Water waves over a muddy seabed

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Allepey, S. India (Mathew, Baba & Kurian, 1995)

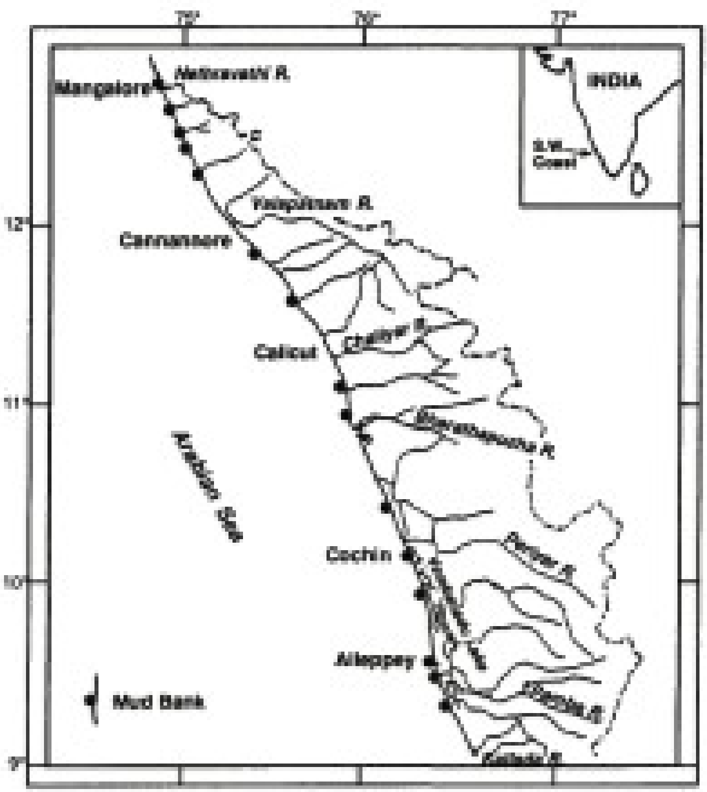


Figure 1. Locations of monsoonal mudbanks along the south-west coast of India.

Kurup (1977); Rao et al. (1983); Murty et al

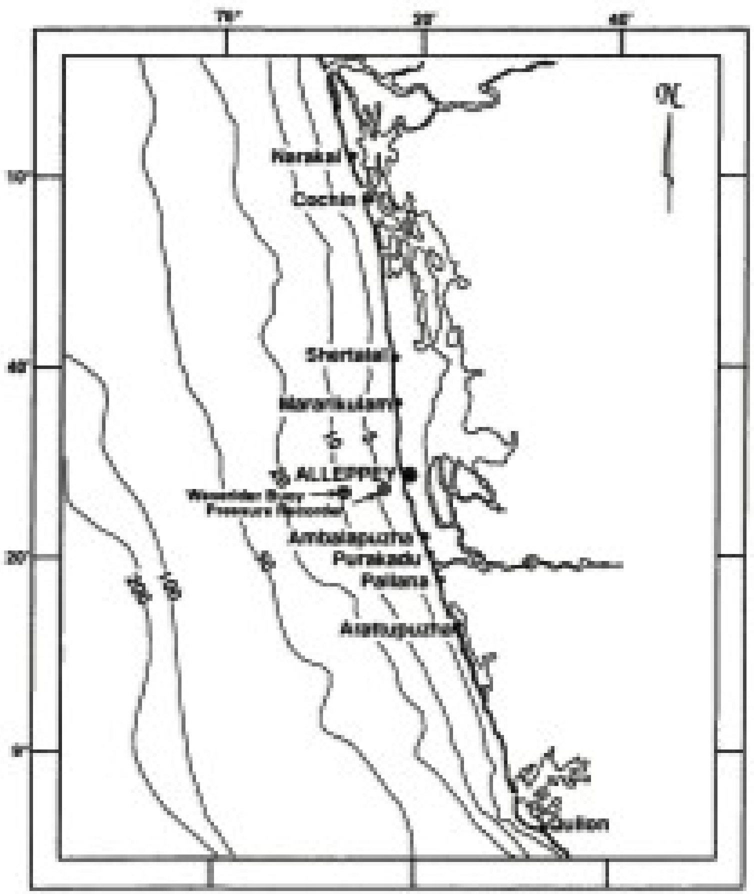
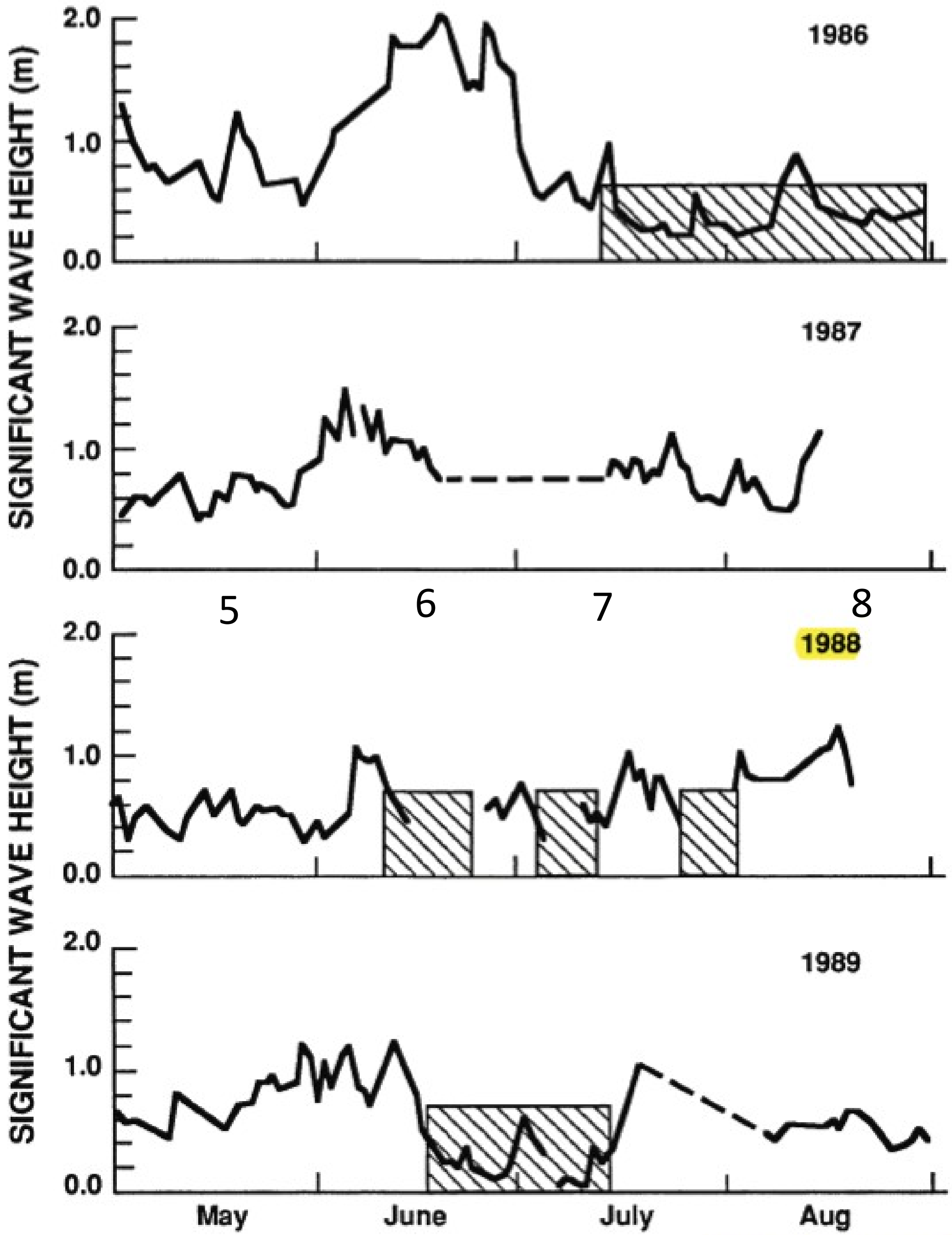


Figure 2. Area of study and locations of wave recording at Alleppey. Depths are in meters.



India, Mathew, Baba, Kurian JCR 1995

1989

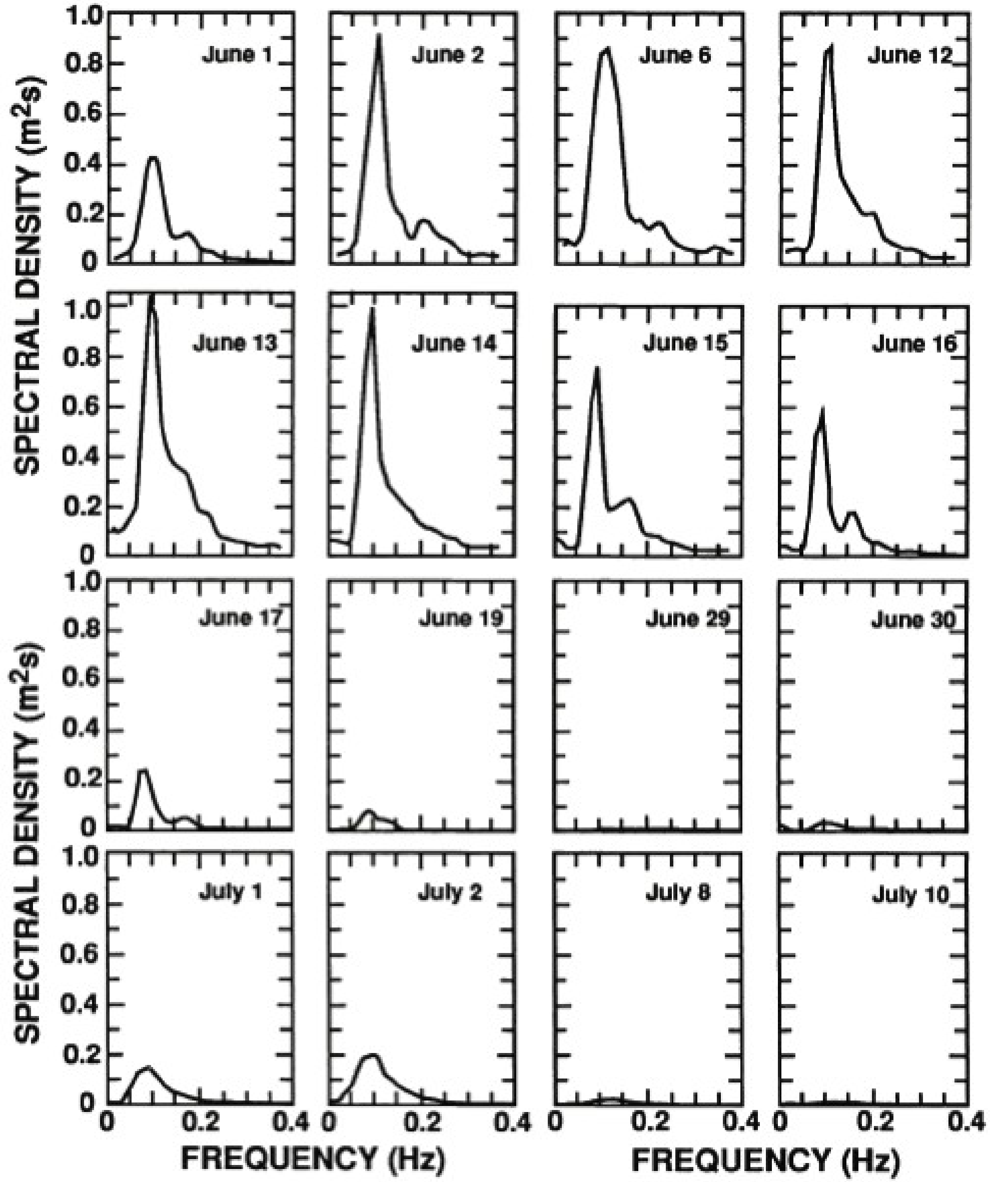
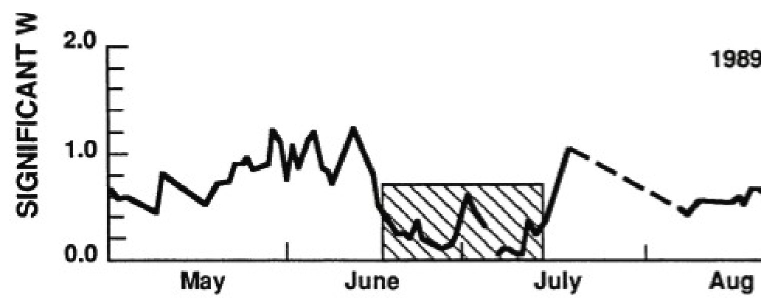


Figure 10. Transformation of nearshore wave spectra during pre-mudbank and mudbank conditions in 1989 (the mudbank dissipated on 20th July due to high wave activity, which damaged the wave recorder, hence no spectra could be obtained to show the dissipation stage).



