Locally compact quantum groups

1. The Haar weights

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Outline of the lecture series

Outline of the series:

- The Haar weights on a locally compact quantum group
- The antipode of a locally compact quantum group
- The main theory
- Duality
- Miscellaneous topics

Lecture notes are being written. The slides, together with some other related documents are available on this website: www.alfons-vandaele.be/fields2013

Outline of the first lecture

Outline of this first lecture:

- Introduction
- The modular theory of weights
- Locally compact quantum groups - The definitions
- From C*-algebras to von Neumann algebras
- Conclusions

Locally compact groups

Consider a locally compact group G. We can associate two C*-algebras:

- The abelian C*-algebra $C_0(G)$ of continuous complex functions on G tending to 0 at infinity.
- The reduced group C*-algebra $C_r^*(G)$.

Each of these carries a natural coproduct:

- For $f \in C_0(G)$ one defines $\Delta(f)(p,q) = f(pq)$ when $p,q \in G$.
- On $C_r^*(G)$ the coproduct is defined by $\Delta(\lambda_p) = \lambda_p \otimes \lambda_p$ where $p \mapsto \lambda_p$ is the canonical imbedding of the group in the multiplier algebra of $C_r^*(G)$.

We can also consider these two coproducts on the von Neumann algebras $L_\infty(G)$ and $VN(G)$.

Locally compact quantum groups

The two C*-algebras $C_0(G)$ and $C_r^*(G)$ are in duality with each other in the sense that the product on one component yields the coproduct on the other one. If the group G is abelian, then the Fourier transform provides a natural isomorphism between $C_r^*(G)$ and $C_0(\widehat{G})$ where $\widehat{G}$ is the Pontryagin dual of G.

This kind of symmetry breaks down in the non-abelian case.

Locally compact quantum groups are introduced to restore this symmetry.The pair $(C_0(G), \Delta)$ is replaced by (A, Δ) where now A is any C*-algebra with a coproduct Δ and $(C_r^*(G), \Delta)$ is replaced by the dual $(\widehat{A}, \widehat{\Delta})$ which is again of the same type.

The terminology comes from physics.The passage from the abelian C*-algebra $C_0(G)$ to a general, possibly non-abelian C*-algebra is like quantization.

Locally compact quantum groups

Some historical notes

- The theory of locally compact quantum groups is a theory developed within the field of operator algebras.
- The main purpose was the generalization of Pontryagin's duality to non-abelian locally compact groups.
- There have been various attempts in a period of several decades.
- A first breakthrough came with the theory of Kac algebras (Kac and Vainerman, Enock and Schwartz).
- New developments in the theory of quantum groups led to new challenges.
- This resulted in the present theory (with contributions by various people: Baaj and Skandalis; Masuda and Nakagami, Woronowicz; Kustermans and Vaes).

Weights on C*-algebras

Definition

Let A be a C*-algebra. A weight on a C*-algebra is a map $\varphi : A^+ \to [0, \infty]$ satisfying

- $\varphi(\lambda a) = \lambda\varphi(a)$ for all $a \in A^+$ and $\lambda \in \mathbb{R}$ with $\lambda \geq 0$,
- $\varphi(a + b) = \varphi(a) + \varphi(b)$ for all $a, b \in A^+$.

Given such a way we have two associated sets.

Notation

We denote

$$\mathfrak{N}_\varphi = \{a \in A \mid \varphi(a^*a) < \infty\}$$

and set $\mathfrak{M}_\varphi = \mathfrak{N}_\varphi^*\mathfrak{N}_\varphi$, *the subspace of* A *spanned by elements of the form* a^*b *with* $a, b \in \mathfrak{N}_\varphi$.

Clearly $\mathfrak{N}_\varphi$ is a left ideal and $\mathfrak{M}_\varphi$ is a *-subalgebra.

