Modeling Large-Scale Atmospheric and Oceanic Flows 1

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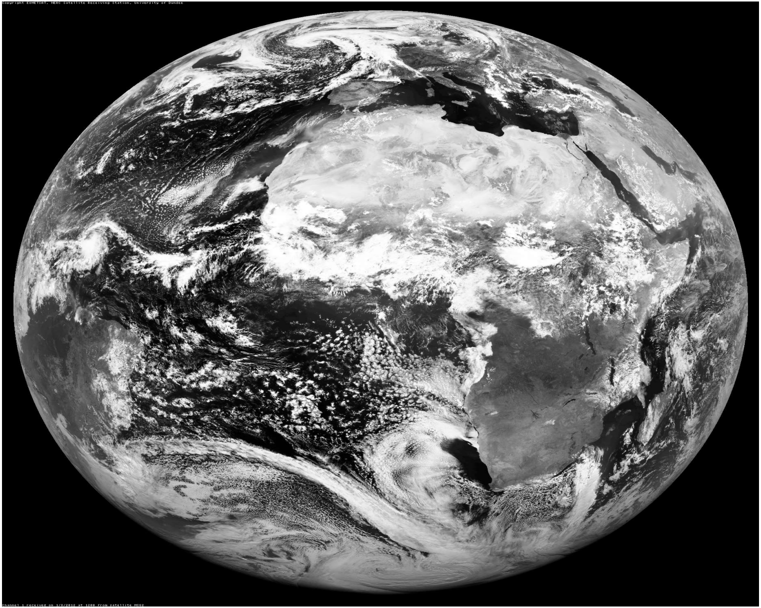
Plan

GFD seen from space



GFD : what is this ?

Hydrodynamics in all its complexity plus :

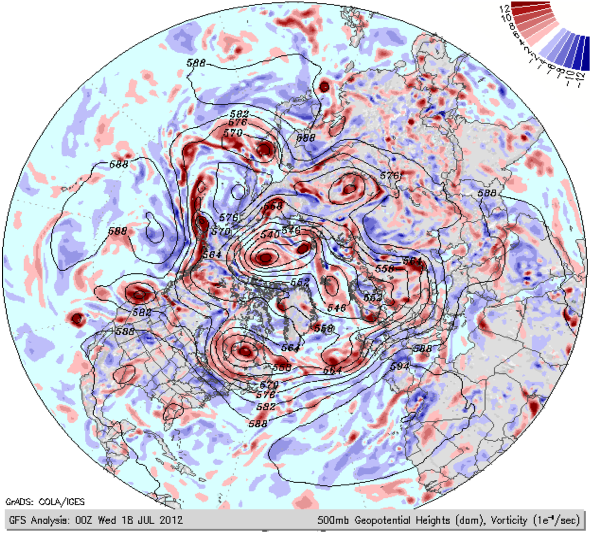
- Rotating frame
- Thermal effects, stratification
- Spherical geometry (large and medium scales)
- Complex domains (coasts, topography/bathymetry)
- Multi-phase fluid (water vapor, ice)

Scales :

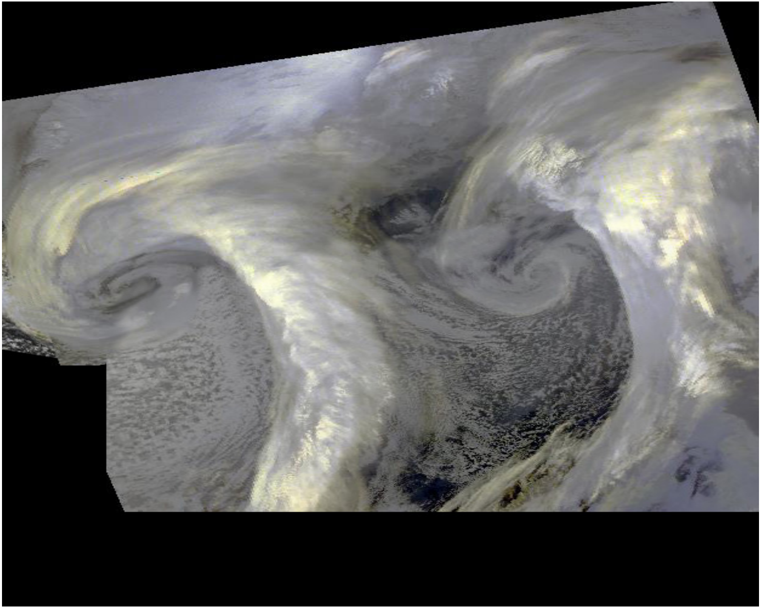
- Large : planetary 10^4 km
- Medium : atmosphere - synoptic, 10^3 km ; ocean - meso-scle $10 - 10^2$ km
- Small : atmosphere - meso-scale $1 - 10$ km ; ocean - sub-mesoscale 1 km
- Very small : meters

Our interest : modeling medium and large scales.

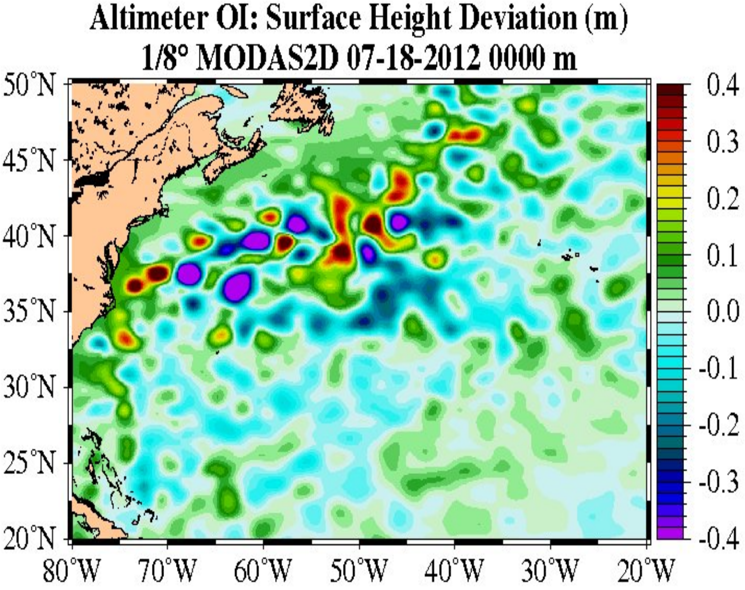
Dynamical actors : vortices, atmosphere



Atmospheric vortices for real



Dynamical actors : vortices, ocean



Dynamical actors : waves, atmosphere

Dynamical actors : waves, ocean

Work plan

We will :

- Remind the fundamentals
- Construct an hierarchy of models of decreasing complexity by
  1. vertically averaging and geting Rotating Shallow Water models
  2. filtering fast wave motions and geting Quasi-Geostrophic models
- Review their basic properties

General reminder

Governing equations for fluid envelopes of the Earth :

- Mechanical system ⇒ local conservation of momentum
- Continuous media ⇒ local conservation of mass
- Thermodynamical system ⇒ equation of state

Main difficulty - nonlinearity

Example of essentially nonlinear process : wave-breaking

Governing equations :

Eulerian description of the perfect fluid in terms of velocity, density and pressure fields : $\vec{v}(\vec{x},t)$, $\rho(\vec{x},t)$, $P(\vec{x},t)$.

