab experiments and (most) DNS: $Re_b \lesssim 1$.
- A & O simulations may have smaller effective Re_b from eddy or numerical viscosity



Brethouwer et al., *J. Fluid Mech.* 585, 343-368, 2007

What we do

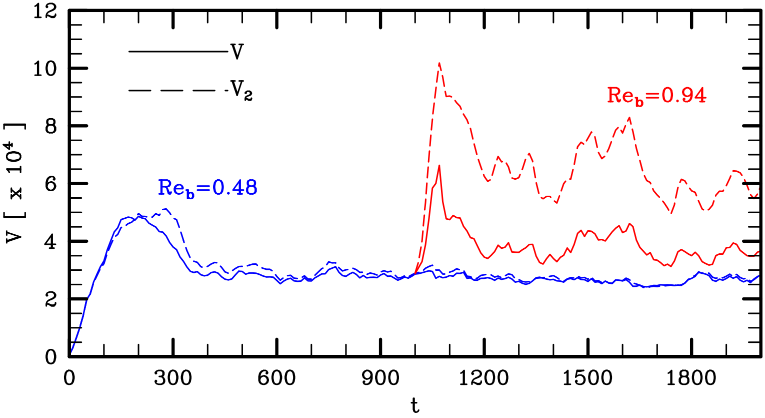
- **Direct numerical simulations of stratified turbulence with $Re_b \lesssim 4$**
- Questions:
  - how important are higher-order contributions to potential enstrophy?
  - test hypothesis that potential enstrophy $\approx$ quadratic only for $Re_b \ll 1$
  - implications for using idealized experiments/simulations as proxy for a & o?

Approach

- Numerical model
  - periodic BCs, constant N
  - spectra, FFT, de-aliased
  - DNS: $\Delta x = \Delta z \lesssim$ Kolmogorov scale
- Experimental set-up: lab-scale units
  - domain: $L = 2\pi$
  - force large-scale vortical modes
  - gives $U \approx 0.02$, $L_h \approx 4$, $T \approx 200$
  - run for 2000 time units; average over 1000-2000
  - set $\kappa = \nu$
- Vary N and ν to get:
  - $0.0004 \leq Fr_h \leq 0.02$
  - $4000 \leq Re \leq 20000$
  - $0.002 \leq Re_b \leq 4 \leftarrow$ not geophysical, but at least $O(1)$
  - resolution: 512^3, 960^3 (SciNet)

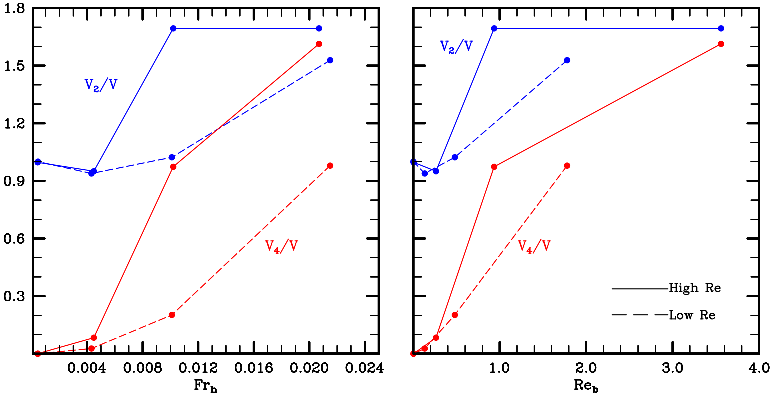
Time series of V and V_2 for $Fr_h = 0.01$

- V and V_2 for $Fr_h = 0.01$ with two different Re_b
  - relative size of V_2 depends on Re_b, even at fixed Fr_h.
  - higher-order terms important for $Re_b \approx 1$



Relative contributions of V_2 and V_4

- V_2/V and V_4/V vs Fr_h and Re_b
  - no collapse with Fr_h
  - see collapse with Re_b for small enough Re_b