Alleppey, S. India (Mathew et al 1995)

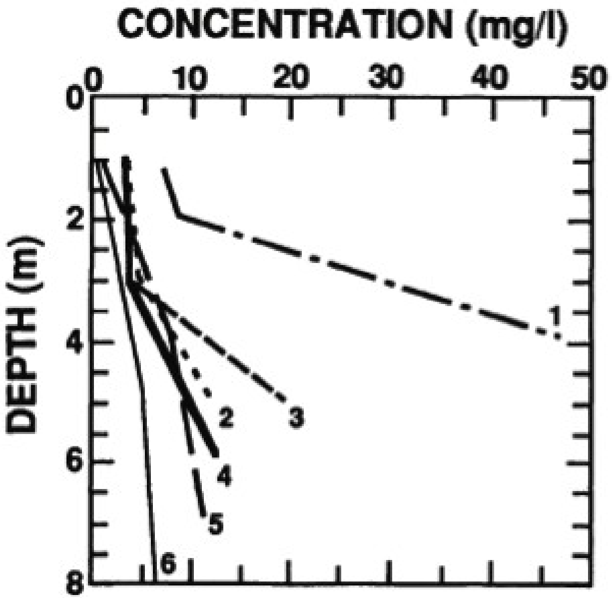


Figure 2. Vertical variation of SSC during a non-mudbank period (July, 1987); nos. 1, 2, 3, 4, 5 and 6 indicate stations, which correspond to depths of 5, 6, 7, 8, 9 and 10 m, respectively.

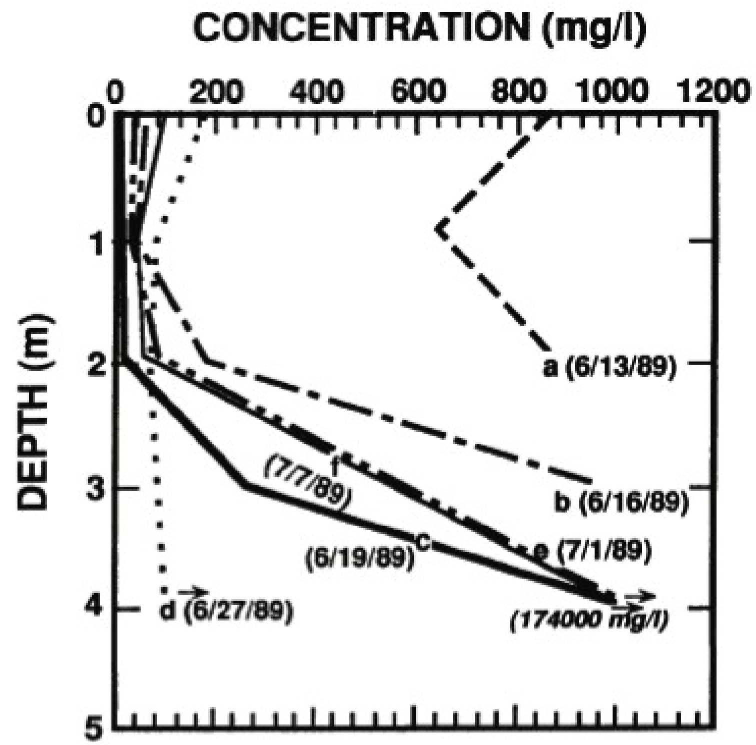


Figure 3. Vertical variation of SSC before and during mudbank formation: a) Before mudbank formation (nearshore wave height, H_s = 1.23 m). b) During mudbank formation (H_s = 0.77 m; suspended sediment concentration at the bottom, SSC_{bottom} = 12,500 mg/l). c) When the mudbank was formed (H_s = 0.31 m; SSC_{bottom} = 174,000 mg/l). d) When the mud was settling and spreading laterally (H_s = 0.10 m; SSC_{bottom} = 94,000 mg/l). e) When the offshore wave activity increased and currents became noticeable in the mudbank area (H_s = 0.50 m; SSC_{bottom} = 161,000 mg/l). f) When settling recommenced in the mudbank area (H_s = 0.45 m; SSC_{bottom} = 81,000 mg/l).

Past works

- Field survey:
  - Surinum: Wells & Colemen 1981
  - India: Mathew, Baba & Kurian, 1995
- Math models:
  - Newtonian mud: Dalrymple & Liu 刘立方 , Liu & Chan
  - Bingham plastic: Mei and Liu 刘格非 , Coussot
  - Simple Kelvin-Voigt viscoelastic: McPherson, Jiang & Mehta, Ng 吴朝安
  - Plasto-viscoelastic: Shibayama

Field mud samples

$$D_{37} = 5\ \mu m$$

$$D_{50} = 90\ \mu m$$

Fluid-mud rheology

Not Newtonian, Not Bingham plastic, but Viscoelastic

$$\text{Dynamic rheology} \quad \tau_{ij} = G\left(\frac{\partial \mathcal{U}_i}{\partial x_j} + \frac{\partial \mathcal{U}_j}{\partial x_i}\right) + \mu\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

$$\mathcal{U}_i: \quad \text{mud displacement}, \quad U_i: \quad \text{mud velocity}, \quad U_i = \frac{\partial \mathcal{U}_i}{\partial t}$$

For simple harmonic motion $\propto e^{-i\omega t}$

$$\tau_{ij} = \left(\mu + i\frac{G}{\omega}\right)\left(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\right)$$

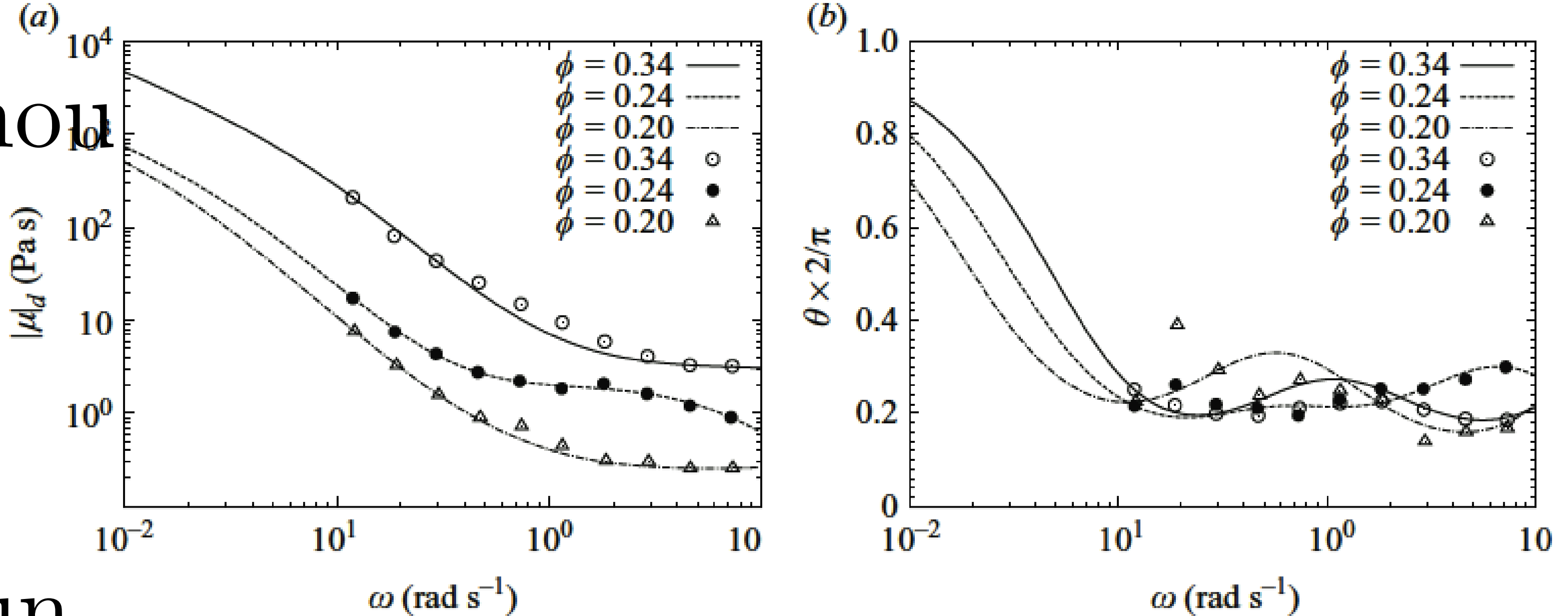
Denote :

$$\mu_c = \mu + i\frac{G}{\omega}, \quad G = G(\omega), \quad \mu = \mu(\omega)$$

