Two dimensional water waves in holomorphic coordinates

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This is joint work with Mihaela Ifrim and John Hunter

The setting:

- inviscid incompressible fluid flow (Euler equations)
- irrotational
- with gravity
- free boundary
- no surface tension
- periodic or nonperiodic setting

Questions:

- Local well-posedness in optimal Sobolev spaces
- Long time solutions

Goals:

- To use Zakharov's formulation of the equations in holomorphic coordinates to provide a simpler approach to the local problem
- To introduce a modified energy method which yields an easier route to long time solutions

The standard formulation

Fluid domain: $\Omega(t)$, free boundary $\Gamma(t)$.
Velocity field u, pressure p, gravity $g = -j$.
Euler equations in $\Omega(t)$:

$$\begin{cases} u_t + u \cdot \nabla u = \nabla p - j \\ \text{div } u = 0 \\ \text{curl } u = 0 \\ u(0,x) = u_0(x) \end{cases}$$

Boundary conditions on Γ_t:

$$\begin{cases} \partial_t + u \cdot \nabla \text{ is tangent to } \bigcup \Gamma_t & \text{(kinematic)} \\ p = 0 \quad \text{on } \Gamma_t & \text{(dynamic)} \end{cases}$$

Velocity potential

$$u = \nabla\phi, \qquad \Delta\phi = 0 \quad \text{in } \Omega_t$$

Dynamic boundary condition:

$$\phi_t + \frac{1}{2}|\nabla\phi|^2 + y = 0 \qquad \text{on } \Gamma_t$$

Holomorphic coordinates:

$$Z : \{\Im z \leq 0\} \to \Omega_t, \qquad Z(\alpha + i\beta) - (\alpha + i\beta) \to 0 \ \text{ at } \ \text{infinity}$$

Holomorphic velocity potential:

$$Q = \phi + i\psi$$

Zakharov's equations

- P - Projection onto negative (or positive) wavenumbers
- α - holomorphic parameter for free surface
- Z - free surface parametrization
- Q - Velocity complex potential

$$\begin{cases} Z_t + FZ_\alpha = 0, \\ Q_t + FQ_\alpha - i(Z - \alpha) + P\left[\dfrac{|Q_\alpha|^2}{J}\right] = 0, \end{cases}$$

where

$$F = P\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right], \qquad J = |Z_\alpha|^2.$$

Boundary condition at infinity:

$$Z(\alpha) - \alpha \to 0 \ \ (nonperiodic) \qquad (Z(\alpha) - \alpha)_{avg} = 0 \ \ (periodic)$$

Equations for $(W = Z - \alpha, Q)$

$$\begin{cases} W_t + F(1 + W_\alpha) = 0, \\ Q_t + FQ_\alpha - iW + P\left[\dfrac{|Q_\alpha|^2}{J}\right] = 0. \end{cases}$$

where

$$F = P\left[\frac{Q_\alpha - \bar{Q}_\alpha}{J}\right], \qquad J = |1 + W_\alpha|^2.$$

Conserved energy (Hamiltonian):

$$E(W, Q) = \int \frac{1}{2}|W|^2 + \frac{1}{2i}(Q\bar{Q}_\alpha - \bar{Q}Q_\alpha) - \frac{1}{4}(\bar{W}^2 W_\alpha + W^2 \bar{W}_\alpha)\, d\alpha$$

Symmetries:

- Translations in α and t.
- Scaling $(W(t,\alpha), Q(t,\alpha)) \to (\lambda^{-2}W(\lambda t, \lambda^2\alpha), \lambda^{-3}Q(\lambda t, \lambda^2\alpha))$

Physical parameters

- $R = \dfrac{Q_\alpha}{1+W_\alpha}$ - velocity field on the free boundary
- $b = 2\Re P\left[\dfrac{R}{1+\bar{W}_\alpha}\right]$ - advection coefficient
- $a = 2\Im P[R\bar{R}_\alpha]$ - normal derivative of the pressure $(1+a)$

Alternate system for $(\mathbf{W} = W_\alpha, R)$:

$$\begin{cases} (\partial_t + M_b\partial_\alpha)\mathbf{W} + P\left[\dfrac{1+\mathbf{W}}{1+\bar{\mathbf{W}}}R_\alpha\right] = G(\mathbf{W}, R) \\ (\partial_t + T_b\partial_\alpha)R - iP\left[\dfrac{(1+a)\mathbf{W}}{1+\mathbf{W}}\right] = K(\mathbf{W}, R) \end{cases} \tag{0.1}$$

where

$$G = (1+\mathbf{W})P\left[\frac{\bar{R}_\alpha}{1+\mathbf{W}} + \frac{R\bar{\mathbf{W}}_\alpha}{(1+\bar{\mathbf{W}})^2}\right] + [P, \mathbf{W}]\left(\frac{R_\alpha}{1+\mathbf{W}} + \frac{\bar{R}\mathbf{W}_\alpha}{(1+\mathbf{W})^2}\right)$$

$$K = -P\left[R\bar{R}_\alpha\right]$$

represent perturbative terms in the equation.