The GNS representation

Proposition

Assume that the weight is lower semi-continuous. Let $\mathcal{F}$ be the set of positive linear functionals ω on A, majorized by φ on A^+. Then for all positive elements a we have:

$$\varphi(a) = \sup_{\omega \in \mathcal{F}} \omega(a)$$

Assume further that the weight φ is semi-finite, lower semi-continuous and faithful.

Proposition

There is a non-degenerate representation π_φ on a Hilbert space $\mathcal{H}_\varphi$ and a linear map $\Lambda_\varphi : \mathfrak{N}_\varphi \to \mathcal{H}$ such that

$$\varphi(b^* a) = \langle \Lambda_\varphi(a), \Lambda_\varphi(b) \rangle$$

and such that $\Lambda_\varphi(ab) = \pi_\varphi(a)\Lambda_\varphi(b)$.

The GNS representation

Proposition

For all ω majorized by φ there is a vector ξ in $\mathcal{H}$ and a bounded operator $\pi'(\xi)$ in the commutant $\pi_\varphi(A)'$ satisfying $\pi'(\xi)\Lambda_\varphi(a) = \pi_\varphi(a)\xi$ for $a \in \mathfrak{N}_\varphi$ and

$$\omega(a) = \langle \pi_\varphi(a)\xi, \xi \rangle$$

for all $a \in A$.

Such a vector is called right bounded.

Proposition

Consider the closure of the subspace of $\mathcal{H}_\varphi$ spanned by the right bounded vectors. This space is invariant under the commutant $\pi_\varphi(A)'$. Denote by p the projection onto this space. Then $p \in \pi_\varphi(A)''$.

Extension of weights

Denote by $\widetilde{A}$ the double dual A^{**} of A. Any element $\omega \in A^*$ has a unique normal extenstion $\widetilde{\omega}$ to $\widetilde{A}$.

Proposition

Let φ be a lower semi-continuous semi-finite weight on A. Define $\widetilde{\varphi} : \widetilde{A}^+ \to [0, \infty]$ by

$$\widetilde{\varphi}(x) = \sup_{\omega \in \mathcal{F}} \omega(x).$$

Then $\widetilde{\varphi}$ is a normal, semi-finite weight on $\widetilde{A}$.

A weight on a von Neumann algebra is defined as a weight on a C*-algebra. It is said to be normal if $\widetilde{\varphi}(x) = \sup \widetilde{\varphi}(x_\alpha)$ for any increasing net $x_\alpha \to x$.

Central weights on C*-algebras

Notation

- *We denote by* e *the support of* $\widetilde{\varphi}$. *It is the smallest projection in* $\widetilde{A}$ *such that* $\widetilde{\varphi}(1-e)=0$.
- *Consider also the unique normal extension of* π_φ *to* $\widetilde{A}$. *It is a normal* **-homomorphism from* $\widetilde{A}$ *onto* $\pi_\varphi(A)''$. *Denote by* f *the support projection of this extension. It belongs to the center of* $\widetilde{A}$

Proposition

- $\pi_\varphi(e)=p$
- f *is the central support of* e *in* $\widetilde{A}$.

Definition

The weight φ is called central if its support e is central in $\widetilde{A}$.

Modular theory for central weights

Assume that φ is a central, faithful, lower semi-continuous, semi-finite weight on the C*-algebra A.

Proposition

Let $\mathfrak{A}$ be $\Lambda_\varphi(\mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi)$. It is a *-algebra with multiplication and involution inherited from $\mathfrak{N}_\varphi^* \cap \mathfrak{N}_\varphi$. The involution is denoted as $\xi \mapsto \xi^\sharp$. Considered with its scalar product from $\mathcal{H}_\varphi$, it is a left Hilbert algebra with associated left von Neumann algebra $\pi_\varphi(A)''$.

The main point is the preclosedness of the $^\sharp$-operation. The argument is based on

$$\langle \pi'(\xi_1)^*\xi_2, \Lambda_\varphi(a)\rangle = \langle \Lambda_\varphi(a^*), \pi'(\xi_2)^*\xi_1\rangle$$

for any pair of right bounded elements ξ_1, ξ_2. Vectors of the form $\pi'(\xi_1)^*\xi_2$ span a dense subspace of $\mathcal{H}_\varphi$ because the weight is assumed to be central.