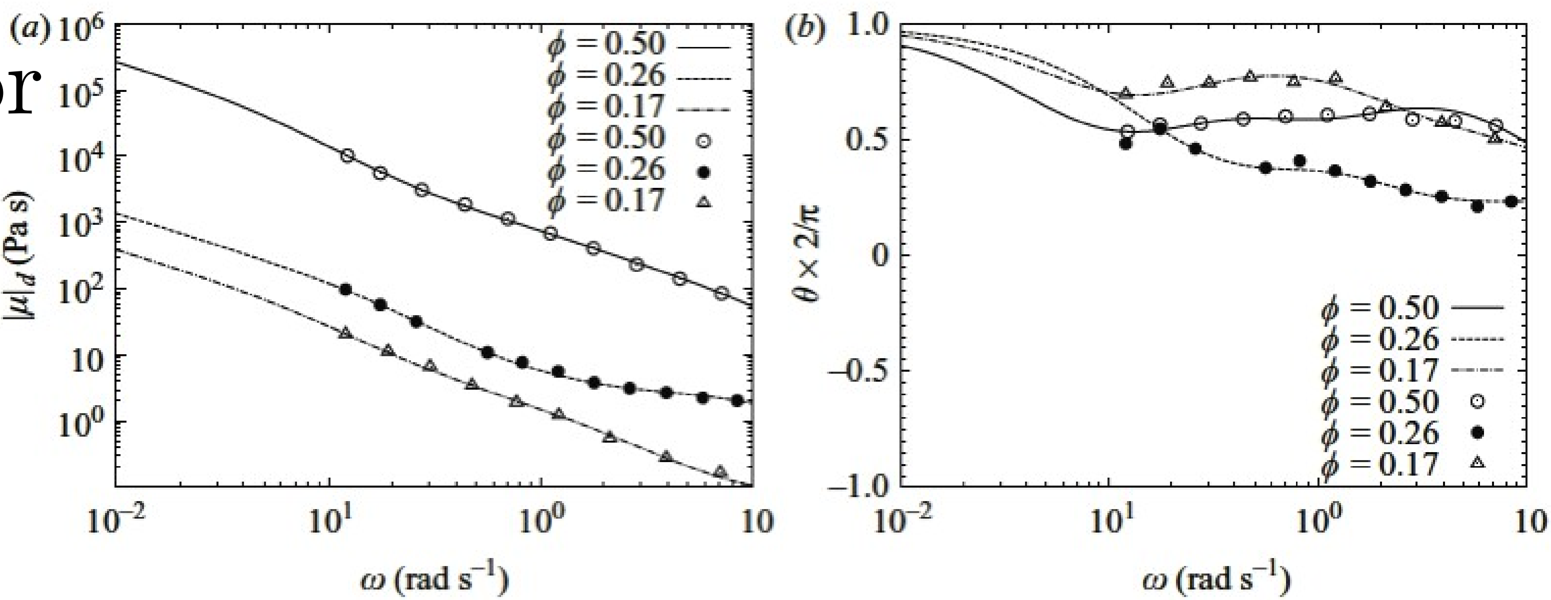
$$\mu_c = |\mu_c|\, e^{i\theta_c}$$

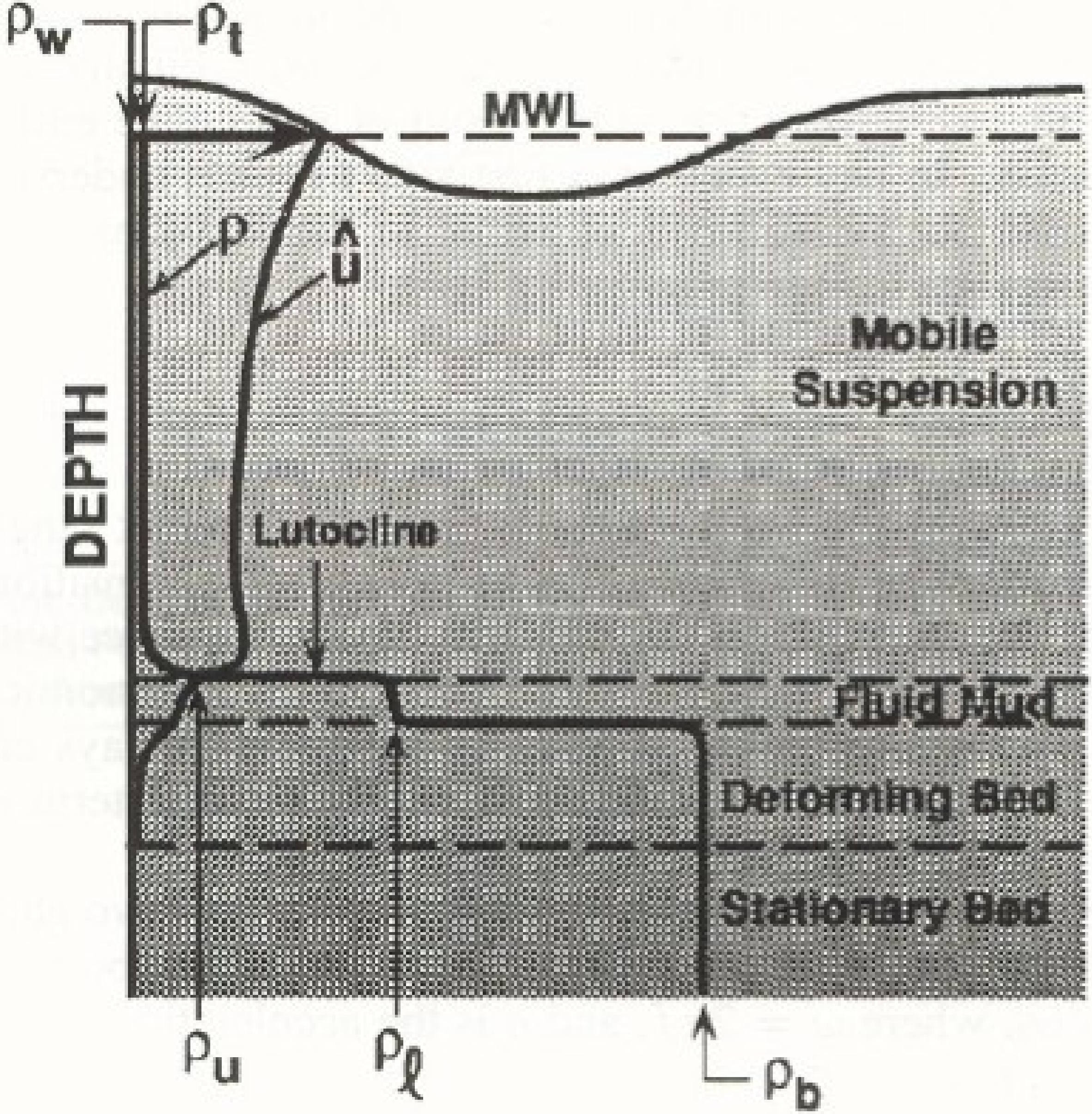
(Huang & Huhe, 1985)

Hangzhou Bay



Lianyun Harbor

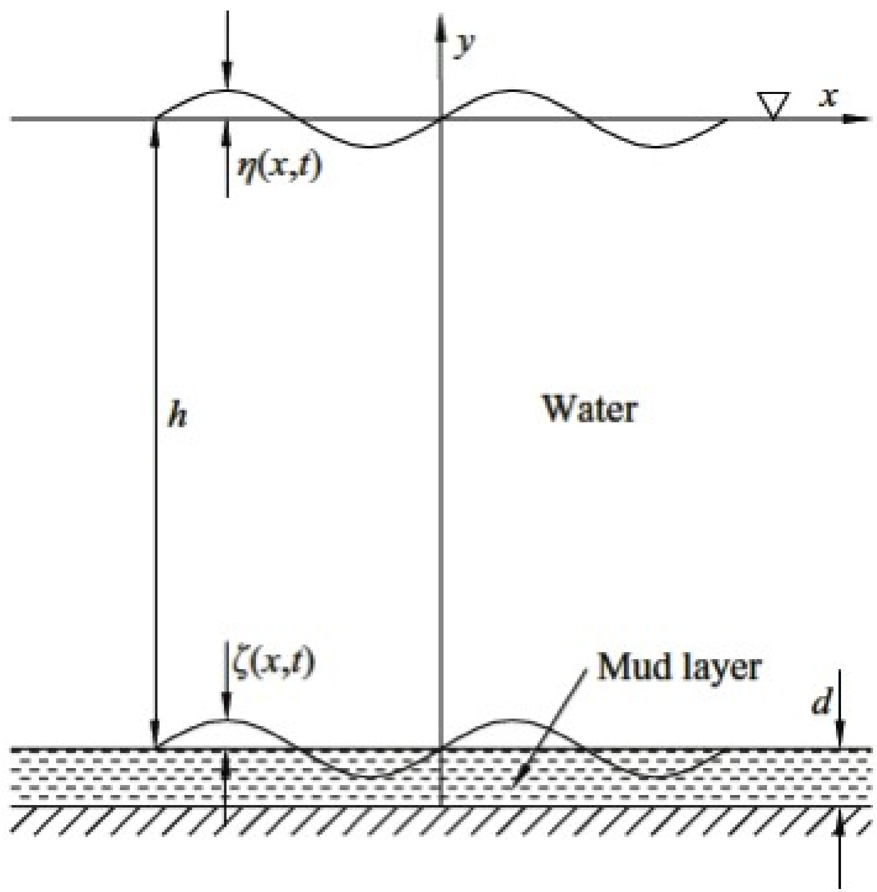



ρw
ρt
MWL
ρ
û
Mobile
Suspension
DEPTH
Lutocline
Fluid Mud
Deforming Bed
Stationary Bed
ρu
ρℓ
ρb

A

Part I: Horizontal seabed, intermediate wave length (M)

Short and long waves over a muddy seabed



Shallow mud layer

$$\eta = O(a_0), \quad k\eta \sim ka_0 = \epsilon \ll 1$$

$$\text{For shallow mud :} \quad \frac{d}{h} = O(\epsilon) \ll 1$$

$$\text{We can show that :} \quad \frac{\text{interface}}{\text{free surface}} = \frac{\zeta}{\eta} = O(\epsilon) \ll 1$$

$$k\zeta = O(\epsilon^2)$$

Scales

Horizontal length scales:

Wave length $1/k \ll$ damping distance,

Time scales:

Wave period $1/\omega, \ll$ Damping time :

Hence multiple scale : $x,\ x_1 = \epsilon t, \quad t, \ \ t_1 = \epsilon t$

Water layer

$$\psi = k_o x - t$$

$$\Phi = \sum_{n=0} \epsilon^n \sum_{m=-n}^{n} \Phi_{nm}\, e^{im\psi},$$

$$\eta = \sum_{n=0} \epsilon^n \sum_{m=-n}^{n} \eta_{nm}\, e^{im\psi}$$

Water at Leading order
Homogeneous BVP on the fast scale

$$O(\epsilon^0):$$

$$\eta_{01} = \frac{1}{2}A(x_1, t_1)\, e^{i\psi} + c.c,$$

$$\Phi_{00} = \Phi_{00}(x_1, t_1)$$

$$\Phi_{01} = -\frac{iA}{2}\frac{\cosh k(z+h)}{\cosh kh}. \qquad k_0 \tanh k_0 h = 1$$

Leading order mud motion

Stokes boundary layer

$$\text{Mud momentum:} \quad \rho_M \frac{\partial^2 u}{\partial t} = - - \frac{\partial p}{\partial x} - \mu \frac{\partial^2 u}{\partial y^2}$$

$$\text{Define vertical coordinate in mud} : Y = \frac{1}{d}(y' + h' + d')$$

$$\text{Mud/water interface} \quad : \quad Y = 1 + \epsilon \frac{a_0}{d_0} \zeta$$

$$-iu_{01} = -\frac{ik_0 \gamma A}{\cosh k_0 h} + \frac{\mu}{Re} \frac{a_0}{d_0} \frac{\partial^2 u_{01}}{\partial Y^2}, \quad 0 < Y < 1$$

$$u_{01} = \frac{\gamma k_0 A}{2 \cosh k_0 h} \{1 - \cosh(\sigma Y - \sinh(\sigma Y) + \tanh \sigma [\cosh(\sigma Y) - 1]\}$$

Interface displacement :

$$\zeta_{01} = \gamma \frac{d}{a_0} \frac{k_0 A}{2 \sinh k_0 h} G(\sigma), \quad G(\sigma) = 1 - \frac{\tanh \sigma}{\sigma}$$

$$\sigma = |\sigma| \exp\left(\frac{\theta}{2} + \frac{\pi}{2}\right), \quad |\sigma| = \sqrt{2} D = \frac{d}{\delta_s},$$

$$\delta_s = \sqrt{\frac{|\mu'|}{\rho_m \omega}}, \quad \text{Stokes layer thickness}$$