Local wellposedness

Energy space: $(W, Q) \in \mathcal{H}^0 = (L^2 \times \dot{H}^{\frac{1}{2}})$.
Higher regularity $\dot{\mathcal{H}}^k$: $(W^{(k)}, R^{(k-1)}) \in \mathcal{H}^0$.

Theorem

The two dimensional water wave equation is locally well-posed in $\dot{\mathcal{H}}^1 \cap \dot{\mathcal{H}}^2$.

Prior work by

- Wu (Lagrangian coordinates)
- Alazard-Burq-Zuily (Eulerian coordinates)

Main ideas:

- Energy estimates for solutions and their derivatives
- Energy estimates for the linearized equation

The linearized equation

In linearized variables (w, q): *non-diagonal degenerate* first order hyperbolic system. Better, use diagonalized variables $(w, r = q - Rw)$:

$$\begin{cases} (\partial_t + M_b \partial_\alpha) w + P\left[\dfrac{1}{1+\bar{W}_\alpha} r_\alpha\right] + P\left[\dfrac{R_\alpha}{1+\bar{W}_\alpha} w\right] = \mathcal{G}(w,r), \\ (\partial_t + M_b \partial_\alpha) r - iP\left[\dfrac{1+a}{1+W_\alpha} w\right] = \mathcal{K}(w,r), \end{cases}$$

Energy

$$E(w,r) = \int_{\mathbb{R}} (1+a)|w|^2 + \Im(\bar{r} r_\alpha) d\alpha \approx \|(w,r)\|_{\mathcal{H}^0}^2$$

Control norms

$$A = \|W_\alpha\|_{L^\infty} + \|D^{\frac{1}{2}} R\|_{L^\infty}, \qquad \text{(scale invariant)}$$

$$B = \|D^{\frac{1}{2}} W_\alpha\|_{BMO} + \|R_\alpha\|_{BMO} \qquad \text{(controlled by } \dot{\mathcal{H}}^2\text{)}$$

Scale invariant energy estimate:

$$\frac{d}{dt} E(w,r) \lesssim_A B E(w,r)$$

Energy estimates

Because of the translation invariance, the pair (W_α, R) solves the linearized equation. Hence by Gronwall,

$$\|(W_\alpha, R)(t)\|_{\dot{\mathcal{H}}^0} \lesssim e^{c(A)Bt} \|(W_\alpha, R)(0)\|_{\dot{\mathcal{H}}^0}$$

The same method gives for all higher k:

$$\|(W^{(k)}, R^{(k-1)})(t)\|_{\dot{\mathcal{H}}^0} \lesssim e^{c(A)Bt} \|(W^{(k)}, R^{(k-1)})(0)\|_{\dot{\mathcal{H}}^0}$$

The case $k = 2$ suffices to control B and leads to the local well-posedness result.
Higher k's lead to results on preservation of higher regularity for as long as B stays bounded.

Normal forms and long time existence

Question: Find improved lifespan estimates for small data solutions.

(i) Equations with quadratic nonlinearities:

$$\frac{d}{dt}E(u) \lesssim \|u\| E(u)$$

For data $\|u(0)\| = \epsilon \ll 1$ this leads by Gronwall to a lifespan $T_\epsilon \approx \epsilon^{-1}$

(ii) Equations with cubic nonlinearities:

$$\frac{d}{dt}E(u) \lesssim \|u\|^2 E(u)$$

For data $\|u(0)\| = \epsilon \ll 1$ this leads by Gronwall to a lifespan $T_\epsilon \approx \epsilon^{-2}$

(iii) Normal form method: transform an equation with a quadratic nonlinearity into one with a cubic one via a normal form transformation,

$$u \to v = u + B(u,u) + \text{higher}$$

Normal forms for water waves

Existence of a normal form transformation is related to the absence of resonant bilinear interactions. For 2-d water waves in holomorphic coordinates, such a normal form transformation exists and is given by

$$\begin{aligned} \tilde{W} &= W - 2M_{\Re W} W_\alpha, \\ \tilde{Q} &= Q - 2M_{\Re W} Q_\alpha, \end{aligned} \tag{0.2}$$

The normal variables solve an equation of the form

$$\begin{cases} \partial_t \tilde{W} + \tilde{Q}_\alpha = \text{ cubic and higher} \\ \partial_t \tilde{Q} - i\tilde{W} = \text{ cubic and higher} \end{cases}$$

However, the cubic and higher nonlinearities also contain higher derivatives, so one cannot close the energy estimates. This is related to the fact that the normal form transformation is not invertible, and further to the fact that the water wave equation is quasilinear, rather than semilinear.

