Counting stationary modes: a discrete view of geometry and dynamics

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Weyl Law at 100, Fields Institute

Outline

- A (sketchy) history of Weyl's law, from 19th century physics to V.Ivrii's proof of the 2-term asymptotics
- resonances of open wave systems : counting them all, vs. selecting only the long-living ones

An urgent mathematical challenge from theoretical physics

In October 1910, Hendrik Lorentz delivered lectures in Göttingen, *Old and new problems of physics*. He mentioned an "urgent question" related with the **black body radiation problem** :

> *Prove that the density of standing electromagnetic waves inside a bounded cavity $\Omega \subset \mathbb{R}^3$ is, at high frequency, independent of the shape of Ω.*

A similar conjecture had been expressed a month earlier by Arnold Sommerfeld, for *scalar* waves. In this case, the problem boils down to counting the solutions of the *Helmholtz equation* inside a bounded domain $\Omega \subset \mathbb{R}^d$:

$$(\Delta + \lambda_n^2) u_n = 0, \qquad u_{n \restriction \partial\Omega} = 0 \text{ (Dirichlet b.c.)}$$

Our central object : the *counting function* $N(\lambda) \stackrel{\text{def}}{=} \#\{0 \leq \lambda_n \leq \lambda\}$.
In 1910, $N(\lambda)$ could be computed only for simple, separable domains (rectangle, disk, ball..) : one gets in the high-frequeny limit ($\lambda \to \infty$)

$$N(\lambda) \sim \frac{|\Omega|}{4\pi} \lambda^2 \quad (2 - dim.), \qquad N(\lambda) \sim \frac{|\Omega|}{6\pi^2} \lambda^3 \quad (3 - dim.)$$

An efficient postdoc

Hermann Weyl (a fresh PhD) was attending Lorentz's lectures. A few months later he had proved the 2-dimensional scalar case,

$$N(\lambda) = \frac{|\Omega|}{4\pi}\lambda^2 + o(\lambda^2), \quad \lambda \to \infty,$$

which was presented by D.Hilbert in front of the Royal Academy of Sciences.

Within a couple of years, Weyl had generalized his result in various ways : 3 dimensions, electromagnetic waves, elasticity waves.

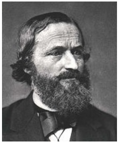
In 1913 he conjectured (based on the case of the rectangle) a 2-term asymptotics, depending on the boundary conditions :

$$N^{(d)}_{D/N}(\lambda) = \frac{\omega_d\,|\Omega|}{(2\pi)^d}\lambda^d \mp \frac{\omega_{d-1}|\partial\Omega|}{4(2\pi)^{d-1}}\lambda^{d-1} + o(\lambda^{d-1})$$

Then he switched to other topics (general relativity, gauge theory etc.)

Why were (prominent) physicists so interested in this question ?

The **black body radiation problem** had puzzled physicists for several decades [KIRCHHOFF'1859].
At thermal equilibrium, a black body emits EM waves with a spectral distribution $\rho(\lambda, T)$, which depends on the density $D(\lambda) = \frac{dN(\lambda)}{d\lambda}$ of stationary waves inside Ω.



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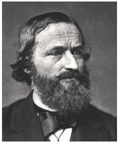

Around 1900, all physicists *took for granted* that the asymptotics for $D(\lambda)$ was independent of the shape.They were confronting a more annoying puzzle : **Ultraviolet catastrophe**

Equipartition of energy $\Longrightarrow \rho(\lambda, T) \propto D(\lambda) T \propto |\Omega| \lambda^2 T$
$\Longrightarrow$ the full emitted power $P(T) = \int_0^\infty \rho(\lambda, T)\, d\lambda$ is infinite as soon as $T > 0$!