$$\theta \to \frac{\pi}{2}, \quad \text{more elastic}$$

$$G(\sigma) = 1 - \frac{\tanh \sigma}{\sigma}$$

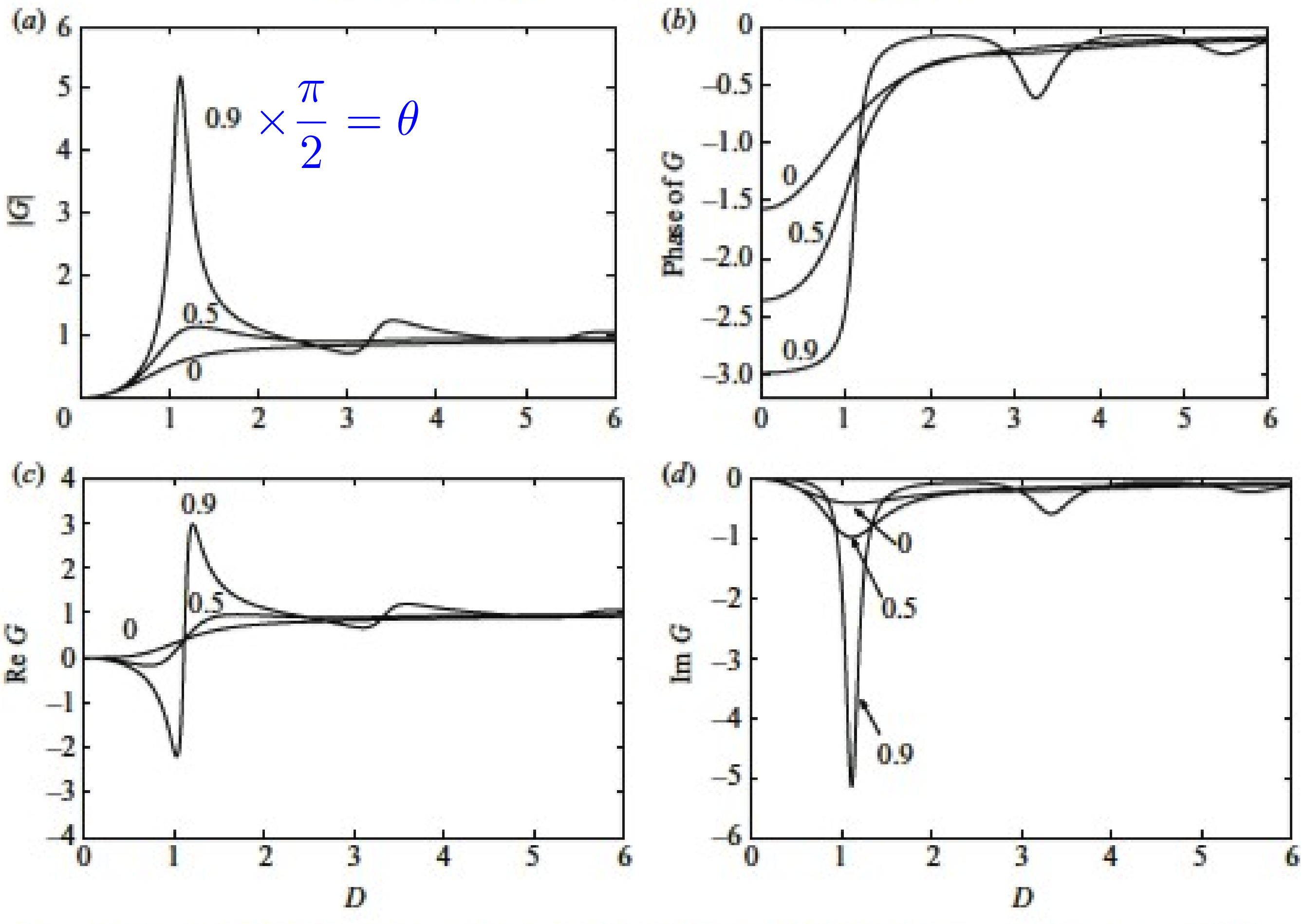


FIGURE 5. (*a*, *b*) Magnitude and phase of *G*. (*c*, *d*) Real and imaginary parts of *G* for different values of $D = \delta/\delta_S$ and the degree of elasticity $\theta \times 2/\pi = (0.0, 0.5, 0.9)$.

Ap

Solvability of order 1

$O(\epsilon)$: Require solvability of the inhomogeneous BVP for Φ_{11} :

$$\frac{\partial A}{\partial t_1} + C_g \frac{\partial A}{\partial x_1} = i k_1 C_g A, \qquad C_g = \text{group velocity},$$

$$k_1 = k_1^r + i k_1^i = -\gamma \frac{d}{a_0} \frac{2k_0^2\, G(\sigma)}{2k_0 h + \sinh 2k_0 h}$$

Narrow banded waves : $x_1 > 0, \quad \Omega = C_g K,$

$$A(x_1, t_1) = A(0) e^{i k_1 x_1} \cos(K x_1 - \Omega t_1)$$

$$= A(0)\, e^{-\epsilon \mathrm{Im}(k_1) x} e^{i \epsilon \mathrm{Re}(k_1) x} \cos\left(\epsilon K x - \epsilon C_g K t\right)$$

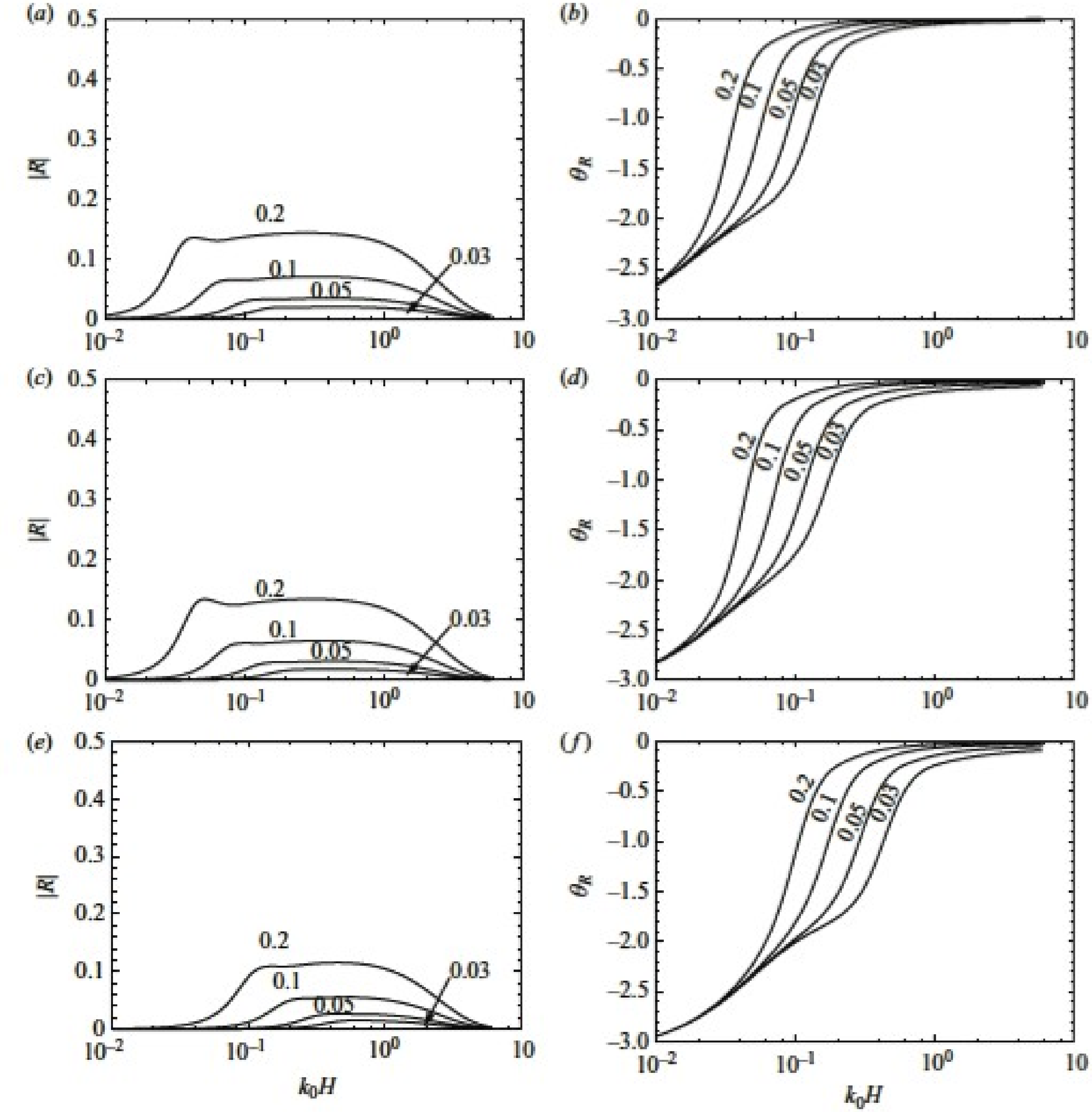


FIGURE 6. Modulus and phase of the ratio of complex vertical displacements $R=\epsilon\zeta_{10}/A$ for Hangzhou Bay mud samples: (*a*, *b*) $\phi=0.20$; (*c*, *d*) $\phi=0.24$; (*e*, *f*) $\phi=0.34$.

Interface displacement

Hangzhou Bay for

$\phi = 0.20, \quad 0.24, \quad 0.34$

d/h=0.03--0.2

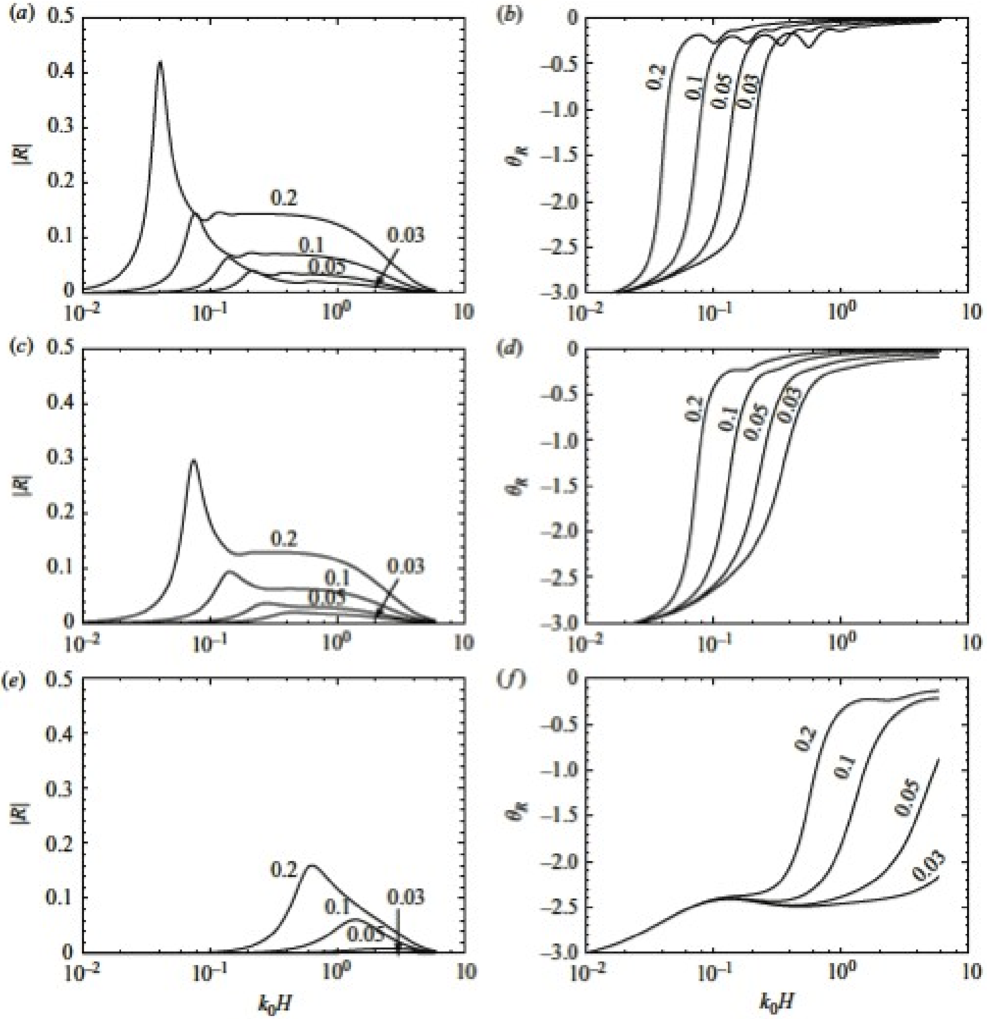


FIGURE 7. Modulus and phase of the ratio of complex vertical displacements R for Lianyungang mud sample: (a, b) $\phi=0.17$; (c, d) $\phi=0.26$; (e, f) $\phi=0.50$.