Equations of motion

- Newton's second law :

$$\rho\left(\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\vec{\nabla}\vec{v}\right)=-\vec{\nabla}P+\vec{F}, \qquad (1)$$

  F - external forces.

- Continuity equation :

$$\frac{\partial\rho}{\partial t}+\vec{\nabla}\cdot(\rho\vec{v})=0. \qquad (2)$$

Closure

Equation of state

- General equation of state (1-phase system) :

$$P = P(\rho, s), \tag{3}$$

  s - mass density of entropy.

- Barotropic fluid :

$$P = P(\rho) \leftrightarrow s = \text{const}, \tag{4}$$

- Baroclinic fluid : :

$$P = P(\rho, s), \Rightarrow \tag{5}$$

  equation for s neccessary. Perfect fluid :

$$\frac{\partial s}{\partial t} + \vec{v} \cdot \vec{\nabla} s = 0. \tag{6}$$

Euler - Lagrange duality

Duality : $\vec{x} \leftrightarrow \vec{X}$, $\vec{X}(\vec{x}, t)$ - positions of fluid parcels.
Lagrangian derivative :

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}. \tag{7}$$

Newton's equations :

$$\rho(\vec{X}, t) \frac{d^2 \vec{X}}{dt^2} = -\vec{\nabla} P(\vec{X}, t) + \vec{F}. \tag{8}$$

Continuity equation :

$$\rho_i(x) d^3\vec{x} = \rho(\vec{X}, t) d^3\vec{X}, \leftrightarrow \rho_i(x) = \rho(\vec{X}, t) \mathcal{J} \tag{9}$$

where ρ_i - initial distribution of density, $\mathcal{J} = \frac{\partial(X,Y,Z)}{\partial(x,y,z)}$ Jacobi determinant (Jacobian). Fluid velocity : $\vec{v}(\vec{X}, t) = \frac{d\vec{X}}{dt} \equiv \dot{\vec{X}}$.

Particular case of barotropic fluid - incompressible fluid :

Volume conservation :

$$\mathcal{J} = 1 \leftrightarrow \vec{\nabla} \cdot \vec{v} = 0 \Rightarrow . \qquad (10)$$

pressure no more independent variable.

1. If, in addition, $\rho = const$:

$$\vec{\nabla} \cdot \left(\vec{v} \cdot \vec{\nabla} \vec{v} \right) = -\frac{1}{\rho} \vec{\nabla}^2 P. \qquad (11)$$

2. Otherwise

$$\frac{d\rho}{dt} = \frac{\partial \rho}{\partial t} + \vec{v} \cdot \vec{\nabla} \rho = 0. \qquad (12)$$

and

$$\vec{\nabla} \cdot \left(\vec{v} \cdot \vec{\nabla} \vec{v} \right) = -\vec{\nabla} \cdot \left(\frac{\vec{\nabla} P}{\rho} \right). \qquad (13)$$

Energy and thermodynamics

1st principle, one-phase system

$$\delta\epsilon = T\delta s - P\delta v, \quad (14)$$

ϵ - internal energy , $v = \frac{1}{\rho}$
Enthalpy per unit mass $h = \epsilon + Pv$:

$$\delta h = T\delta s + v\delta P. \quad (15)$$

Energy density of the fluid :

$$e = \frac{\rho \vec{v}^2}{2} + \rho\epsilon. \quad (16)$$

Local conservation of energy :

$$\frac{\partial e}{\partial t} + \vec{\nabla} \cdot \left[\rho \vec{v} \left(\frac{\vec{v}^2}{2} + h\right)\right] = 0. \quad (17)$$

Dissipation : molecular fluxes

Dissipation : correction of macroscopic fluxes of

- momentum
- mass
- internal energy (heat)

with corresponding molecular fluxes calculated from relations flux - gradient :

$$\vec{f}_A = -k_A \vec{\nabla} A, \tag{19}$$

A - a thermodynamical variable, $\vec{f}_A$ - corresponding molecular flux .

"Corrected" equations

Viscosity, incompressible case (Navier -Stokes equation

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} = -\frac{\vec{\nabla} P}{\rho} + \nu \vec{\nabla}^2 \vec{v}, \; \vec{\nabla} \cdot \vec{v} = 0. \tag{20}$$

Diffusivity : continuity equation

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = D \vec{\nabla}^2 \rho. \tag{21}$$

Thermoconductivity : heat/temperature equation

$$\frac{\partial T}{\partial t} + \vec{v} \cdot \vec{\nabla} T = \chi \vec{\nabla}^2 T. \tag{22}$$

Non-dimensional numbers

Reynolds : $Re = UL/\nu$, U, L - scales of the flow. Peclet : $\nu \to D$ or χ.

Euler equations in the rotating frame + gravity :

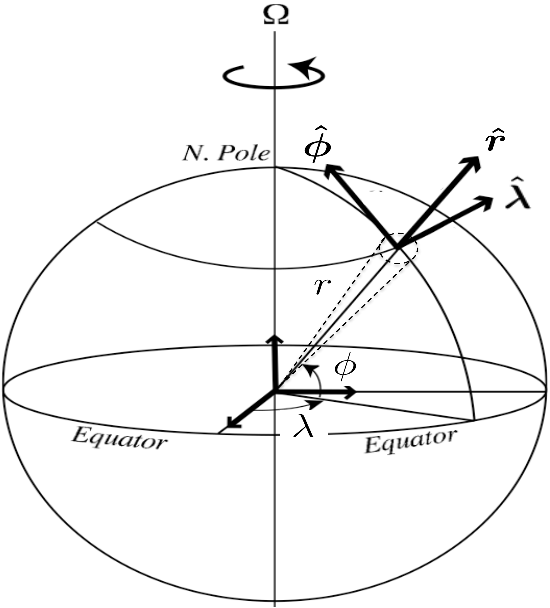
Coriolis force :

$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} + 2\vec{\Omega} \wedge \vec{v} - \vec{g}^* = -\frac{\vec{\nabla} P}{\rho} \tag{23}$$

Effective gravity :

$$\vec{g}^* = \vec{g} + m\vec{\Omega} \wedge \left(\vec{\Omega} \wedge \vec{r}\right) \tag{24}$$

Spherical coordinates



Euler and continuity equations

$$\frac{dv_r}{dt} - \frac{v_\lambda^2 + v_\phi^2}{r} - 2\Omega\cos\phi v_\lambda + g^* = -\frac{1}{\rho}\partial_r P,$$

$$\frac{dv_\lambda}{dt} + \frac{v_r v_\lambda - v_\phi v_\lambda \tan\phi}{r} + 2\Omega\left(-\sin\phi v_\phi + \cos\phi v_r\right) = -\frac{1}{\rho r}\partial_\lambda P,$$

$$\frac{dv_\phi}{dt} + \frac{v_r v_\phi + v_\lambda^2 \tan\phi}{r} + 2\Omega\sin\phi v_\lambda = -\frac{1}{\rho r\sin\theta}\partial_\phi P,$$

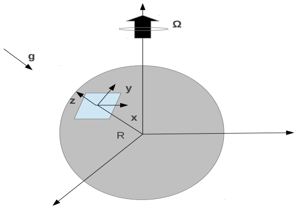
$$\frac{d\rho}{dt} + \rho\left[\frac{1}{r^2}\frac{\partial(r^2 v_r)}{\partial r} + \frac{1}{r\cos\phi}\left(\frac{\partial(\cos\phi v_\phi)}{\partial\phi} + \frac{\partial v_\lambda}{\partial\lambda}\right)\right],$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + v_r\partial_r + \frac{v_\theta}{r}\partial_\theta + \frac{v_\phi}{r\sin\theta}\partial_\phi$$

Traditional approx. : green + red $\to$ out, $r \to R = \mathrm{const}$

Non-traditional approx : green $\to$ out.