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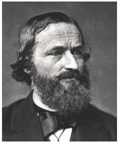

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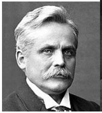

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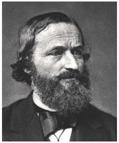

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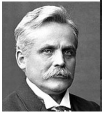

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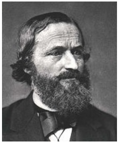

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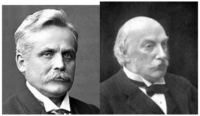

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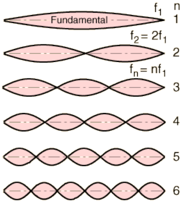
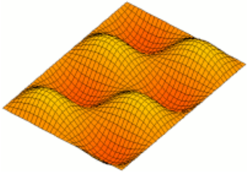
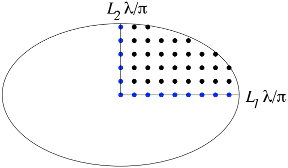
Rectangular cavities







The (Dirichlet) stationary modes of strings or rectangles are explicit :

$$\text{string :}\quad u_n(x) = \sin(\pi x n/L), \quad \lambda = \frac{\pi n}{L},\ n \geq 1 \Longrightarrow N_D(\lambda) = [\lambda L/\pi]$$

$$\text{rectangle :}\quad u_{n_1,n_2}(x,y) = \sin(\pi x n_1/L_1)\sin(\pi y n_2/L_2), \quad \lambda = \pi\sqrt{\left(\frac{n_1}{L_1}\right)^2 + \left(\frac{n_2}{L_2}\right)^2},$$

$\rightsquigarrow$ Gauss's lattice point problem in a 1/4-ellipse. Leads to the 2-term asymptotics, in any dimension $d \geq 2$:

$$N_{D/N}(\lambda) = \frac{L_1 L_2}{4\pi}\lambda^2 \mp \frac{2(L_1+L_2)}{4\pi}\lambda + o(\lambda) \quad (2-dim)\,.$$

NB : even in this case, estimating the remainder is a difficult task.

What if Ω is not separable ?

If the domain Ω doesn't allow separation of variables, the eigenmodes/values are not known explicitly (true PDE problem).
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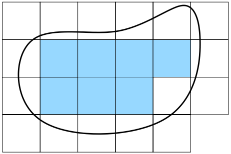
Pave Ω with (small) rectangles. Then,



$$\sum_{\square} N_{\square,D}(\lambda) \leq N_{\Omega}(\lambda) \leq \sum_{\square+\square} N_{\square,N}(\lambda)$$

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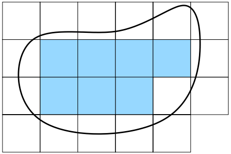

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[COURANT'1924] : this variational method can be improved to give a remainder $\mathcal{O}(\lambda \log \lambda)$, but no better.

What to do next ?

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    Semiclassical analysis : in the semiclassical limit $\hbar \to 0$, deduce properties of $H_\hbar$ from those of the classical Hamiltonian $H(x,\xi) = \frac{|\xi|^2}{2m} + V(x)$ and the flow Φ^t it generates on the phase space $T^*\mathbb{R}^d$.

    Weyl's law : count the eigenvalues of $H_\hbar$ in a fixed interval $[E_1, E_2]$, as $\hbar \to 0$:

$$N_\hbar([E_1,E_2]) = \frac{1}{(2\pi\hbar)^d}\,\mathrm{Vol}\left\{(x,\xi)\in T^*\mathbb{R}^d,\ \ H(x,\xi)\in[E_1,E_2]\right\} + o(\hbar^{-d})$$

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$H_\hbar = -\frac{\hbar^2\Delta}{2}$: back to the geometric setting, Φ^t = geodesic flow, $\lambda \sim \hbar^{-1}$.

Alternative to variational method : mollifying $N(\lambda)$

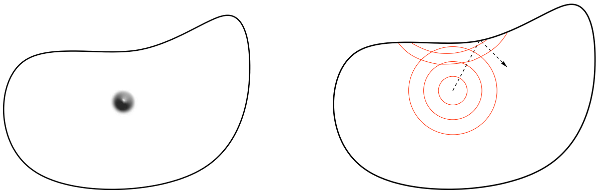
The spectral density can be expressed as a **trace** :

$$N(\lambda) = \operatorname{Tr} \Theta(\lambda - \sqrt{-\Delta})\,.$$

Easier to analyze operators given by *smooth* functions of Δ or $\sqrt{-\Delta}$

- resolvent $(z+\Delta)^{-1}$ defined for $z \in \mathbb{C} \setminus \mathbb{R}$ [CARLEMAN'34]
- heat semigroup $e^{t\Delta}$. Heat kernel $e^{t\Delta}(x,y)$ = diffusion of a Brownian particle. $\operatorname{Tr} e^{t\Delta} = \sum_n e^{-t\lambda_n^2}$ is a smoothing of $N(\lambda)$, with $\lambda \sim t^{-1/2}$ [MINAKSHISUNDARAM-PLEIJEL'52]
- Wave group $e^{-it\sqrt{-\Delta}}$, solves the wave equation. Propagates at unit speed.