Interface displacement

Lianyun Harbor

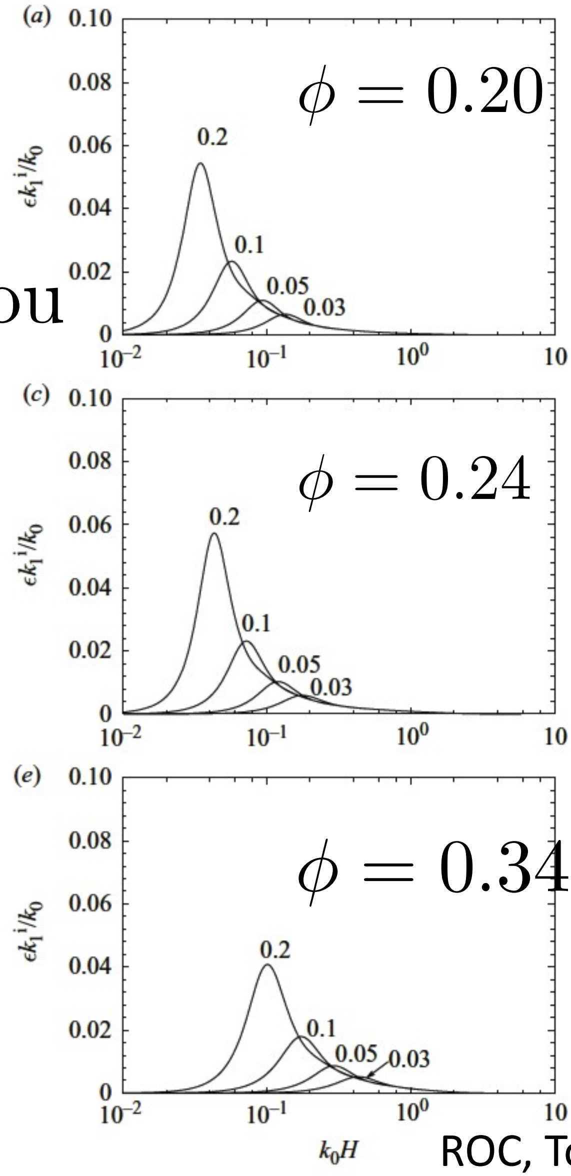
$\phi = 0.17, \; 0.26, \; 0.50$

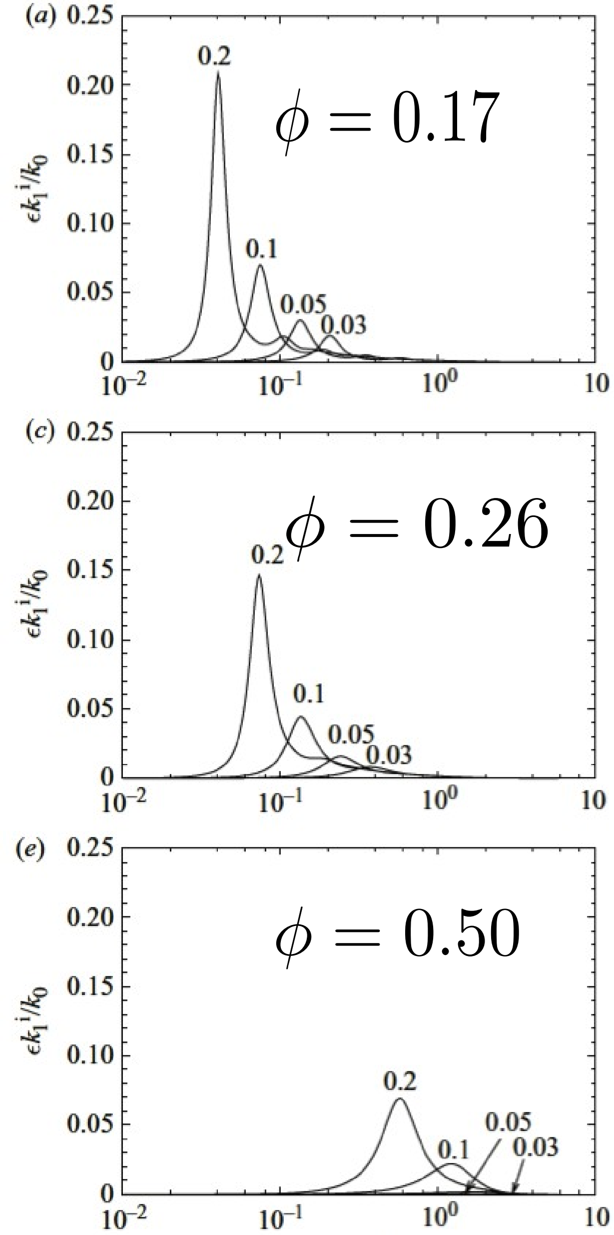
for

d/h=0.03—0.2

Damping coefficient

Hanzhou Bay



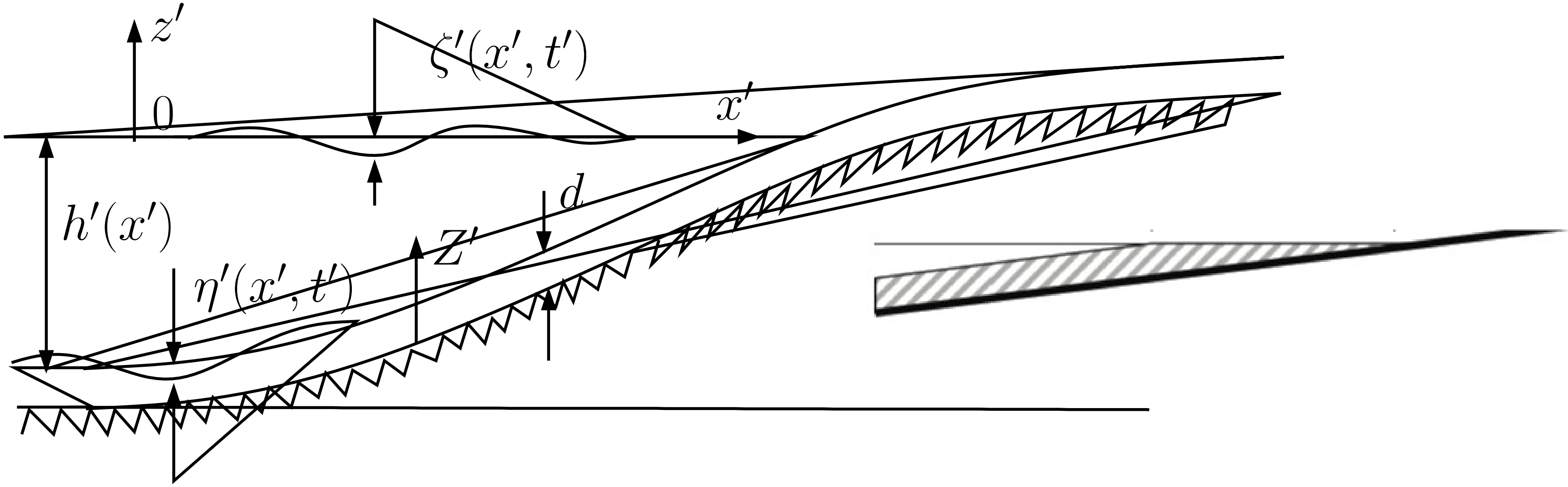


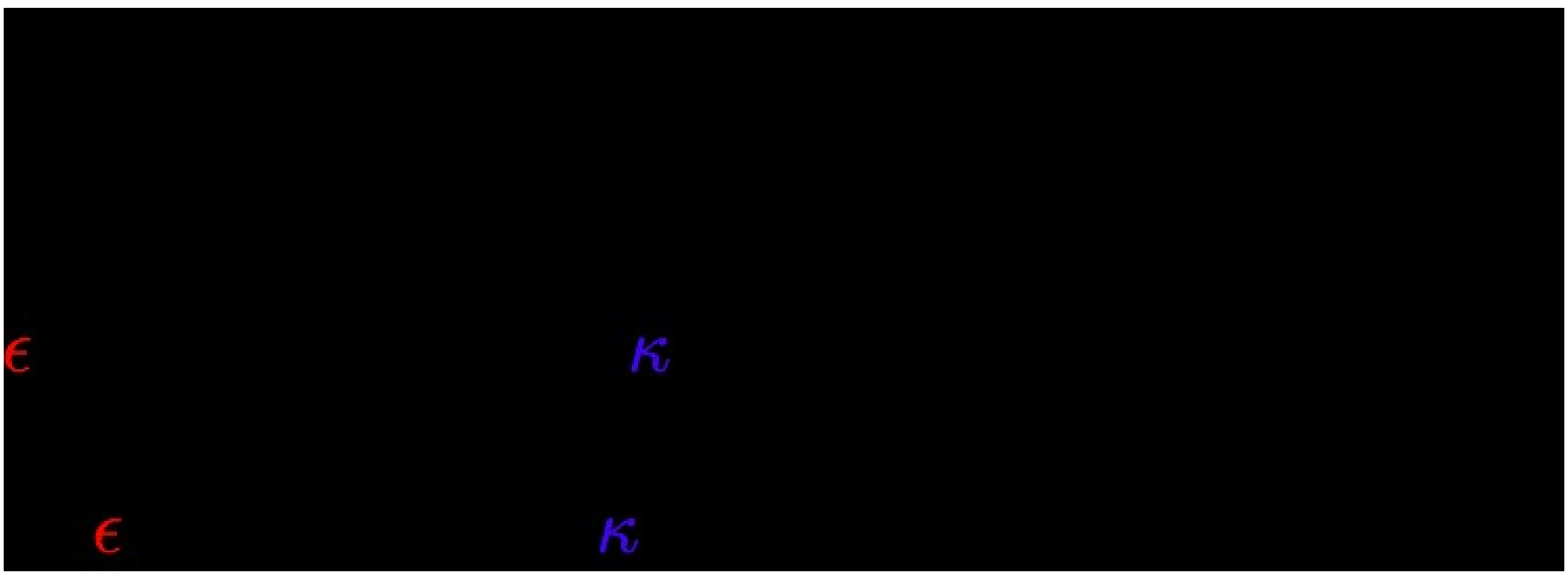
Lianyun Harbor

Part II

Shallow water long waves on a sloping beach

Garnier, Huang & Mei, J. Fluid Mech. 2013,(718), 371-397.





approximation

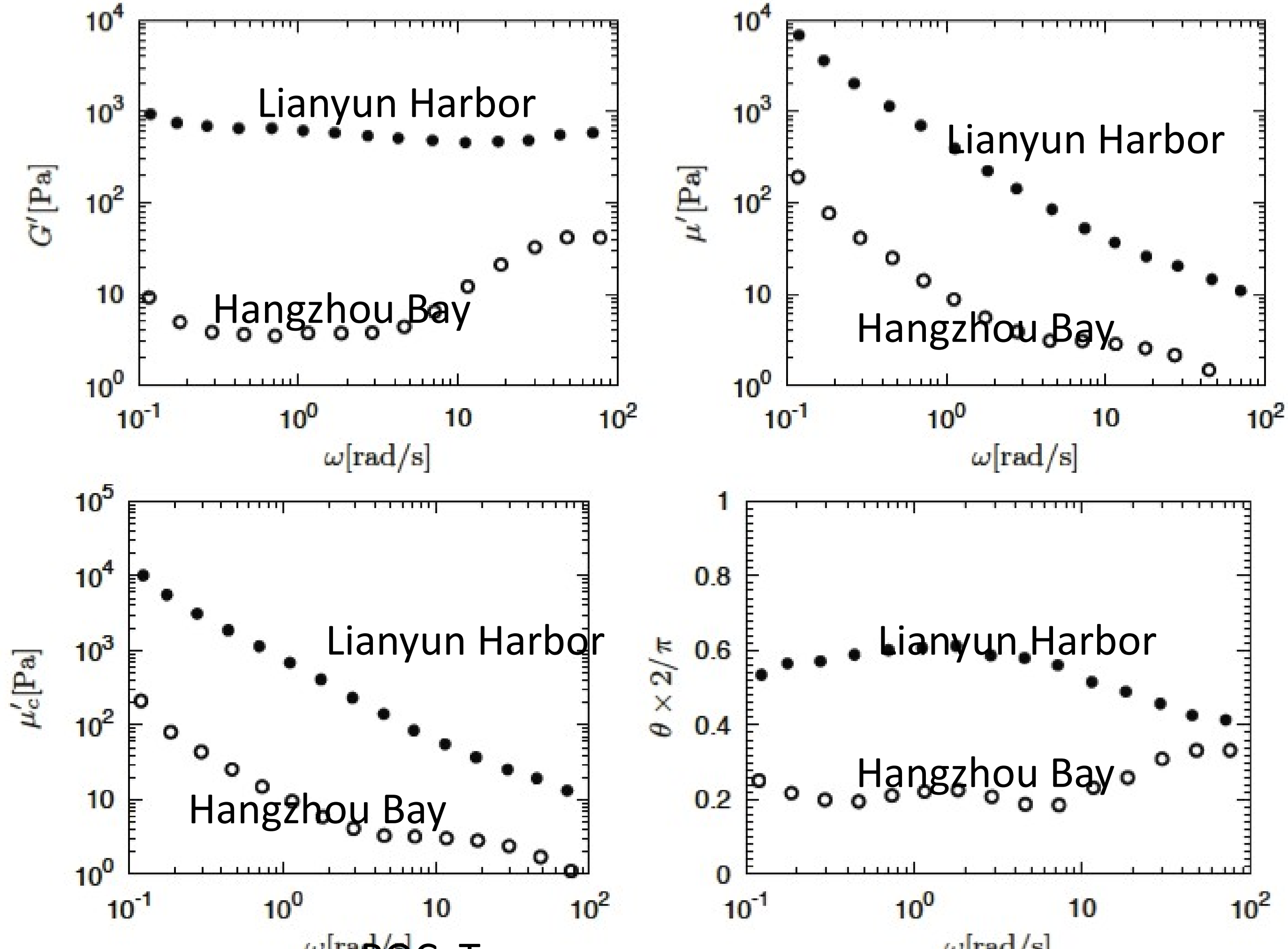
Shallow water, very shallow mud:

$$\text{Water velocity}: (u, w, p), \quad \text{Mud velocity}(U, W, P)$$
$$\text{Water depth}: h(X), \quad X = \kappa^2 x$$

$$\text{Water surface}: \ \zeta, \quad \text{Mud surface}: \ \eta;$$
$$\frac{\eta}{\zeta} \sim \frac{W}{w} \sim \frac{d}{h_0} \ll 1$$

$$\text{Mud surface displacement} \ll \text{water surface displacement}$$

Fluid mud from Hangzhou Bay and Lianyun Harbor $\rho_M = 1590\ kg/m^3$ (Huang & Huhe



Water layer: Long wave equations (dimensionless and Boussinesq)

$$\frac{\partial \zeta}{\partial t} - \delta \frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x}\left[(h + \epsilon\zeta)\, u\right] = 0$$

$$\frac{\partial u}{\partial t} + \epsilon u \frac{\partial u}{\partial x} + \frac{\partial \zeta}{\partial x} - \frac{\kappa^2}{3} h^2 \frac{\partial^3 u}{\partial x^2 \partial t} = 0$$

$$p = \frac{h(X)}{\epsilon} + \zeta - \delta\eta$$

$$\text{Slow coordinate :} \quad X = \kappa^2 x$$

Crudest approximation

Linear approximation:

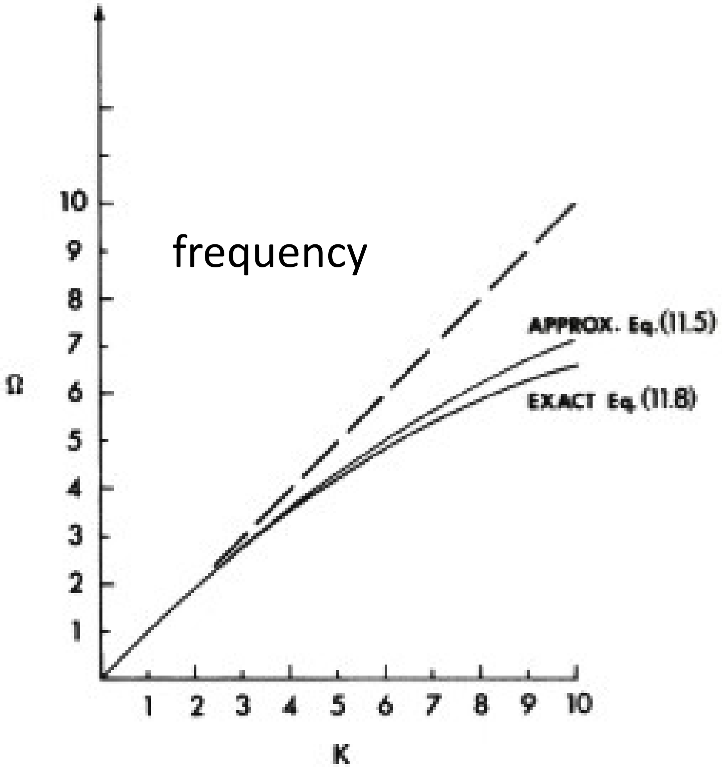
$$\frac{\partial \zeta}{\partial t} + \frac{\partial (h(X)u)}{\partial x} = 0$$

$$\frac{\partial \zeta}{\partial t} + \frac{\partial \zeta}{\partial x} = 0$$

Solution :

$$\zeta \propto e^{\pm i\xi}, \quad \xi = \frac{1}{\kappa^2}\int^X \frac{dX}{\sqrt{h(X)}} - t$$

Harmonic generation in shallow water



frequency

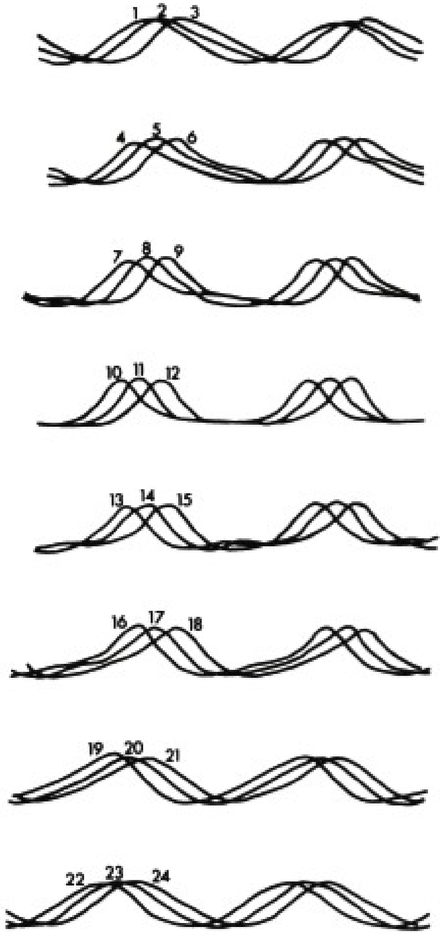
$$\text{dispersion}: \quad \omega_1 = f(k_1);$$

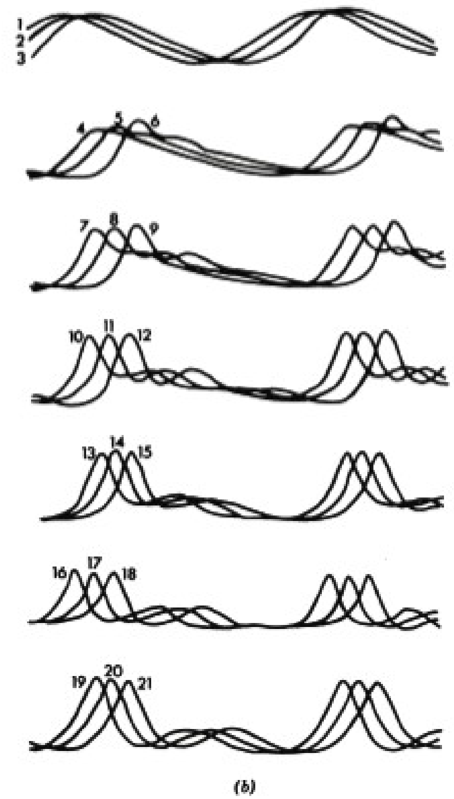
$$\omega_2 = 2\omega, \quad \omega_2 = f(k_2),$$

$$k_2 \approx 2k_1$$

Wave number

Barbara Karakiewicz, 1972





(a) A = 5 cm: H = 30 cm, T = 1:90 s, L = 3:23 m (Ur = 0:45);

(b) A = 5 cm: H = 20 cm, T = 2:75 s, L= 3:86 m (Ur = 2:35)

Periodic surface waves

$$\text{Slow coordinate}: X = \kappa^2 x$$

Let $\quad \xi = \frac{1}{\kappa^2}\int^X \frac{dX}{\sqrt{h(X)}} - t$

$$\zeta = \frac{1}{2}\sum_{m=-\infty}^{\infty} A_m(X)e^{im\xi}, \quad \eta = \frac{1}{2}\sum_{m=-\infty}^{\infty} B_m(X)e^{im\xi},$$

$$\sqrt{h}\frac{dA_m}{dX} + \frac{h_X}{4\sqrt{h}}A_m = -\frac{im^3}{6}hA_m +$$

$$\frac{\epsilon}{\kappa^2}\frac{3im}{8h}\left(\sum_{\ell=1}^{\infty} 2A_\ell^* A_{m+\ell} + \sum_{\ell=!}^{[m/2]} \alpha_\ell A_\ell A_{m-\ell}\right) + \frac{\delta}{\kappa^2}\frac{im}{2}B_m$$

cf: Nonlinear optics

Mud motion at leading order

New coordinate: from the mud surface

$$Z = z + h(X) + d$$

$$\text{Mass}: \quad \frac{1}{\sqrt{h}}\frac{\partial U^{(0)}}{\partial \xi} + \frac{\kappa^2}{\delta}\frac{dh}{dX}\frac{\partial U^{(0)}}{\partial Z} + \frac{\partial W^{(0)}}{\partial Z} = 0, \quad 0 < Z < 1$$

$$\text{Momentum:} \quad \frac{1}{R}\frac{\partial \tau_{xz}^{(0)}}{\partial Z} + \frac{\partial U^{(0)}}{\partial \xi} = \frac{\gamma}{\sqrt{h}}\frac{\partial \zeta^{(0)}}{\partial \xi}, \quad 0 < Z < 1$$

$$\text{Mud surface}: \quad -\frac{\partial \eta^{(0)}}{\partial \xi} = W^{(0)} + \frac{\kappa^2}{\delta}\frac{dh}{dX}U^{(0)}, \quad Z = 1.$$

$$\frac{\partial U^{(0)}}{\partial Z} = 0, \quad Z = 1$$

$$\text{Rigid bed}: \quad U^{(0)} = W^{(0)} = 0, \quad Z = 0$$

Mud Harmonics

$$U^{(0)} = \frac{1}{2}\sum_{m=1}^{\infty} U_m^{(0)}(Z)e^{\imath m\xi} + c.c., \quad W^{(0)} = \frac{1}{2}\sum_{m=1}^{\infty} W_m^{(0)}(Z)e^{\imath m\xi} + c.c.$$

$$\tau_{xz}^{(0)} = \frac{1}{2}\sum_{m=1}^{\infty} (\tau_{xz}^{(0)})_m(Z)e^{\imath m\xi} + c.c. \quad \eta = \frac{1}{2}\sum_{m=1}^{\infty} B_m e^{\imath m\xi} + c.c.,$$

$$\text{Will get :} \quad B_m(X) = \frac{\gamma}{h}\left(1 - \frac{\tanh \sigma_m}{\sigma_m}\right)A_m(X) = \frac{\gamma}{h}G(\sigma_m)A_m(X)$$

$$\text{where} \quad \sigma_m = \sqrt{-\imath \frac{mR}{\mu_m}} = \sqrt{-\imath m \frac{\rho_M \omega d^2}{\mu_c'}}, \qquad \mu_m = \mu(m\omega)$$