Tangent plane approximation



$$\frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v} + f \hat{z} \wedge \vec{v} + \vec{g} = -\frac{\vec{\nabla} P}{\rho}$$

f - plane : $f = const$; β - plane : $f = f + \beta y$; f - Coriolis parameter : $f = 2\Omega \sin \phi$

Ocean : observations

- Typical density profile :

$$\rho(\vec{x}, t) = \rho_0 + \rho_s(z) + \sigma(x, y, z; t), \quad \rho_0 \gg \rho_s \gg \sigma.$$

- Mesoscale motions close to hydrostatics :

$$g\rho + \partial_z P = 0, \Rightarrow P = P_0 + P_s(z) + \pi(x, y, z; t),$$

- Water $\approx$ incompressible

$$\vec{\nabla} \cdot \vec{v} = 0,$$

Equations of motion :

$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}_h + f \hat{z} \wedge \vec{v}_h = -\vec{\nabla}_h \phi, \qquad (25)$$

$\vec{v} = \vec{v}_h + \hat{z} w$, $\phi = \frac{\pi}{\rho_0}$ - geopotential.

$$\partial_t \rho + \vec{v} \cdot \vec{\nabla} \rho = 0, \quad \vec{\nabla} \cdot \vec{v} = 0. \qquad (26)$$

Boundary conditions (no dissipation) :
Rigid lid/flat bottom :

$$w|_{z=0} = w|_{z=H} = 0 \qquad (27)$$

Non-trivial bathymetry : $w|_{z=b} = \frac{db}{dt}$
Forcing/dissipation : external forces, viscosity - in (25) ; mass sources/sinks, diffusivity - in (26)

Atmosphere : Observations

- Mean pressure - monotonic with height,
- Synoptic motions - close to hydrostatics,
- Vertical velocities - small
- Potential temperature $\theta = e^s$ mostly advected (dry situation)

Pressure as vertical coordinate + hydrostatics $\Rightarrow$

- r.h.s. of the horizontal momentum eqns $\rightarrow$ gradient of geopotential
- velocity incompressible $\vec{\nabla} \cdot \vec{v} = 0$

Additional change of vert. coord. ("pseudo-height") + smallness of the vertical velocity $\rightarrow$ hydrostatic relation standard, up to a sign.

Equations of motion

In the absence of forcing/dissipation :

$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}_h + f \hat{z} \wedge \vec{v}_h = -\vec{\nabla}_h \phi, \quad (28)$$

$$-g \frac{\theta}{\theta_0} + \frac{\partial \phi}{\partial \bar{z}} = 0, \quad (29)$$

$$\frac{\partial \theta}{\partial t} + \vec{v} \cdot \vec{\nabla} \theta = 0; \quad \vec{\nabla} \cdot \vec{v} = 0. \quad (30)$$

Identical to oceanic primitive equations with $\sigma \to -\theta$.
Forcing/dissipation : external forces + viscosity in (28), thermal sources + thermoconductivity in (30)

Conservative form and verical averaging

Equations of horizontal motion

$$\partial_t(\rho u) + \partial_x(\rho u^2) + \partial_y(\rho v u) + \partial_z(\rho w u) - f\rho v = -\partial_x p, \quad (31)$$

$$\partial_t(\rho v) + \partial_x(\rho u v) + \partial_y(\rho v^2) + \partial_z(\rho w v) + f\rho u = -\partial_y p, \quad (32)$$

Integration between two material surfaces $z_{1,2}$.

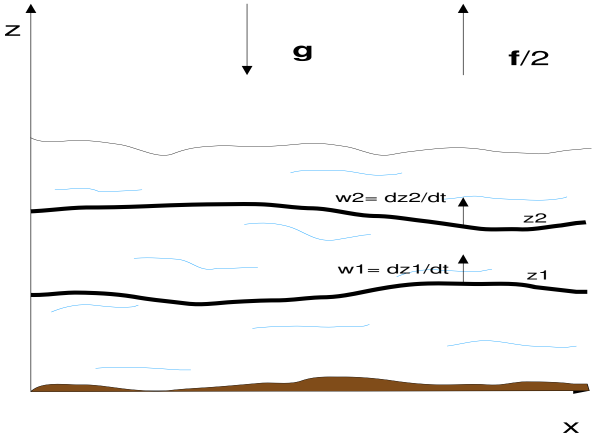
By definition :

$$w|_{z_i} = \frac{dz_i}{dt} = \partial_t z_i + u\partial_x z_i + v\partial_y z_i, \quad i = 1, 2. \quad (33)$$

Leibnitz formula :

$$\int_{z_1}^{z_2} dz \partial_x F = \partial_x \int_{z_1}^{z_2} dz F - \partial_x z_2 \, F|_{z_2} + \partial_x z_1 \, F|_{z_1} \quad (34)$$

Motion of material surfaces



Integrated momentum equations

Using (33), (34) we get :

$$\partial_t \int_{z_1}^{z_2} dz \rho u + \partial_x \int_{z_1}^{z_2} dz \rho u^2 + \partial_y \int_{z_1}^{z_2} dz \rho uv$$
$$- \quad f \int_{z_1}^{z_2} dz \rho v = -\partial_x \int_{z_1}^{z_2} dz p - \partial_x z_1 \ p\big|_{z_1} + \partial_x z_2 \ p\big|_{z_2} .$$

$$\partial_t \int_{z_1}^{z_2} dz \rho v + \partial_x \int_{z_1}^{z_2} dz \rho uv + \partial_y \int_{z_1}^{z_2} dz \rho v^2$$
$$+ \quad f \int_{z_1}^{z_2} dz \rho u = -\partial_y \int_{z_1}^{z_2} dz p - \partial_y z_1 \ p\big|_{z_1} + \partial_y z_2 \ p\big|_{z_2} .$$

Integrated continuity equation :

$$\partial_t \int_{z_1}^{z_2} dz \rho + \partial_x \int_{z_1}^{z_2} dz \rho u + \partial_y \int_{z_1}^{z_2} dz \rho v = 0. \qquad (35)$$

Integrated density + hydrostatics :

$$\mu = \int_{z_1}^{z_2} dz \rho = -\frac{1}{g} \left(p|_{z_2} - p|_{z_1} \right), \qquad (36)$$