Once one has a good control on the trace of either of these operators, get estimates on $N(\lambda)$ through some *Tauberian theorem*.



Using the wave equation - X without boundary

The **wave group** provides the most precise estimates for $R(\lambda)$ (Fourier transform is easily inverted)

Uncertainty principle : control $e^{-it\sqrt{-\Delta}}$ on a time scale $|t| \leq T \Longleftrightarrow$ control $D(\lambda)$ smoothed on a scale $\delta\lambda \sim \frac{1}{T}$.

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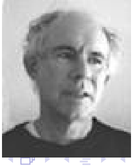
- [CHAZARAIN'73,DUISTERMAAT-GUILLEMIN'75] Tr $e^{-it\sqrt{-\Delta}}$ has singularities at $t = T_\gamma$ the lengths of closed geodesics $\rightsquigarrow R(\lambda) = o(\lambda^{d-1})$, provided the set of periodic points has measure zero.





Periodic orbits as oscillations of $D(\lambda)$

Around 1970, (some) physicists want to understand the oscillations of $D(\lambda)$.
Motivations : nuclear physics, semiconductors
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In the case of a classically chaotic system, the *Gutzwiller trace formula* relates quantum and classical informations :

$$D(\lambda) = \overline{D(\lambda)} + D^{fl}(\lambda) \overset{\lambda\to\infty}{\sim} \sum_{j\geq 0} A_{0,j}\lambda^{d-j} + \mathrm{Re} \sum_{\gamma \ per.\ geod.} e^{i\lambda T_\gamma} \sum_{j\geq 0} A_{\gamma,j}\lambda^{-j}$$

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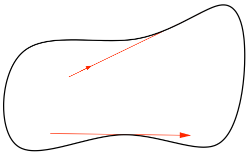
• 1 application : (X, g) negatively curved, lower bound for $R(\lambda)$, in terms of the full set of per. orbits [JAKOBSON-POLTEROVICH-TOTH'07]

Domains with (smooth) boundaries

Even if $\partial\Omega$ is smooth, describing the wave propagation in presence of boundaries is difficult, mostly due to gliding or grazing rays. Lots of progresses in the 1970s [MELROSE, SJÖSTRAND, TAYLOR]



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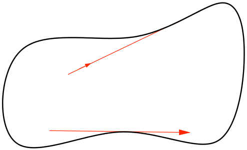

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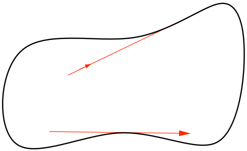

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[IVRII'80, MELROSE'80] : FINALLY, Weyl's 2-term asymptotics

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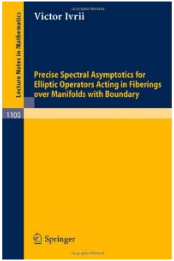

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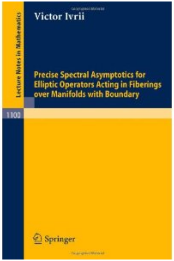

238 pages

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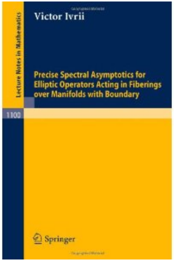

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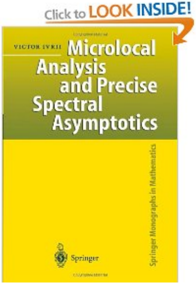

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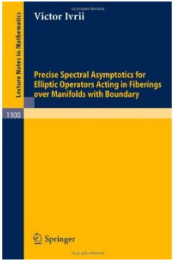

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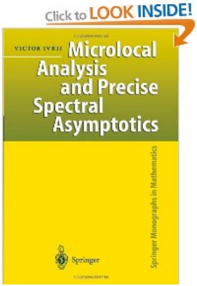

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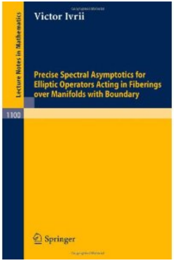