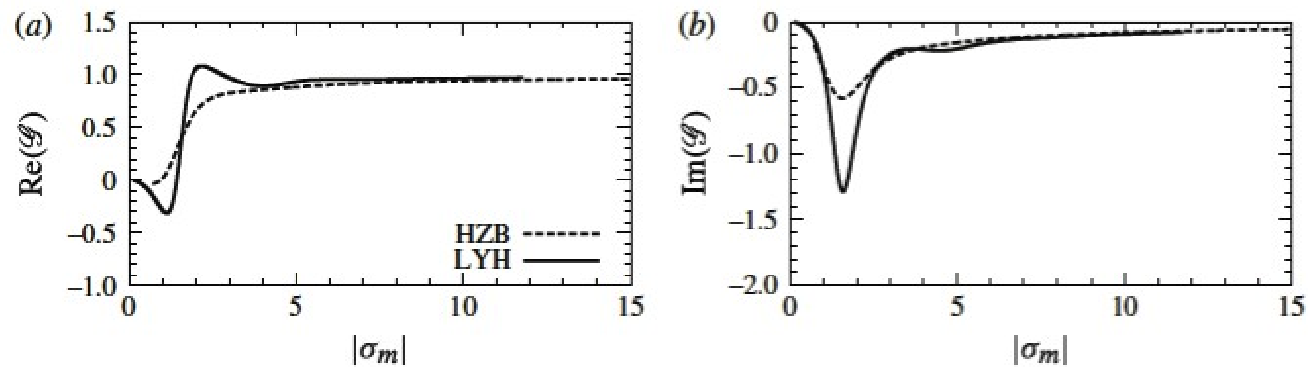
$$|\sigma_m| = \frac{d}{\delta_m}$$

Wave harmonics

$$\sqrt{h}\frac{dA_m}{dX}+\frac{h_X}{4\sqrt{h}}A_m = -\frac{im^3}{6}hA_m+\frac{\epsilon}{\kappa^2}\frac{3im}{8h}\left(\sum_{\ell=1}^{\infty}2A_\ell^*A_{m+\ell}+\sum_{\ell=!}^{[m/2]}\alpha_\ell A_\ell A_{m-\ell}\right)$$
$$+\frac{\delta}{\kappa^2}\frac{im}{2}\frac{\gamma A_m}{h}\left(1-\frac{\tanh\sigma_m}{\sigma_m}\right)$$

Numerical integration with initial coniditions : $A_m(0)$ prscribed

Energy: $$\frac{d}{dX}\left[\sqrt{h}\sum_1^{\infty}|A_m|^2\right]=\frac{\gamma}{h}\frac{\delta^2}{\kappa^2}\sum_1^{\infty}\mathrm{Im}\left[m\left(1-\frac{\tanh\sigma_m}{\sigma_m}\right)\right]|A_m|^2$$



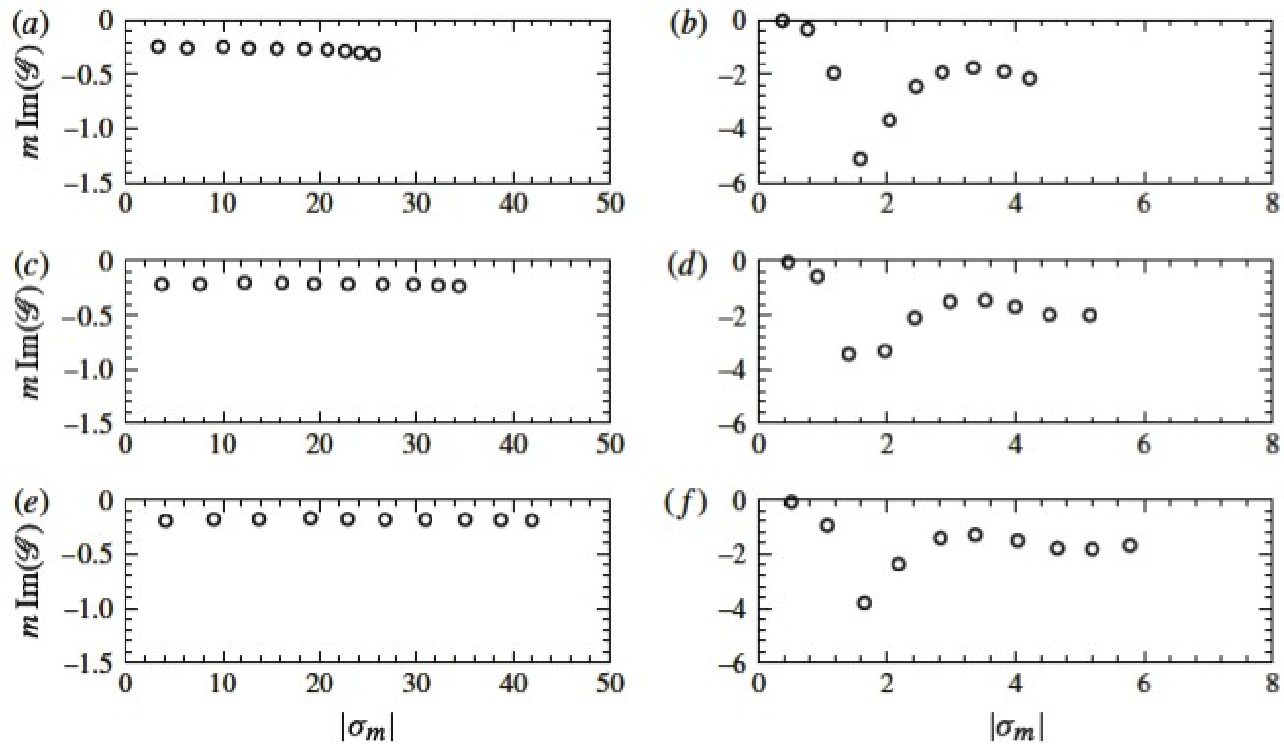
Hangzhou Bay

Lianyun Harbor

d=0.5 m

d=0.75 m

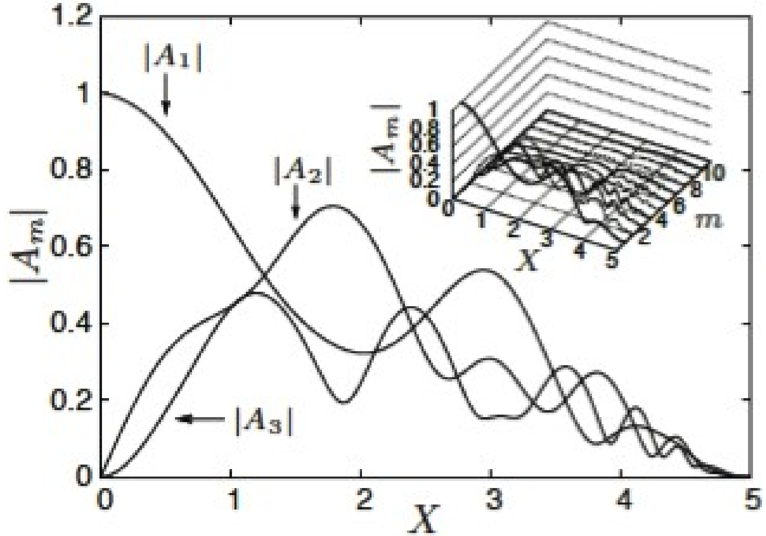
d=1.0 m



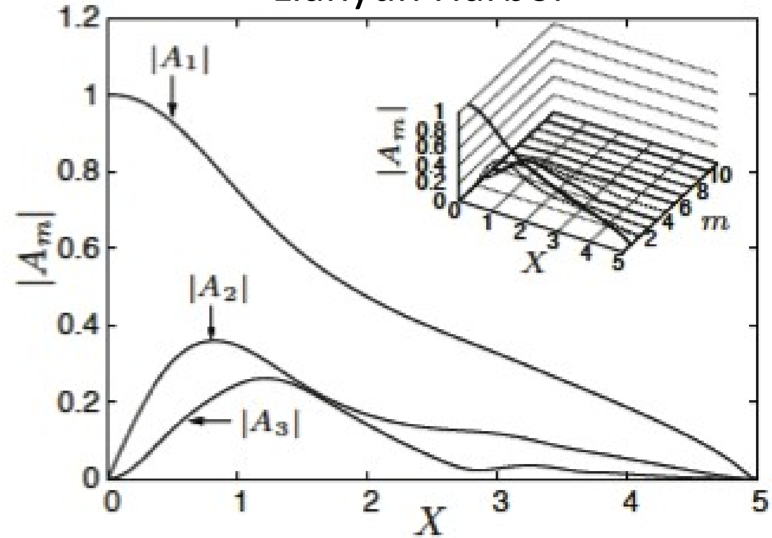
Surface wave harmonics

$$A_1(0) = 1, \quad A_2(0) = A_2(0) = \cdots = 0$$

Hangzhou Bay



Lianyun Harbor

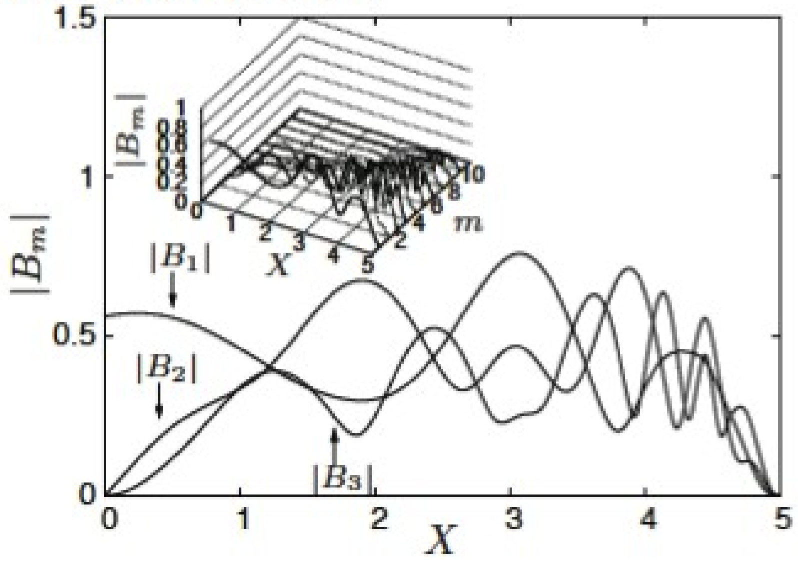


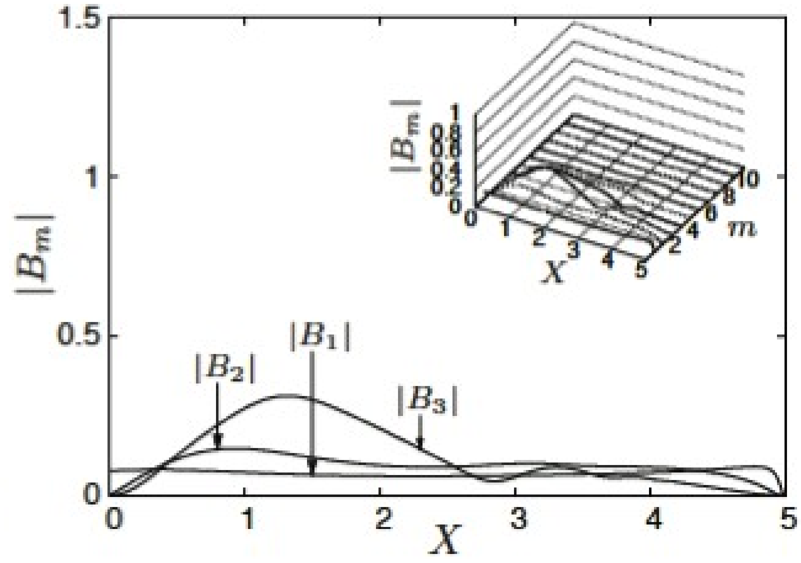
$$\epsilon = 0.1, \quad s = 0.2, \quad \delta = 0.15, \quad \kappa^2 = 0.0986$$

Interface harmonics

$$\epsilon = 0.1, \quad s = 0.2, \quad \delta = 0.15, \quad \kappa^2 = 0.0986$$

re shown in the inset.