Left Hilbert algebras

Definition

Let $\mathfrak{A}$ be a *-algebra equipped with an inner product. The involution is denoted as $\xi \mapsto \xi^\sharp$. It is assumed that:

- for each ξ, the map $\eta \mapsto \xi\eta$ is bounded,
- for each ξ, η, ζ we have $\langle \xi\eta, \zeta \rangle = \langle \eta, \xi^\sharp \zeta \rangle$,
- the map $\xi \mapsto \xi^\sharp$ is preclosed as a conjugate linear map on the Hilbert space,
- the subspace $\mathfrak{A}^2$ is dense in $\mathfrak{A}$.

Then $\mathfrak{A}$ is called a left Hilbert algebra.

We use $\mathcal{H}$ for the Hilbert space completion of $\mathfrak{A}$ and $\pi(\xi)$ for the operator on this Hilbert space extending left multiplication by ξ. Clearly π is a non-degenerate *-representation of $\mathfrak{A}$ by bounded operators. The von Neumann algebra $\pi(\mathfrak{A})''$ is called the left von Neumann algebra of $\mathfrak{A}$.

Full left Hilbert algebras

Start with a given Hilbert algebra $\mathfrak{A}$. Denote by S the closure of the $^\sharp$-operation and by F its adjoint. Define $\mathfrak{A}'$ as the space of right bounded elements ξ in $\mathcal{H}$, that are in the domain of F.

Proposition

$\mathfrak{A}'$ is a right Hilbert algebra if we define $\xi_2\xi_1 = \pi'(\xi_1)\xi_2$ and $\xi^\flat = F\xi$. Its Hilbert space is again $\mathcal{H}$ and its right von Neumann algebra $\pi'(\mathfrak{A}')''$ is the commutant $\pi(\mathfrak{A})'$.

We can now define left bounded elements η by the assumption that there is a bounded operator $\pi(\eta)$ satisfying $\pi(\eta)\xi = \pi'(\xi)\eta$ for all $\xi \in \mathfrak{A}'$. And we can define $\mathfrak{A}''$ as the space of left bounded elements η that are in the domain of S. This will give again a left Hilbert algebra. It contains the original one $\mathfrak{A}$ and it is full (or achieved) in the sense that repeating the procedure gives nothing new.

The relation with the extension of weights

We have seen that a central weight on a C*-algebra gives rise to a left Hilbert algebra. There is also a converse.

Theorem

Assume that $\mathfrak{A}$ is an full left Hilbert algebra with left von Neumann algebra M. Define $\varphi : M^+ \to [0,\infty]$ by

$$\varphi(x^*x) = \|\eta\|^2 \qquad \text{if } \eta \in \mathfrak{A} \text{ and } x = \pi(\eta).$$

Define $\varphi(x^*x) = \infty$ if x is not of that form. Then φ is a faithful, normal semi-finite weight on the von Neumann algebra M.

If we start with a central weight on a C*-algebra, associate the left Hilbert algebra $\mathfrak{A}$, then the restriction of $\widetilde{\varphi}$ to $\pi_\varphi(A)''$ is precisely the weight associated as above to the full left Hilbert algebra $\mathfrak{A}''$.

Locally compact quantum groups - C*-setting

Definition

Let A be a C*-algebra with a coproduct $\Delta : A \to M(A \otimes A)$. Assume that elements of the form

$$(\omega \otimes \iota)\Delta(a) \qquad \text{and} \qquad (\iota \otimes \omega)\Delta(a)$$

each span a dense subspace of A. Assume that there is a left Haar weight φ and a right Haar weight ψ on (A, Δ). Then this pair is called a locally compact quantum group.