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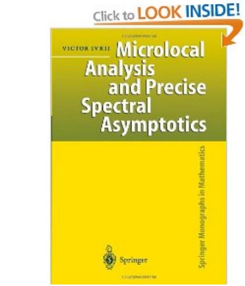

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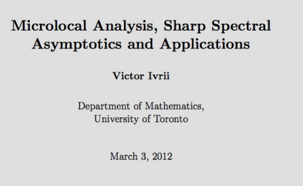
Microlocal Analysis, Sharp Spectral Asymptotics and Applications

Victor Ivrii

Department of Mathematics,
University of Toronto

March 3, 2012

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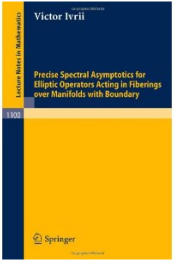

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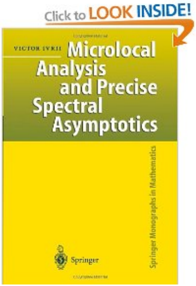

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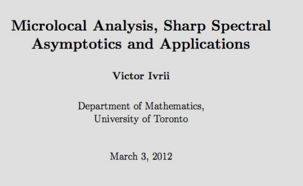

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so far, 2282 pages...

How did Victor manage to find these tricks ?

How did Victor manage to find these tricks ?

Working hard...

How did Victor manage to find these tricks ?

Working hard...

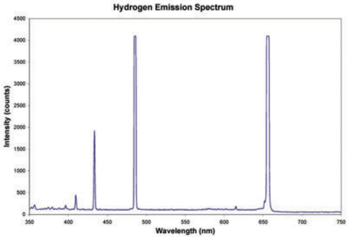
...in an inspiring environment

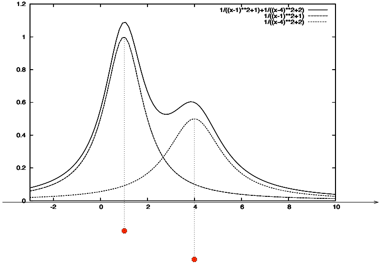
From eigenvalues to resonances

In quantum or wave physics, stationary modes are most often a mathematical idealization.

A system always interacts with its environment (measuring device, absorption, spontaneous emission. . .) $\rightsquigarrow$ each excited state has a finite lifetime τ_n.

$\Longrightarrow$ the spectrum is not a sum of δ peaks, but rather of Lorentzian peaks centered at $E_n \in \mathbb{R}$, of widths $\Gamma_n = \frac{1}{\tau_n}$ (decay rates).

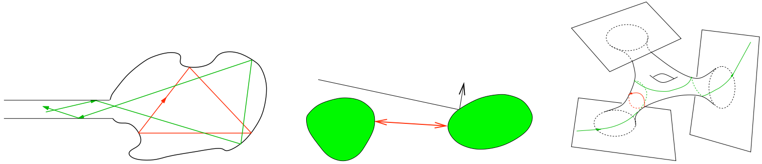




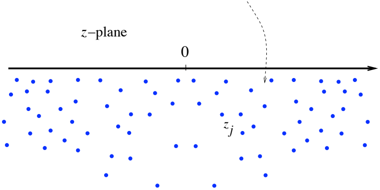
Each Lorentzian $\leftrightarrow$ a complex resonance $z_n = \lambda_n - i\Gamma_n$.

Clean mathematical setting : geometric scattering

Open cavity with waveguide.
Obstacle / potential $V \in C_c(\mathbb{R}^d)$ / (X, g) Euclidean near infinity

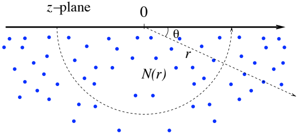


$-\Delta_\Omega$ (or $-\Delta + V$) has abs. cont. spectrum on $\mathbb{R}^+$. Yet, the cutoff resolvent $R_\chi(z) \stackrel{\text{def}}{=} \chi(\Delta + z^2)^{-1}\chi$ admits (in *odd* dimension) a meromorphic continuation from $\operatorname{Im} z > 0$ to $\mathbb{C}$, with isolated poles of finite multiplicities $\{z_n\}$, the resonances (or scattering poles) of Δ_X.