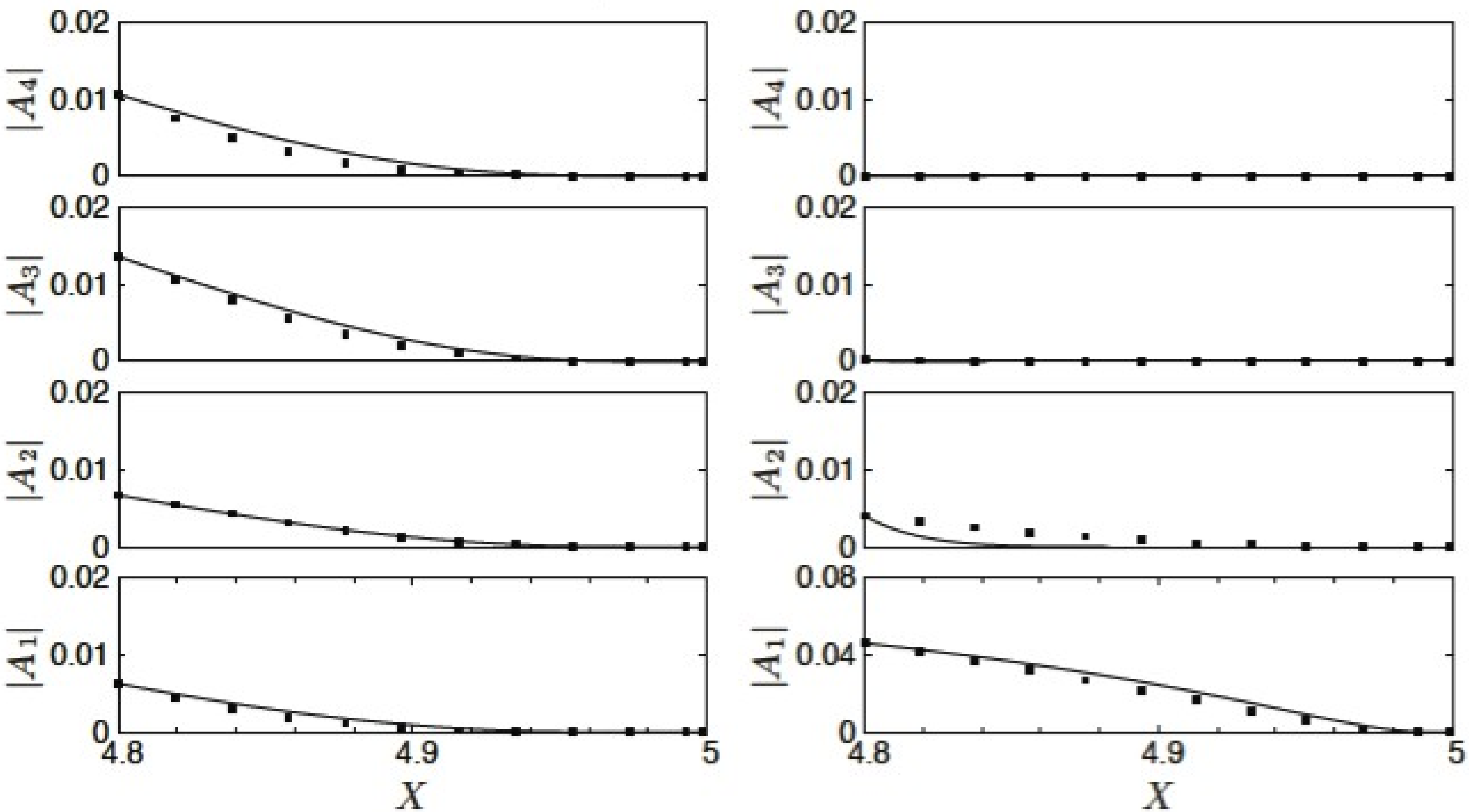


Hangzhou Bay

Lianyun Harber

Near the shoreline

$$\frac{|A_m(X_0)|^2}{|A_m(X)|^2} = \sqrt{\frac{h(X_0)}{h(X)}}\,\frac{\exp\left(\frac{2\alpha_m}{s\sqrt{h(X_0)}}\right)}{\exp\left(\frac{2\alpha_m}{s\sqrt{h(X)}}\right)}$$



Hangzhou Bay

Lianyun Harbor

Steady displacement (cf: Acoustic streaming)

$$\text{No wave}: \ \mathcal{U} = \frac{\rho_M - \rho}{2G_0} Z(Z - 2d)$$

$$\text{Under waves}:$$

$$\frac{1}{R}\frac{\partial \langle \tau_{xz}^{(1)} \rangle}{\partial Z} = \frac{\epsilon}{\kappa^2} \left\langle \frac{\kappa^2}{\delta} \frac{dh}{dX} U^{(0)} \frac{\partial U^{(0)}}{\partial Z} + W^{(0)} \frac{\partial U^{(0)}}{\partial Z} \right\rangle$$

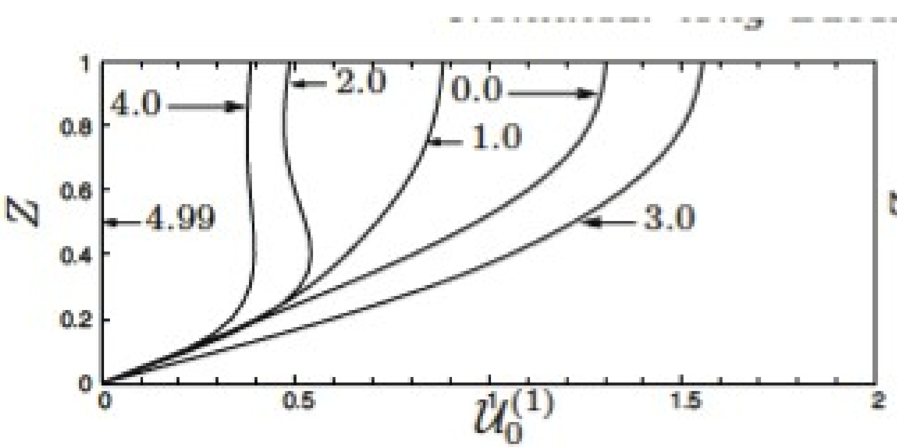
$$\left\langle \tau_{xz}^{(1)} \right\rangle = G_0 \frac{\partial \mathcal{U}^{(1)}}{\partial Z}$$

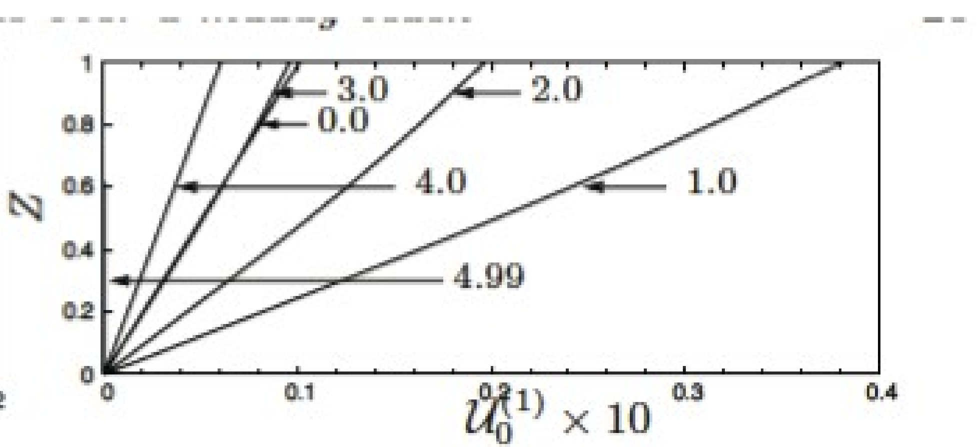
$$\text{Boundary conditions}:$$

$$\left(\tau_{xz}^{(1)} \right)_0 = -\frac{\epsilon}{\kappa^2} \left(\eta^{(0)} \frac{\partial \tau_{xz}^{(0)}}{\partial Z} \right)_0, \quad Z = 1.$$

$$\mathcal{U}^{(1)} = 0, \quad Z = 0$$

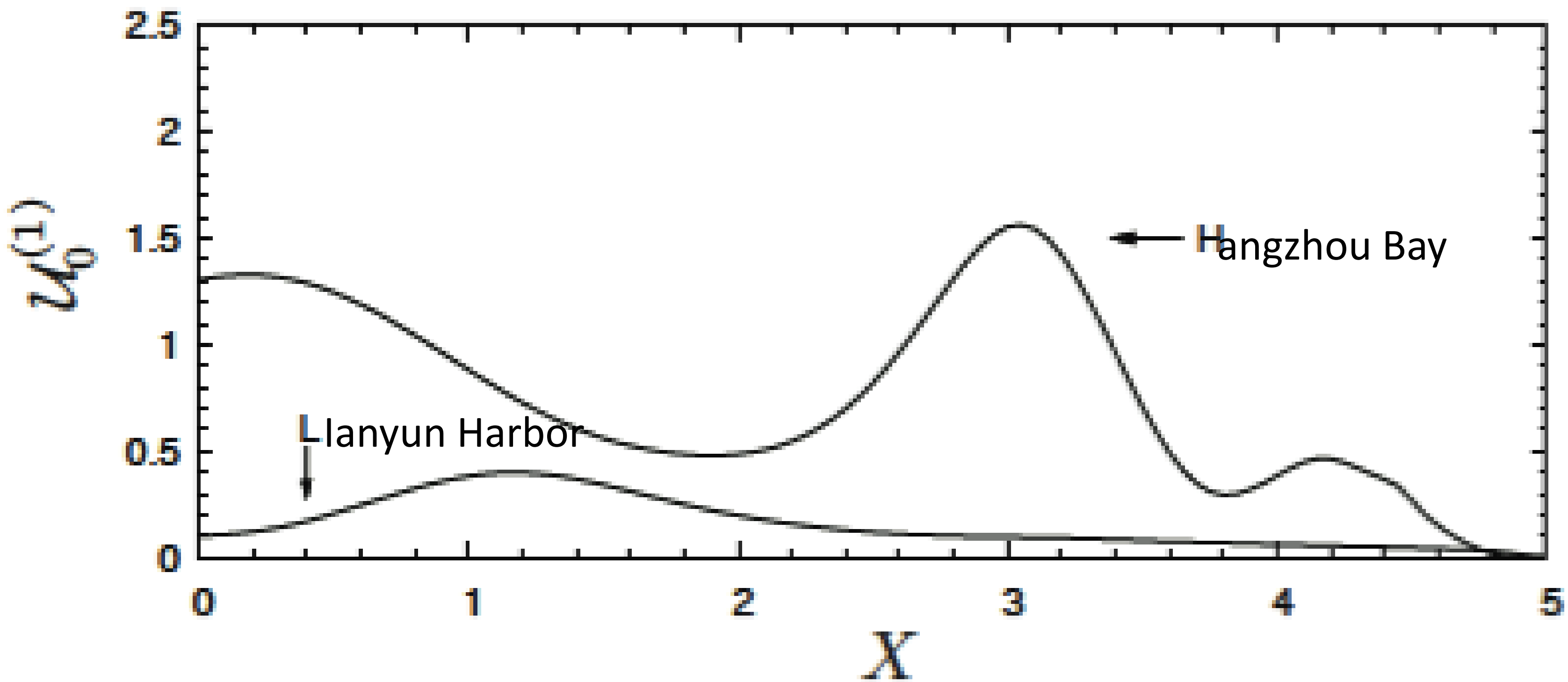
Mean mud displacement





Hangzhou Bay Lianyun Harbor

Profile of mud surface displacement



Work needed

- Short-range:
  - Measurement of fluid mud rheology with a dynamic rheometer (RMS 605?)
  - Laboratory experiments on a sloping beach
- Fundamental:
  - Transition between consolidated and fluidized mud
  - Transition between fluid mud and clear water (lutocline)

شكراً جزيلاً

Thank you

Merci ,   Gracias

Vielen Dank

ありがとうございます。

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Спасибо   ευχαριστώ