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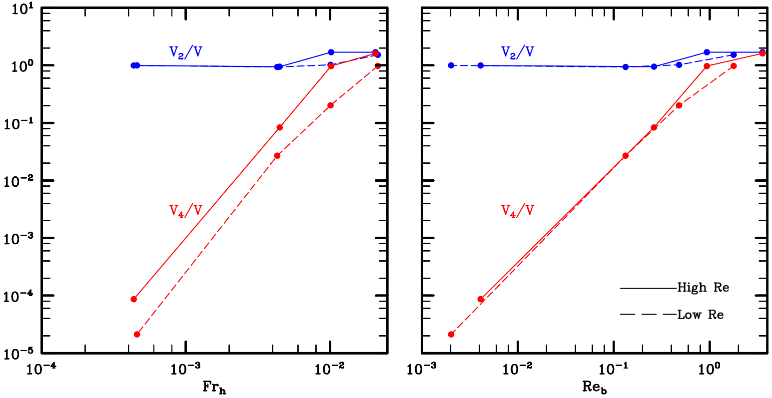
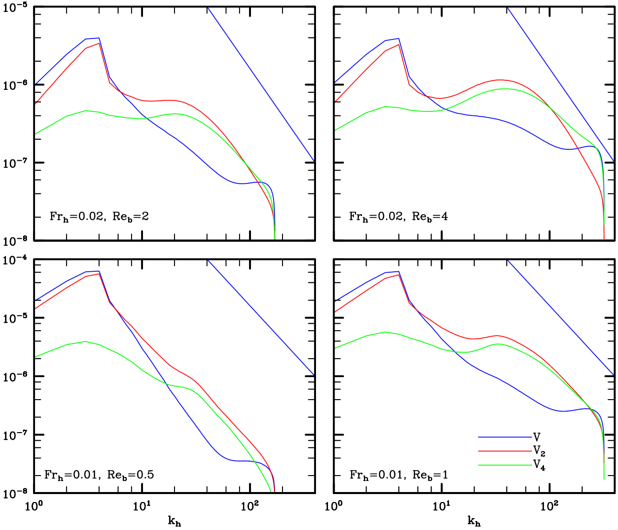
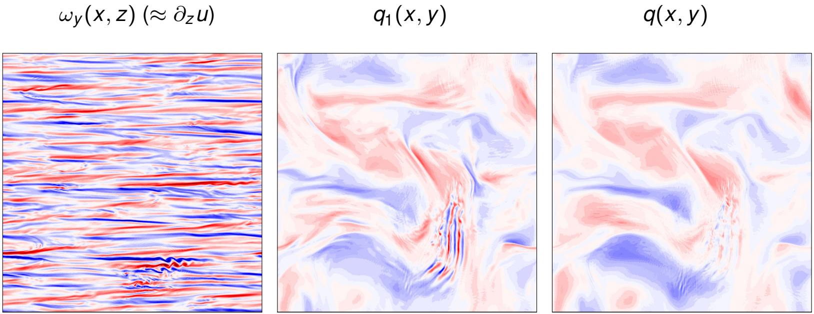


Potential enstrophy spectra

- Horizontal wavenumber spectra of V, V_2, and V_4



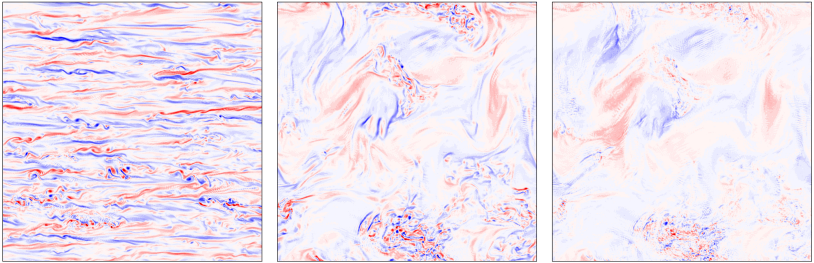
Snapshots: $Fr_h = 0.01$, $Re_b = 0.5$



- Intermittent KH instabilities (as in Laval, McWilliams & Dubrulle 2003, etc.)
  - show up in ω_z field, which contributes to q_1
  - but not (much) in q field
  - larger Re_b: more KH, transitions to small-scale 3D turb

Larger $Re_b = 2$

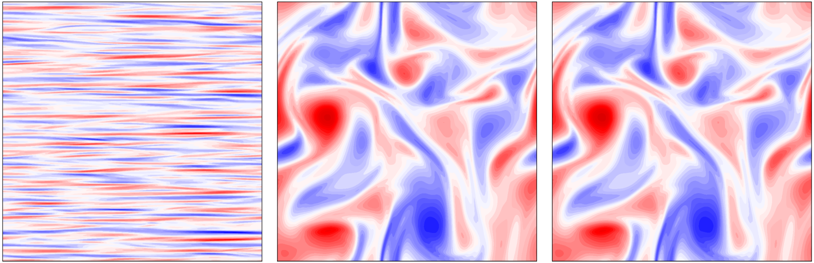
$\omega_y(x,z)$ ($\approx \partial_z u$) $\quad$ $q_1(x,y)$ $\quad$ $q(x,y)$



- Intermittent KH instabilities (as in Laval, McWilliams & Dubrulle 2003, etc.)
  - show up in ω_z field, which contributes to q_1
  - but not (much) in q field
  - larger Re_b: more KH, transitions to small-scale 3D turb

Smaller $Re_b = 0.1$

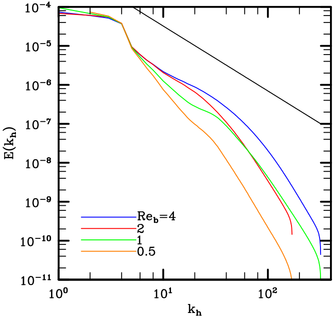
$\omega_y(x, z)$ ($\approx \partial_z u$) | $q_1(x, y)$ | $q(x, y)$