The modified energy method

Idea: Modify the energy rather than the equation in order to get cubic energy estimates.

Step 1: Construct a cubic normal form energy

$$E^n_{NF}(W,Q) = quadratic + cubic(\|\tilde{W}^{(n)}\|^2_{L^2} + \|\tilde{Q}^{(n)}\|^2_{\dot{H}^{\frac{1}{2}}})$$

Then

$$\frac{d}{dt}E^n_{NF}(W,Q) = quartic + higher$$

Here higher derivatives arise on the right, making it impossible to close.

Step 2: Switch $E^n_{NF}(W,Q)$ to diagonal variables $E^n_{NF}(W_\alpha, R)$.

Step 3: To account for the fact that the equation is quasilinear, replace the leading order terms in $E^n_{NF}(W_\alpha, R)$ with their natural quasilinear counterparts to obtain a good quasilinear energy $E^n(W_\alpha, R)$. Clue: look at the quasilinear energy for the linearized equation.

Cubic estimates for the linearized equation

Modified energy:

$$\tilde{E}(w,r) = \int (1+a)|w|^2 + \Im(\bar{r} r_\alpha) + 2(\Im[\bar{R} w r_\alpha] - \Re[\bar{W}_\alpha w^2]) \, d\alpha$$

Then the solutions to the linearized equation satisfy

$$\frac{d}{dt} \tilde{E}(w,r) \lesssim_A AB\tilde{E}(w,r)$$

Higher order counterpart:

Theorem

There exists a modified energy functional $E^n(W_\alpha, R)$ so that

$$E^n(W_\alpha, R) \approx \|W^{(n)}, R^{(n-1)}\|^2_{\dot{\mathcal{H}}^0}$$

and

$$\frac{d}{dt} E^n(W_\alpha, R) \lesssim_A ABE^n(W_\alpha, R)$$

The periodic case

Using the energy estimates in the previous theorem for $n = 1, 2$ in the periodic case we obtain

Theorem

Consider the two dimensional water wave equation with initial data in $\dot{\mathcal{H}}^1 \cap \dot{H}^2$, of size ϵ,

$$\|(W_\alpha, R)\|_{\dot{\mathcal{H}}^0} + \|(W_{\alpha\alpha}, R_\alpha)\|_{\dot{\mathcal{H}}^0} \lesssim \epsilon$$

Then the solutions have a lifespan of at least

$$T_\epsilon \approx \epsilon^{-2}$$

- The cubic energy estimates for higher norms show that bounds for higher norms also propagate on the same timescale.
- The same result applies to the water wave problem on the real line. However on the real line one can get better results using dispersive decay if in addition the data is spatially localized.

Water waves on the real line: heuristics

Dispersive decay for the linear equation

$$\begin{cases} W_t + Q_\alpha = 0 \\ Q_t - iW = 0 \end{cases}$$

with smooth localized data of size ϵ:

$$|W| + |Q| \lesssim t^{-\frac{1}{2}}$$

This shows that one would expect our control norms to decay like

$$A, B \approx \epsilon t^{-\frac{1}{2}}$$

Hence

$$\frac{d}{dt} E \lesssim \epsilon^2 t^{-1} E$$

which provides uniform bounds up to

$$T_\epsilon = e^{c\epsilon^{-2}}$$

Almost global solutions

Idea: Obtain pointwise decay from the scaling and translation symmetries.

Theorem

*Assume that the initial data for the water wave equation satisfies***:*

$$\sum_{0 \leq j+k \leq 7} \| S^j \partial_\alpha^k (W, Q)(0) \|_{\dot{\mathcal{H}}^0} \lesssim \epsilon$$

Then the solution exists up to time $T_\epsilon = e^{c\epsilon^{-2}}$, with uniform bounds in the above norms.

- Result by Wu (with much more regularity)
- Recent global result by Pusateri-Ionescu

Outline of proof

Control norm:

$$C = \sum_{0 \le j+k \le 2} \|S^j \partial_\alpha^k W\|_{L^\infty} + \sum_{0 \le j+k \le 1} \|S^j \partial_\alpha^k R\|_{L^\infty}$$

Elliptic bounds for water waves (W, Q):

$$\sup_{[1,T]} t^{\frac{1}{2}} C \lesssim \sup_{[1,T]} \sum_{0 \le j+k \le 7} \|S^j \partial_\alpha^k (W, Q)(t)\|_{\dot{\mathcal{H}}^0}$$

Modified energy method: there exist energy functionals $E^{j,k}$ such that

$$\frac{d}{dt} \sum_{j+k \le 7} E^{j,k} \lesssim C^2 \sum_{j+k \le 7} E^{j,k}$$