A coproduct Δ on A is a non-degenerate *-homomorphism from A to $M(A \otimes A)$ where the minimal tensor product is taken. It is assumed to be coassociative: $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.
A left Haar weight is a proper central weight φ that is left invariant:

$$\varphi((\omega \otimes \iota)\Delta(a)) = \omega(1)\varphi(a)$$

for all positive ω in A^* and all $a \in A^+$ such that $\varphi(a) < \infty$.

Extension to $\widetilde{A}$

Proposition

The coproduct extends to a unital and normal *-homomorphism $\widetilde{\Delta} : \widetilde{A} \to \widetilde{A} \otimes \widetilde{A}$ (where now we consider the von Neumann tensor product). The normal extensions $\widetilde{\varphi}$ and $\widetilde{\psi}$ are still left, respectively right invariant weights on $(\widetilde{A}, \widetilde{\Delta})$.

The extension of the coproduct is standard. We have a unital imbedding of $M(A \otimes A)$ in $\widetilde{A} \otimes \widetilde{A}$ and hence a unique normal extension $\widetilde{\Delta} : \widetilde{A} \to \widetilde{A} \otimes \widetilde{A}$. Coassociativity on this level is a consequence of the uniqueness of normal extensions.
The left invariance of the extension $\widetilde{\varphi}$ is less trivial. The modular theory of weights is used as the main tool. Similarly for the right invariance of $\widetilde{\psi}$.
Important remark: The extended weights $\widetilde{\varphi}$ and $\widetilde{\psi}$ are (in general) no longer faithful.

Uniqueness of the supports

Let e and f be the supports of $\widetilde{\varphi}$ and $\widetilde{\psi}$. They are central projections by assumption.

Proposition

The support projections e and f are equal. Hence they are independent of the choices of the weights φ and ψ on (A, Δ).

Proof (sketch): From the left invariance of $\widetilde{\varphi}$, it follows that $\widetilde{\Delta}(1-e)(1 \otimes (1-e)) = 0$. Multiply with $\widetilde{\Delta}(x)$ where $x \in \widetilde{A}^+$ and $\widetilde{\psi}(x) < \infty$. Then apply right invariance of $\widetilde{\psi}$ to arrive at $\widetilde{\psi}(x(1-e)) = 0$. This implies that $f(1-e) = 0$. Similarly we find $e(1-f) = 0$ and so $e = f$.

Remark

It is this result that makes it possible to pass, from the very beginning, to the study of locally compact quantum groups in the von Neumann algebra framework.

Restriction to the von Neumann algebra $\pi_\varphi(A)''$

Now we restrict the coproduct and the Haar measures again.

Theorem

Denote $M = \widetilde{A}e$ and define $\Delta_0 = M \to M \otimes M$ by $\Delta_0(x) = \widetilde{\Delta}(x)(e \otimes e)$. Then Δ_0 is a coproduct on M. The restrictions φ_0 and ψ_0 of the weights $\widetilde{\varphi}$ and $\widetilde{\psi}$ are now faithful invariant functionals. Hence, the pair (M, Δ_0) is a locally compact quantum group in the von Neumann algebraic setting.

Most of this is now straightforward. One needs the equality $\widetilde{\Delta}(e)(1 \otimes e) = (1 \otimes e)$ to obtain that Δ_0 is still unital.
Also observe that M is isomorphic with $\pi_\varphi(A)''$ because we cut down to the support of π_φ in $\widetilde{A}$.

Conclusions

We conclude:

- The starting point is a pair (A, Δ) of a C*-algebra A with a coproduct Δ satisfying certain density conditions.
- It is assumed that there are a left and a right Haar weight and that these weights are central.
- Then the coproduct and the weights are extended to the double dual von Neumann algebra $\widetilde{A}$ and it is shown that the extensions of the weights are still invariant.
- It turns out that there is a unique central projection $e \in \widetilde{A}$ that is the common support of all these invariant weights.
- Then there is a unique von Neumann algebra $M = \widetilde{A}e$, the coproduct restricts to this von Neumann algebra and the weights restrict to faithful invariant weights.
- The result is a unique locally compact von Neumann algebraic quantum group associated with the C*-algebraic locally compact quantum group.

References

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