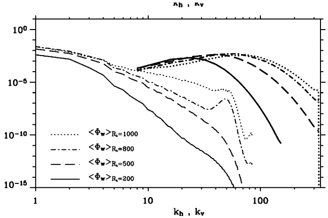


- Intermittent KH instabilities (as in Laval, McWilliams & Dubrulle 2003, etc.)
  - show up in ω_z field, which contributes to q_1
  - but not (much) in q field
  - larger Re_b: more KH, transitions to small-scale 3D turb

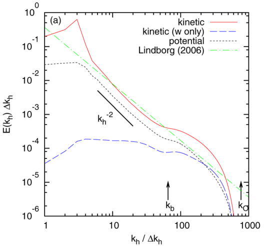
Energy spectra $E(k_h)$

- Bumps due to KH inst (Laval et al 2003)
- Position of bump at $k_h \approx N/U$ (Waite 2011)





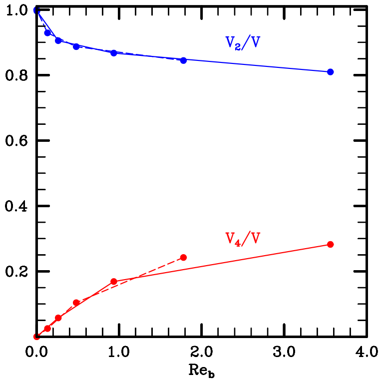
Laval, McWilliam & Dubrulle, *Phys. Rev. E* 68, 03608, 2003



Waite, *Phys. Fluids* 23, 06602, 2011

Relative contributions of V_2 and V_4 from large scales only

- Compute potential enstrophy from large horizontal scales (filter out KH billows)
  - nice collapse when plotted against Re_b
  - for small Re_b, $V \approx V_2$
  - higher-order contributions to V grow with increasing Re_b
  - consistent with KH interpretation, since $L_b/L_h \sim \sqrt{Re_b/Re}$



Discussion

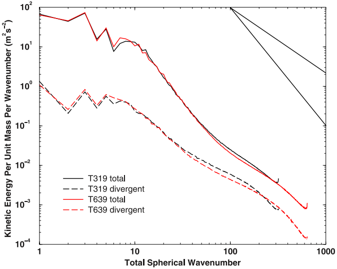
- Quadratic potential enstrophy **is not** a good approximation when $Re_b \gtrsim 1$, even for $Fr_h \ll 1$
  - regime of weakly (or marginally) viscous stratified turbulence
  - layerwise structure with KH instabilities and small-scale turbulence
  - breakdown of quadratic approximation occurs at small horizontal scales: KH instabilities?
- Quadratic potential enstrophy **is** a good approximation when $Re_b < 0.4$
  - regime of viscously coupled layerwise “pancakes”
  - no KH instabilities or transition to small-scale turbulence
  - likely that Aluie & Kurien (2011) is in this regime
- But Re_b does not tell the whole story
  - V_2/V does not collapse w.r.t. Re_b unless small scales are filtered

More info: Waite (2013), Potential enstrophy in stratified turbulence, *JFM* 722, R4.

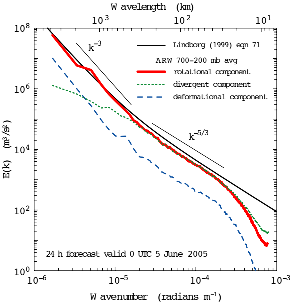
Discussion

- Implications for atmospheric and ocean:
  - back-of-the-envelope: atmospheric meso $Re_b \sim 10^4$, oceanic sub-meso $Re_b \sim 10^2$-10^3
  - quadratic approx seems doubtful here
- Atmospheric models may have small *effective* Re_b
  - Brune & Becker (2013) computed mesoscale $U/N \approx 80$ m, not resolved
  - artificially small mesoscale $Fr_v \Rightarrow$ quadratic V?
  - mesoscale cascade in these models probably not stratified turbulence
  - lack of consensus on decomposition of mesoscale spectrum into waves and vortical modes

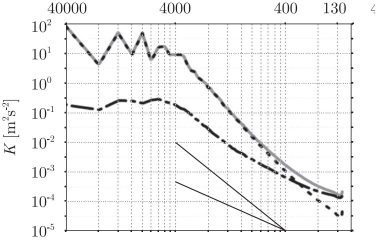
Discussion



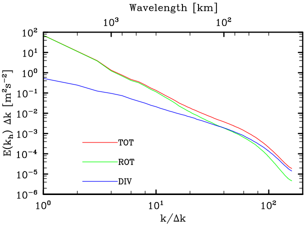
Hamilton, Takahashi & Ohfuchi, *GRL* 113, 2008



Skamarock & Klemp, *JCP* 227, 2008



Brune & Becker, *JAS* 70, 2013



Waite & Snyder, *JAS* 70, 2013

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