Introducing density-weighted vertical average :

$$\langle F \rangle = \frac{1}{\mu} \int_{z_1}^{z_2} dz \rho F. \qquad (37)$$

Equations for the averages :

$$\partial_t (\mu\langle u\rangle) + \partial_x \left(\mu\langle u^2\rangle\right) + \partial_y (\mu\langle uv\rangle) - f\mu\langle v\rangle = - \partial_x \int_{z_1}^{z_2} dzp - \partial_x z_1 \left. p\right|_{z_1} + \partial_x z_2 \left. p\right|_{z_2}, \quad (38)$$

$$\partial_t (\mu\langle v\rangle) + \partial_x (\mu\langle uv\rangle) + \partial_y \left(\mu\langle v^2\rangle\right) + f\mu\langle u\rangle = - \partial_y \int_{z_1}^{z_2} dzp - \partial_y z_1 \left. p\right|_{z_1} + \partial_y z_2 \left. p\right|_{z_2}, \quad (39)$$

$$\partial_t \mu + \partial_x (\mu\langle u\rangle) + \partial_y (\mu\langle v\rangle) = 0. \quad (40)$$

Pressure and mean-field approximation

Expression for pressure

Pressure inside the layer (z_1, z_2) in terms of pressure at the lower surface and position :

$$p(x, y, z, t) = -g \int_{z_1}^{z} dz' \rho(x, y, z', t) + p|_{z_1} . \qquad (41)$$

Closure hypothesis :

Weak variations in the vertical (columnar motion), correlations decoupled :

$$\langle uv \rangle \approx \langle u \rangle \langle v \rangle, \ \langle u^2 \rangle \approx \langle u \rangle \langle u \rangle, \ \langle v^2 \rangle \approx \langle v \rangle \langle v \rangle. \qquad (42)$$

Remark : corrections may be intruduced via turbulent viscosity/diffusivity.

Mean density and pressure

Mean density

$$\bar{\rho} = \frac{1}{(z_2 - z_1)} \int_{z_1}^{z_2} dz\, \rho, \quad \mu = \bar{\rho}(z_2 - z_1). \tag{43}$$

Pressure in terms of $\bar{\rho}$:

$$p(x, y, z, t) \approx -g\bar{\rho}(z - z_1) + p|_{z_1} . \tag{44}$$

Hypothesis : $\bar{\rho} = const$ ($\bar{\rho}(x, y, t)$ also possible $\rightarrow$ Ripa's equations).

Master equations

For any pair of material surfaces :

Momentum :

$$\bar{\rho}(z_2 - z_1)(\partial_t \langle \mathbf{v}_h \rangle + \langle \mathbf{v}_h \rangle \cdot \nabla_h \langle \mathbf{v}_h \rangle + f \hat{\mathbf{z}} \wedge \langle \mathbf{v}_h \rangle) =$$
$$-\nabla_h \left(-g\bar{\rho}\frac{(z_2 - z_1)^2}{2} + (z_2 - z_1)\, p|_{z_1} \right)$$
$$-\nabla_h z_1\, p|_{z_1} + \nabla_h z_2\, p|_{z_2} . \qquad (45)$$

Mass :

$$(z_2 - z_1)_t + \nabla_h \cdot ((z_2 - z_1)\langle \mathbf{v}_h \rangle) = 0. \qquad (46)$$

Multi-layer Rotating Shallow Water models

Workflow

- Choose N material surfaces $z_1, z_2, ..., z_N$
- Write down the master equations for each layer (z_{i+1}, z_i), $i = 1, 2, N-1$
- Apply appropriate boundary conditions at $z_{1,N}$
- Require continuity of pressure across each interface

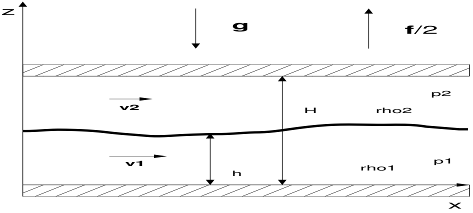
Generalizations

- Non-constant $\bar{\rho} = \bar{\rho}(x, y, t) \Rightarrow$ advection of $\bar{\rho}$ + additional term in the pressure gradient
- Deviations from the mean-field and/or molecular dissipation/diffusion $\Rightarrow$ terms $\propto \nabla_h^2 \mathbf{v}_h$, $\nabla_h^2 (z_{i+1} - z_i)$ in the momentum and mass equations
- Additional fluxes across the interfaces (convection, exchanges with boundary layers) : to be added while expressing w_i in terms of dz_i/dt.

Example : rotating shallow water (RSW), 2 layers

Configuration 2 layers, rigid lid

Application of equations (45) to the fluid between the flat bottom $z_1 = 0$ and the lid $z_3 = H$. Choose a material surface $z = z_2(x, y, t) \equiv h(x, y, t)$ inside the fluid,
$\vec{\nabla}_h \to \vec{\nabla}$, $\vec{v}_h \to \mathbf{v}$. Vertical boundaries - material surfaces .
Generalization to non-trivial topography : $z_1 \to b(x, y)$.



Equations of motion

$\mathbf{v}_{1(2)}, \bar{\rho}_{1(2)}$ - velosity and density in the lower (upper) layer.

$$\partial_t \mathbf{v}_2 + \mathbf{v}_2 \cdot \nabla \mathbf{v}_2 + f \hat{\mathbf{z}} \wedge \mathbf{v}_2 = -\frac{1}{\bar{\rho}_2} \nabla \left. p \right|_H \tag{47}$$

$$\partial_t \mathbf{v}_1 + \mathbf{v}_1 \cdot \nabla \mathbf{v}_1 + f \hat{\mathbf{z}} \wedge \mathbf{v}_1 = -\frac{1}{\bar{\rho}_1} \nabla \left. p \right|_H - g \frac{\bar{\rho}_1 - \bar{\rho}_2}{\bar{\rho}_1} \nabla h, \tag{48}$$

$$\partial_t h + \nabla \cdot (\mathbf{v}_1 h) = 0 \,, \tag{49}$$

$$\partial_t (H - h) + \nabla \cdot (\mathbf{v}_2 (H - h)) = 0 \,, \tag{50}$$

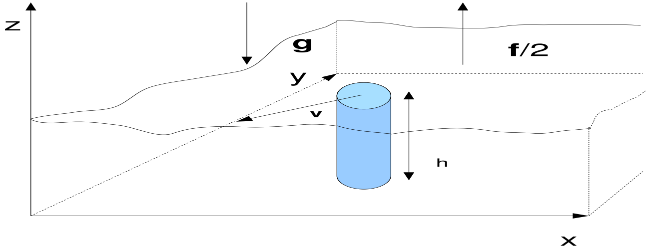
Classical one-layer RSW model

2-layer RSW in the limit $\bar{\rho}_2 \to 0 \Rightarrow$

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} + f \hat{\mathbf{z}} \wedge \mathbf{v} + g \nabla h = 0 \,, \qquad (51)$$

$$\partial_t h + \nabla \cdot (\mathbf{v} h) = 0 \;\Rightarrow \qquad (52)$$

Motion of fluid columns :



Conservation laws - RSW model

Energy

By construction, equations (51), (52) express the local momentum and mass conservation. Energy density :

$$e = h\frac{\mathbf{v}^2}{2} + g\frac{h^2}{2} \tag{53}$$

obeys the conservation law :

$$\partial_t e + \nabla \cdot \left(\mathbf{v} h \left(\frac{\mathbf{v}^2}{2} + gh \right) \right) = 0, \tag{54}$$

and total energy, $E = \int dxdy\, e$, is constant for isolated system.