Counting (all) resonances

Can we estimate $N(r) \stackrel{\text{def}}{=} \#\{j\,;\,|z_j| \leq r\}$?
Cannot use selfadjoint methods (minimax)
$\Longrightarrow$ *Upper bounds* are much easier to obtain than lower bounds.



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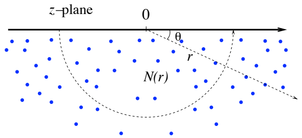

Main tool : **complex analysis**.
Construct an entire function $d(z)$ which vanishes at the resonances.
$d(z) = \det(I - K(z))$ with $K(z)$ holomorphic family of compact ops.

- Control the growth of $d(z)$ when $|z| \to \infty$ (count *singular values* of $K(z)$, use self-adjoint Weyl's law)

$\stackrel{Jensen}{\Longrightarrow} N(r) \leq C\, r^d, \quad r \to \infty$ [MELROSE,ZWORSKI,VODEV,SJÖSTRAND-ZWORSKI...]

Connection with a *volume* : $C = c_d a^d$ if Supp $V \subset B(0, a)$
[ZWORSKI'87,STEFANOV'06]

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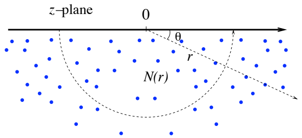

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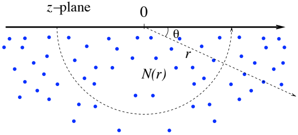

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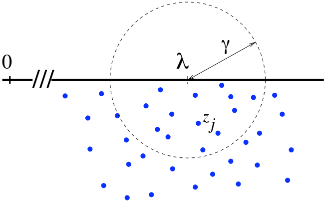
• [CHRISTIANSEN'10] Distribution in *angular sectors*, higher density near $\mathbb{R}$.
[SJÖSTRAND'12] Semiclassical setting, potential with a small random perturbation : Weyl law for resonances in a thin strip below $\mathbb{R}$.

Counting "long living" resonances

From a physics point of view, the resonances with $|\operatorname{Im} z| \gg 1$ are not very significant (very small lifetime) $\rightsquigarrow$ rather count resonances of bounded decay rates :
$N(\lambda, \gamma) \stackrel{\text{def}}{=} \#\{j\,;\, |z_j - \lambda| \le \gamma)\}$, $\gamma > 0$ fixed, $\lambda \to \infty$.



This counting gives informations on the classical dynamics on the trapped set

$$K \stackrel{\text{def}}{=} \left\{(x, \xi) \in S^*X\,;\, \Phi^t(x, \xi) \text{ uniformly bounded for all } t \in \mathbb{R}\right\}$$

(compact subset of S^*X, invariant through Φ^t).

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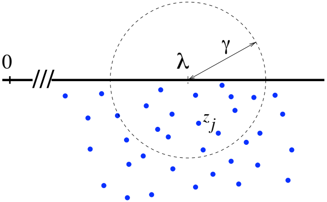

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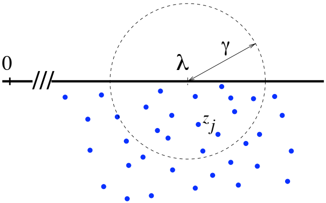

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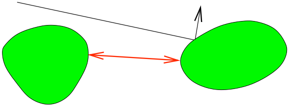
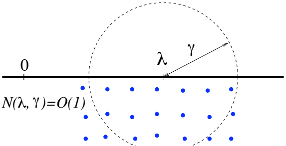
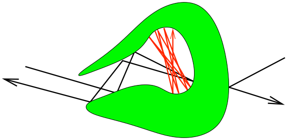
- $K = \emptyset \implies$ no long-living resonance
- K= a single hyperbolic orbit. Resonances form a (projected) deformed lattice, encoding the length and Lyapunov exponents of the orbit [IKAWA'85,GÉRARD'87]

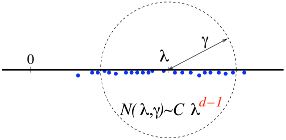




Counting "long living" resonances (2)

- K contains an elliptic periodic orbit $\Rightarrow$ many resonances with $\mathrm{Im}\, z = \mathcal{O}(\lambda^{-\infty}) \Longrightarrow N(\lambda, \gamma) \asymp \lambda^{d-1}$ [Popov, Vodev, Stefanov]