Potential vorticity - RSW model

Specific Lagrangian conservation law : of potential vorticity q (PV), which is built from relative vorticity $\zeta = v_x - u_y$, Coriolis parameter f, and the fluid depth h.

$$q = \frac{\zeta + f}{h}. \quad (55)$$

Here $\zeta + f$ -absolute vorticity, f - planetary vorticity.

Lagrangian conservation :

$$\frac{dq}{dt} \equiv (\partial_t + \mathbf{v} \cdot \nabla)\, q = 0, \qquad (56)$$

is obtained by combining equations of vorticity :

$$\frac{d(\zeta + f)}{dt} + (\zeta + f)\nabla \cdot \mathbf{v} = 0, \qquad (57)$$

and continuity

$$\frac{dh}{dt} + h\nabla \cdot \mathbf{v} = 0 \; : \qquad (58)$$

$$\frac{d}{dt}\frac{\zeta + f}{h} = \frac{1}{h}\frac{d}{dt}(\zeta + f) - \frac{\zeta + f}{h^2}\frac{d}{dt}h = 0, \qquad (59)$$

Eulerian expression

Conservation of PV is expressed as time - independence of any integral :

$$\int dxdy\, h\mathcal{F}(q), \tag{60}$$

over the domain of the flow, shere $\mathcal{F}$ is arbitrary function.

Qualitative view of the RSW dynamics :

Two-dimensional motion of fluid columns of variable depth, each preserving its potential vorticity.

Spectrum of small perturbations - RSW model

Linearised equations :

Perturbations about the state of rest $\mathbf{v} = 0$, $h = H_0 = const$ on the f-plane :

$$\begin{aligned} u_t - fv + g\eta_x &= 0, \\ v_t + fu + g\eta_y &= 0, \\ \eta_t + H_0(u_x + v_y) &= 0, \end{aligned} \tag{61}$$

Fourier-transform

Solutions - harmonic waves :

$$(u, v, \eta) = (u_0, v_0, \eta_0)e^{i(\omega t - \mathbf{k}\cdot\mathbf{x})}, \tag{62}$$

with ω, $\mathbf{k}$ - frequency and wavenumber, respectively.$\Rightarrow$ algebraic system for (u_0, v_0, η_0).

Dispersion relation

Solvability condition :

$$\det \begin{pmatrix} i\omega & -f & -igk_x \\ f & i\omega & -igk_y \\ -iH_0k_x & -iH_0k_y & i\omega \end{pmatrix} = 0, \Rightarrow \qquad (63)$$

$$\omega\left(\omega^2 - gH_0\mathbf{k}^2 - f^2\right) = 0. \qquad (64)$$

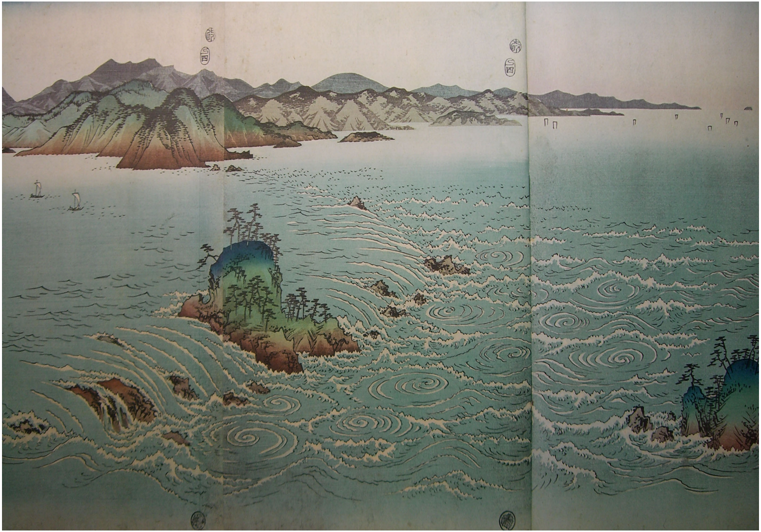
Three roots :

- Stationary solutions $\omega = 0$
- Propagative waves with dispersion relation, Inertia-gravity waves :

$$\omega = \sqrt{gH_0\mathbf{k}^2 + f^2} \geq f. \qquad (65)$$

Preliminary conclusions.

- Two dynamical actors : vortices and waves
- Vortex motions : slow, related to Lagrangian conservation of PV ; zero frequency in lineair approximation .
- Wave motions : fast
- Frequencies of waves and vortices are separated by the spectral gap.



General equations of horizontal motion

$$\frac{\partial \vec{v}_h}{\partial t} + \vec{v} \cdot \vec{\nabla} \vec{v}_h + f \hat{z} \wedge \vec{v}_h = -\vec{\nabla}_h \Phi. \qquad (66)$$

$$f = f_0(1 + \beta y), \quad \Phi = \Phi_0 + \phi = g(H_0 + h) \qquad (67)$$

h - geopotential height.

Scaling for vortex motions

- Velocity $\vec{v}_h = (u, v)$, $u, v \sim U$, $w \sim W << U$
- Unique horizontal scale$L$,
- Vertical scale $H << L$,
- Time-scale : turnover time $T \sim L/U$.

Geostrophic equilibrium :

Equilibrium between the Coriolis and pressure forces :

$$f \hat{\mathbf{z}} \wedge \mathbf{v}_g = -\nabla_h \Phi \qquad (68)$$

Characteristic parameters of horizontal motions

Intrinsic scale : deformation (Rossby) radius :

$$R_d = \frac{\sqrt{gH_0}}{f_0} \tag{69}$$

Non-dimensional parameters :

- Rossby number : $Ro = \frac{U}{f_0 L}$,
- Burger number : $Bu = \frac{R_d^2}{L^2}$,
- Characteristc nonlinearity : $\lambda = \Delta H / H_0$, where ΔH is a typical value of h,
- Non-dimensional gradient of f : $\tilde{\beta} \sim \beta L$

Non-dimensional RSW quations

$$Ro\,(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}) + (1+\tilde{\beta}y)\hat{\mathbf{z}}\wedge\mathbf{v} = -\frac{\lambda Bu}{Ro}\nabla\eta\,, \tag{70}$$

$$\lambda\partial_t\eta + \nabla\cdot(\mathbf{v}(1+\lambda\eta)) = 0\,. \tag{71}$$

Examples of dynamical regimes close to geostrophy : $Ro \equiv \epsilon \ll 1$

- Quasi-geostrophic(QG) : small nonlinearity :

$$\lambda \sim Ro, \Rightarrow Bu \sim 1, \Rightarrow L \sim R_d,\ \tilde{\beta} \sim Ro \tag{72}$$

- Frontal geostrophic (FG) : strong nonlinearity :

$$\lambda \sim 1, \Rightarrow Bu \sim Ro, \Rightarrow L \gg R_d,\ \tilde{\beta} \sim Ro \tag{73}$$

Dearivation of 1-layer QG equations

$$\epsilon\left(\partial_t \mathbf{v} + \mathbf{v}\cdot\nabla\mathbf{v}\right) + (1+\epsilon y)\hat{\mathbf{z}}\wedge\mathbf{v} = -\nabla\eta\,, \qquad (74)$$

$$\epsilon\partial_t\eta + \nabla\cdot(\mathbf{v}(1+\epsilon\eta)) = 0\,. \qquad (75)$$

Asymptotic expansions in *Ro* :

$$\mathbf{v} = \mathbf{v}^{(0)} + \epsilon\mathbf{v}^{(1)} + \epsilon^2\mathbf{v}^{(2)} + \ldots \qquad (76)$$

Order ϵ^0 - geostrophy :

$$u^{(0)} = -\partial_y\eta, \quad v^{(0)} = \partial_x\eta \quad \Rightarrow \quad \partial_x u^{(0)} + \partial_y v^{(0)} = 0, \quad (77)$$

$$\frac{d^{(0)}}{dt}\cdots = \partial_t\ldots + u^{(0)}\partial_x\ldots + v^{(0)}\partial_y\ldots \equiv \partial_t\cdots + \mathcal{J}(\eta,\ldots). \qquad (78)$$

$$\mathcal{J}(A,B) \equiv \partial_x A\partial_y B - \partial_y A\partial_x B. \qquad (79)$$

Order ϵ^1 - quasi-geostrophy :

$$u^{(1)} = -\frac{d^{(0)}}{dt} v^{(0)} - y u^{(0)}, \quad v^{(1)} = \frac{d^{(0)}}{dt} u^{(0)} - y v^{(0)}, \Rightarrow \quad (80)$$

$$\partial_x u^{(1)} + \partial_y v^{(1)} = -\frac{d^{(0)}}{dt} \vec{\nabla}^2 \eta - v^{(0)}, \Rightarrow \quad (81)$$

$$\frac{d^{(0)}}{dt}\left(\eta - \vec{\nabla}^2 \eta\right) - \partial_x \eta = 0 \leftrightarrow \frac{d^{(0)}}{dt}\left(\eta - \vec{\nabla}^2 \eta - y\right) = 0. \quad (82)$$

Restituted dimensions

$$\frac{d^{(0)}}{dt}\left(\frac{f_0^2}{gH_0}\left(\frac{gh}{f_0}\right) - \vec{\nabla}^2\left(\frac{gh}{f_0}\right) - f_0(1 + \beta y)\right) = 0.$$

QG equation on the β- and f -planes :

β - plane

$$\partial_t \eta - \partial_t \vec{\nabla}^2 \eta - \mathcal{J}(\eta, \vec{\nabla}^2 \eta) - \partial_x \eta = 0. \qquad (83)$$

Physical meaning : conservation of PV in QG approximation.
Formal linearisation :

$$\partial_t \eta - \partial_t \vec{\nabla}^2 \eta - \partial_x \eta = 0, \Rightarrow \qquad (84)$$

Waves : $\eta \propto exp^{i(kx+ly-\omega t)} \rightarrow$ dispersion : $\omega = -\frac{k}{k^2+l^2+1} \rightarrow$ Rossby waves.

f-plane

$$\partial_t \eta - \vec{\nabla}^2 \partial_t \eta - \mathcal{J}(\eta, \vec{\nabla}^2 \eta) = 0. \qquad (85)$$

$\Leftrightarrow$ 2D Euler equations for incompressible fluid with streamfunction η and modified streamfunction - vorticity relation : $\zeta = -\eta + \vec{\nabla}^2 \eta$. $R_d \rightarrow \infty$ - standard 2D Euler.

Adding dissipation and forcing

Molecular viscosity

Non-dimensional Navier-Stokes : Euler + $\frac{1}{Re}\nabla^2\vec{v} \Rightarrow$ Vorticity equation : Euler + $\frac{1}{Re}\nabla^2\zeta$.

Interaction of free QG flow with boundary layer

Viscosity $\Rightarrow$ boundary layer. Rotating fluid : Ekman layer. Small Rossby numbers $\Leftrightarrow$ QG regime : vertical velocity on top of the boundary layer : $w(x, y, t) \propto \zeta \Rightarrow$ term $-r\zeta$, $r =$ const in the r.h.s. of the vorticity equation.

Forced-dissipative QG equation :

$$\frac{d_{QG}\zeta}{dt} = -r\zeta + \frac{1}{Re}\nabla^2\zeta + F, \tag{86}$$

where $\frac{d_{QG}\zeta}{dt} \cdots = \partial_t \cdots + \mathcal{J}(\eta, \dots)$ - QG advection, $\zeta = -\frac{1}{R_d^2}\eta + \vec{\nabla}^2\eta + \beta y$ in dimensionful terms.

Parameters and scales of the 2-layer RSW

Parameters :

- Rossby number : $Ro = \frac{U}{f_0 L}$
- Non-dimensional typical deviation of the interface : λ
- Non-dimensional gradient of Coriolis parameter : $\tilde{\beta}$
- Aspect ratio : $d = \frac{H_1}{H_2} = \frac{D_1}{D_2}$, $D_{1,2} = \frac{H_{1,2}}{H}$
- Stratification parameter : $N = 2\frac{\rho_2 - \rho_1}{\rho_2 + \rho_1}$
- Burger number : $Bu = \frac{R_d^2}{L^2}$, $R_d^2 = \frac{NgH}{f_0^2}$

Characteristic scales :

Baroclinic deformation radius : $R_d^2 = \frac{g' H}{f_0}$, g' - reduced gravity $g' = gN$; Prssures in the layers : $P_i \sim \rho_i U L f_0$.

Non-dimensional equations

$$\epsilon \frac{d_i}{dt}\mathbf{v}_i + (1+\tilde{\beta}y)\hat{z}\wedge \mathbf{v}_i = -\vec{\nabla}\pi_i, \quad i = 1,2. \qquad (87)$$

$$\begin{aligned} -\lambda \frac{d_1}{dt}\eta + (D_1 - \lambda\eta)\vec{\nabla}\cdot\mathbf{v}_1 &= 0 \\ \lambda \frac{d_2}{dt}\eta + (D_2 + \lambda\eta)\vec{\nabla}\cdot\mathbf{v}_2 &= 0 \end{aligned} \qquad (88)$$

$$\pi_2 - \pi_1 + \frac{N}{2}(\pi_2 + \pi_1) = \frac{\lambda Bu}{2\epsilon}\eta. \qquad (89)$$

$$\frac{d_i}{dt} = \partial_t + \mathbf{v}_i \cdot \nabla \qquad (90)$$

QG regime

$$\lambda \sim \tilde{\beta} \sim \epsilon \ll 1, \quad \Rightarrow \quad L \sim R_d \tag{91}$$

Asymptotic expansion in $\epsilon \Rightarrow$

$$\begin{aligned} u_i &= u_i^{(0)} - \epsilon \left[\partial_t v_i^{(0)} + \mathcal{J}(\pi_i, v_i^{(0)}) + y u_i^{(0)} \right] + \dots \\ v_i &= v_i^{(0)} + \epsilon \left[\partial_t u_i^{(0)} + \mathcal{J}(\pi_i, u_i^{(0)}) - y v_i^{(0)} \right] + \dots \end{aligned} \tag{92}$$