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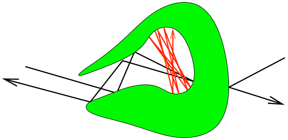
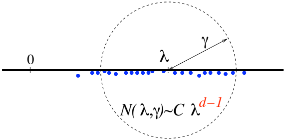
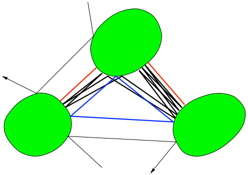
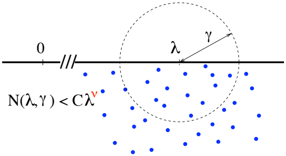




- K a fractal subset carrying a chaotic (hyperbolic) flow. *Quantum chaos*





Fractal Weyl upper bound
[SJÖSTRAND,SJÖSTRAND-ZWORSKI,N-SJÖSTRAND-ZWORSKI]

$$\forall \gamma > 0, \exists C_\gamma > 0, \quad N(\lambda, \gamma) \leq C_\gamma \, \lambda^{\nu}, \quad \lambda \to \infty,$$

where $\dim_{Mink}(K) = 2\nu + 1$ (so that $0 < \nu < d - 1$).

Fractal Weyl law ?

$$N(\lambda, \gamma) \leq C_\gamma \lambda^\nu, \quad \lambda \to \infty,$$

This bound also results from a volume estimate : count the number of quantum states "living" in the $\lambda^{-1/2}$-neighbourhood of K.

Fractal Weyl Law conjecture : this upper bound is sharp, at least at the level of the power ν.

Several numerical studies confirm the conjecture [LU-SRIDHAR-ZWORSKI,GUILLOPÉ-LIN-ZWORSKI,SCHOMERUS-TWORZYDŁO].

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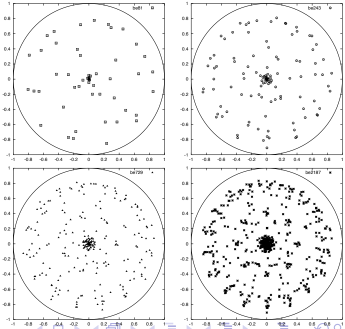
Only proved for a discrete-time toy model (*quantum baker's map*) [N-ZWORSKI]
A chaotic open map $B : \mathbb{T}^2 \to \mathbb{T}^2$ is quantized into a family $(B_N)_{N \in \mathbb{N}}$ of subunitary $N \times N$ matrices, where $N \equiv \hbar^{-1}$.

Fractal Weyl law in this context :
$\# \mathrm{Spec}(B_N) \cap \{e^{-\gamma} \leq |z| \leq 1\} \sim C_\gamma N^\nu$ as $N \to \infty$, where $\nu = \frac{\dim(\text{trapped set of } B)}{2} < 1$.



Fractal Weyl law galore

FWL for quantum maps $\rightsquigarrow$ search for FWL in certain families $(M_N)_{N\to\infty}$ of large matrices

Eur. Phys. J. B **75**, 299–304 (2010)
DOI: 10.1140/epjb/e2010-00144-0

THE EUROPEAN PHYSICAL JOURNAL B

Regular Article

Ulam method and fractal Weyl law for Perron-Frobenius operators

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Eur. Phys. J. B **79**, 115–120 (2011)
DOI: 10.1140/epjb/e2010-10774-7

THE EUROPEAN PHYSICAL JOURNAL B

Regular Article

Fractal Weyl law for Linux Kernel architecture

L. Ermann[1], A.D. Chepelianskii[2], and D.L. Shepelyansky[1,a]

[1] Laboratoire de Physique Théorique (IRSAMC), Université de Toulouse, UPS-CNRS, 31062 Toulouse, France
[2] LPS, Université Paris-Sud, CNRS, UMR8502, 91405 Orsay, France

PHYSICAL REVIEW E **81**, 056109 (2010)

Spectral properties of the Google matrix of the World Wide Web and other directed networks

Bertrand Georgeot, Olivier Giraud,* and Dima L. Shepelyansky
Laboratoire de Physique Théorique (IRSAMC), Université de Toulouse-UPS, F-31062 Toulouse, France
and LPT (IRSAMC), CNRS, F-31062 Toulouse, France
(Received 17 February 2010; published 25 May 2010)

Experimental studies on microwave billiards. [KUHL *et al.*'12]
Major difficulty : extract the ”true” resonances from the signal.