Geostrophy :

$$u_i^{(0)} = -\partial_y \pi_i, \quad v_i^{(0)} = \partial_x \pi_i. \tag{93}$$

Divergence :

$$\partial_x u_i^{(1)} + \partial_y v_i^{(1)} = -\left[\partial_t \vec{\nabla}^2 \pi_i + \mathcal{J}(\pi_i, \vec{\nabla}^2 \pi_i) + \partial_x \pi\right] \tag{94}$$

Equations for η :

$$\partial_t \eta + \mathcal{J}(\pi_i, \eta) - (-1)^i D_i \left[\partial_t \vec{\nabla}^2 \pi_i + \mathcal{J}(\pi_i, \vec{\nabla}^2 \pi_i) + \partial_x \pi\right] = 0, \quad i = 1, 2. \tag{95}$$

2-layer QG equations

Equations for the pressure layerwise :

$$\frac{d_i^{(0)}}{dt}\left[\nabla^2\pi_i - (-1)^i D_i^{-1}\eta + y\right] = 0\,,\ i = 1, 2. \qquad (96)$$

where

$$\frac{d_i^{(0)}}{dt}(\ldots) := \partial_t(\ldots) + J(\pi_i, \ldots)\,,\ i = 1, 2 \qquad (97)$$

Standard limit : weak stratification $\rightarrow \rho_2 \rightarrow \rho_1 \Rightarrow$ $\eta = \pi_2 - \pi_1$

Remarks : 1) on the f - plane - coupled 2D Euler equations with modified streamfunction - vorticity relation ; 2) forcing and dissipation are introduced as in 1-layer case.

RSW equations at small *Ro* and 2 temporal scales

Hypotheses :

- f- plane, open domain,
- unique spatial scale L,
- small Rossby numberϵ, regime QG : $\lambda \sim \epsilon$,
- rapide $t \sim f_0^{-1}$ and slow $t_1 \sim (\epsilon f_0)^{-1}$ time-scales

Non-dimensional equations :

$$(\partial_t + \epsilon \partial_{t_1})\mathbf{v} + \epsilon(\mathbf{v} \cdot \nabla \mathbf{v}) + \hat{\mathbf{z}} \wedge \mathbf{v} + \nabla h = 0\,, \qquad (98)$$

$$(\partial_t + \epsilon \partial_{t_1})h + (1 + \epsilon h)\nabla \cdot \mathbf{v} + \epsilon \mathbf{v} \cdot \nabla h = 0\,, \qquad (99)$$

$$\partial_t Q + \epsilon \mathbf{v} \cdot \nabla Q = 0\,, \quad Q = \epsilon \frac{\zeta - h}{1 + \epsilon h} - \text{PV anomaly.} \qquad (100)$$

Geostrophic adjustement

Cauchy problem with localised i.c.

$$u\big|_{t=0} = u_I \ , \ v\big|_{t=0} = v_I \ , \ h\big|_{t=0} = h_I \ . \qquad (101)$$

Multi-scale asymptotic expansions

$$\begin{aligned} \mathbf{v} &= \mathbf{v}_0(x,y;t,t_1,...) + \epsilon \mathbf{v}_1(x,y;t,t_1,...) + ... \qquad (102) \\ h &= h_0(x,y;t,t_1,...) + \epsilon h_1(x,y;t,t_1,...) + ..., \end{aligned}$$

Slow-fast decomposition order by order in ϵ :

$$h_i = \bar{h}_i(x,y;t_1,...) + \tilde{h}_i(x,y;t,t_1,...), \ i = 0,1,2,... \qquad (103)$$

$$\bar{h}_i(x,y;t_1,...) = \lim_{T\to\infty} \frac{1}{T}\int_0^T h_i(x,y,t,t_1,...)\,dt, \qquad (104)$$

Approximation ϵ^0

$$\partial_t \mathbf{v}_0 + \hat{\mathbf{z}} \wedge \mathbf{v}_0 = -\nabla h_0 \,, \tag{105}$$

$$\partial_t (\zeta_0 - h_0) = 0 \,, \tag{106}$$

where $\zeta_0 = \hat{\mathbf{z}} \cdot \nabla \wedge \mathbf{v}_0$ - relative vorticity, and PV equation is used. I.c. :

$$u_0|_{t=0} = u_I \ , \ v_0|_{t=0} = v_I \ , \ h_0|_{t=0} = h_I \ . \tag{107}$$

Re-writing (105) in terms of relative vorticity ζ and divergence $D = \nabla \cdot \mathbf{v}_0$:

$$\partial_t \zeta_0 + D_0 = 0 \,, \tag{108}$$

$$\partial_t D_0 - \zeta_0 = -\nabla^2 h_0 \,. \tag{109}$$

Integration of (106) in fast time t :

$$\zeta_0 - h_0 = \Pi_0 \,, \tag{110}$$

where Π_0 is yet unknown function of x, y, t_1 (integration "constant").

Approximation ϵ^0 - *contd*

Elimination of ζ_0 et D_0- linear inhomogeneous equation for h_0 :

$$-\frac{\partial^2 h_0}{\partial t^2} - h_0 + \nabla^2 h_0 = \Pi_0(x, y; t_1, t_2, \ldots)\,. \qquad (111)$$

Solution - fast + slow :

$$h_0 = \tilde{h}_0(x, y; t, \ldots) + \bar{h}_0(x, y; t_1, \ldots) \qquad (112)$$

$$-\frac{\partial^2 \tilde{h}_0}{\partial t^2} - \tilde{h}_0 + \nabla^2 \tilde{h}_0 = 0\,; \qquad (113)$$

$$-\bar{h}_0 + \nabla^2 \bar{h}_0 = \Pi_0 \qquad (114)$$

Klein - Gordon (KG) and Helmholtz equations.
Π_0 : geostrophic PV constructed with the help of slow component $\bar{h}_0$.

Initialisation problem :

How to separate i.c. in slow/fast ?

Response (unique at $\epsilon \to 0$)

- By definition :

$$\Pi_0(x,y;0) = \partial_x v_I - \partial_y u_I - h_I \equiv \Pi_I(x,y) \qquad (115)$$

- Determination of initial value $\bar{h}_{0I}$ de $\bar{h}_0$ by inversion :

$$-\bar{h}_{0I} + \nabla^2 \bar{h}_{0I} = \Pi_I, \Rightarrow \bar{h}_{0I} = -(\nabla^2 - 1)^{-1}\Pi_I. \qquad (116)$$

- Determination of initial value $\tilde{h}_{0I}$ de $\tilde{h}_0$:

$$\tilde{h}_{0I} = h_I - \bar{h}_{0I}. \qquad (117)$$

- Second i.c.for $\tilde{h}_0$ (PV and ζ - D eqns) :

$$\partial_t \tilde{h}_0 \Big|_{t=0} = -D_I \equiv \partial_x u_I + \partial_y v_I \,. \qquad (118)$$

Approximation ϵ^0 - *contd*

Decomposition for $\mathbf{v}$:

$$\mathbf{v}_0 = \tilde{\mathbf{v}}_0(x, y; t, ...) + \bar{\mathbf{v}}_0(x, y; t_1, ...)\,, \qquad (119)$$

slow components verify geostrophic relation :

$$\bar{\mathbf{v}}_0 = \hat{\mathbf{z}} \wedge \nabla \bar{h}_0 \qquad (120)$$

and slow ones obey the equations :

$$\partial_t \tilde{\mathbf{v}}_0 + \hat{\mathbf{z}} \wedge \tilde{\mathbf{v}}_0 = -\nabla \tilde{h}_0 \qquad (121)$$

with i.c. :

$$\tilde{u}_I^{(0)} = u_I - \bar{u}_{0I}\,; \;\; \tilde{v}_I^{(0)} = v_I - \bar{v}_{0I}\,, \qquad (122)$$

where $\bar{u}_{0I}, \bar{v}_{0I}, \bar{h}_{0I}$ verify (120). Linearized PV $\tilde{\zeta}_0 - \tilde{h}_0$ of the fast component is identically zero.

Approximation ϵ^0 - *contd*

Fast component solution for h :

Inertia-gravity waves propagating out of initial perturbation ; generated by its non-balanced part $\tilde{u}_I^{(0)}, \tilde{v}_I^{(0)}, \tilde{h}_{0I}$:

$$\tilde{h}_0(\mathbf{x}; t) = \sum_{\pm} \int d\mathbf{k}\, H_0^{(\pm)}(\mathbf{k}) e^{i(\mathbf{k}\cdot\mathbf{x} \pm \Omega_{\mathbf{k}} t)}\,, \tag{123}$$

where

$$H_0^{(\pm)}(\mathbf{k}) = \frac{1}{2}\left(\hat{\tilde{h}}_{0I}(\mathbf{k}) \pm i\, \frac{\hat{D}_I(\mathbf{k})}{\Omega_{\mathbf{k}}}\right), \tag{124}$$

and notation $\hat{...}$ is used for Fourier-transforms.

Approximation ϵ^0 - *end*

Résumé of the first approximation

- Fast and slow components are defined unambiguously
- Slow and fast motions are dynamically split (non-interacting)
- Fast part is completely resolved :inertia-gravity waves propagate out of initial perturbation
- Evolution of the slow component remains to be determined

Approximation ϵ^1

Momentum equations :

$$\partial_t \mathbf{v}_1 + \hat{\mathbf{z}} \wedge \mathbf{v}_1 = -\nabla h_1 - (\partial_{t_1} + \mathbf{v}_0 \cdot \nabla)\, \mathbf{v}_0 \,. \qquad (125)$$

PV equation, first order :

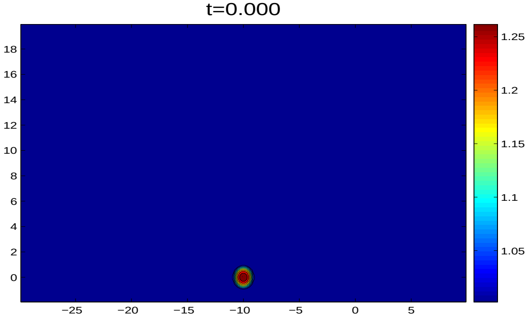
$$\partial_t (\zeta_1 - h_1) - \Pi_0\, \partial_t \tilde{h}_0 + \tilde{u}^{(0)} \partial_x \Pi_0 + \tilde{v}^{(0)} \partial_y \Pi_0 = -\partial_{t_1} \Pi_0 - J(\bar{h}_0, \Pi_0) \,. \qquad (126)$$

Integrability condition $\leftrightarrow$ averaging in t :

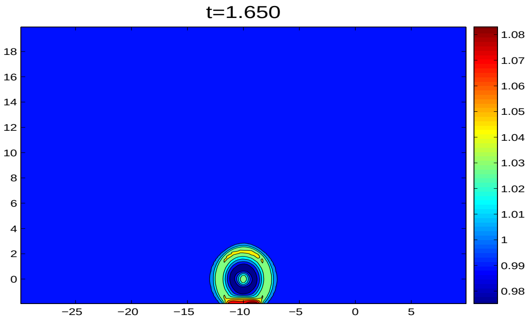
$$\partial_{t_1} \Pi_0 + J(\bar{h}_0, \Pi_0) \equiv \partial_{t_1} (\nabla^2 \bar{h}_0 - \bar{h}_0) + J(\bar{h}_0, \nabla^2 \bar{h}_0) = 0 \,. \qquad (127)$$

$\Rightarrow$ QG equation . Originates from elimination of resonances for the fast component at order 1.

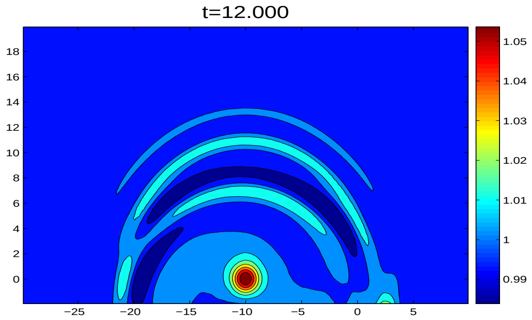
Numerical simulations of the geostrophic ajustement. Initial perturbation of h.



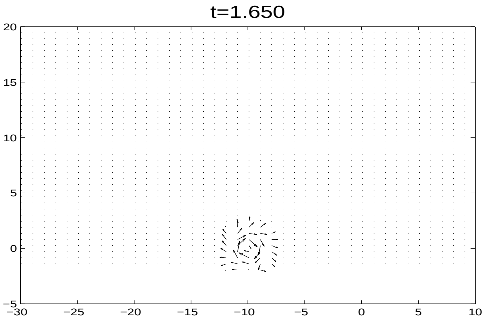
Initial stage of adjustement, *h* field.



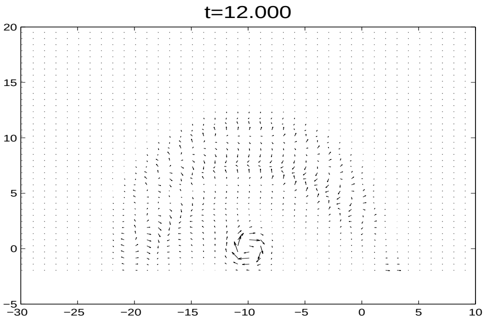
Initial stage of adjustement, *h* field.



Initial stage of adjustement, velocity field.



Advanced stage of adjustement, velocity field.



Preliminary conclusions : QG model(s)

In the limit $Ro \to 0$, and with the choce of vortex (slow) time-scale

- Inertia-gravity waves are filtered out
- Resulting equations for vortex motions in the f-plane approximation are 2D Euler equations with modified streamfunction-vorticity relation
- Specific strongly anysotropic vortex waves (Rossby waves) are present in the β - plane approximation
- QG dynamics $\Leftrightarrow$ fast-time averaging of the full equations.

Hierarchy of simplified models :

- Large-scale atmospheric and oceanic motions : same primitive equations up to changes of variables
- Typical horizontal scale $\gg$ vertical scales $\rightarrow$ vertical averaging $\Rightarrow$ rotating shallow water equations
- Wave-vortex dichotomy ; vortex - slow, waves - fast
- Fast-time averaging at small Rossby numbers - quasi-geostrophic vortex dynamics equations $\approx$ 2D Euler/Navier -Stokes

Literature

Presentation partially based on : *Nonlinear Dynamics of Rotating Shallow Water : Methods and Advances*, V. Zeitlin, ed., Elsevier, 2007, 391p.