Noncommutative geometry and time-frequency analysis

Franz Luef

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Noncommutative geometry

Hermitian structure:

Let $\mathcal{A}$ be a unital C^*-algebra. Then a vector space V is a **left Hilbert $\mathcal{A}$-module**, i.e. $(A,g) \mapsto A \cdot g$ is a map from $V \times \mathcal{A} \to \mathcal{A}$, with a pairing ${}_{\mathcal{A}}\langle .,. \rangle$ such that for all $f,g,h \in V$:

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- Suppose compatible connections ∇_1, ∇_2 are given on V and ∂_1, ∂_2 are derivations on $\mathcal{A}$. Then a **gauge connection** on V is defined by these two compatible connections and its **curvature** is defined by

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Then the **twisted group algebra** $\ell^1(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$ is $\ell^1(\alpha\mathbb{Z} \times \beta\mathbb{Z})$ with twisted convolution $\natural$ as multiplication and $*$ as involution.

- **Twisted convolution** of **a** and **b** is defined by

$$\mathbf{a}\natural\mathbf{b}(k,l) = \sum_{m,n} a(m,n)b(k-m,l-n)c(m,n)),(k-m,l-n))$$

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Spectrally invariant subalgebras of noncommutative tori

$$\mathcal{A}^1_s(\alpha\mathbb{Z}\times\beta\mathbb{Z},c)=\{A=\sum_\lambda a(k,l)\pi(\alpha k,\beta l):\|\mathbf{a}\|_{\ell^1_s}<\infty\}$$

with $\mathbf{a}\|_{\ell^1_s}=\sum_{k,l}|a(k,l)|(1+|k|^2+|l^2|)^{s/2}$

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Theorem:

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The result about the smooth noncommutative torus is due to Connes and the one for the twisted group algebra was proved by Gröchenig-Leinert and later by Rosenberg.

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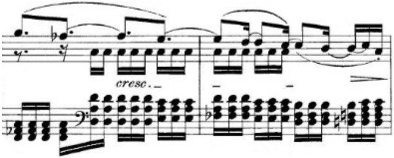

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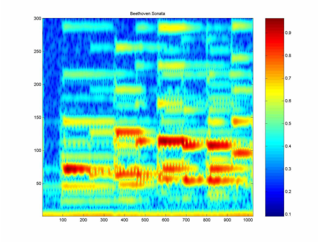

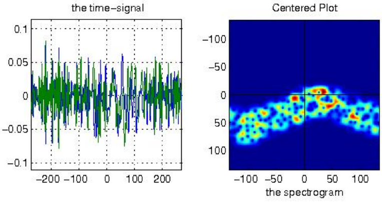

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Wireless communication – OFDM

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Basic Idea:

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- Transmitter: Assuming N suchannels, a bandwidth of W Hz, symbol length of a_T seconds, and subchannel separation $b_F := W/N$, the transmitter of a general OFDM system uses the following waveforms $g_l(t) = g(t)e^{2\pi i l b_F t}$ for $l = 0, ..., N-1$.
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- OFDM – the functions $\{h_{k,l}\}$ are designed to be such that $g_{kl} = h_{kl}$.
- If the two systems $\{g_{k,l}\}$ and $\{h_{k,l}\}$ are biorthogonal, then one refers to it as **biorthogonal frequency division multiplexing (BFDM)**.

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- Transmitter sends a superposition of individual symbols:

$$s(t) = \sum_{k,l \in \mathbb{Z}} c_{kl} g_l(t - ka_T) = \sum_{k,l \in \mathbb{Z}} c_{kl} e^{2\pi i t l b_F} g(t - ka_T)$$

- The OFDM receiver is of a similar structure as the transmitter waveforms $g_{kl}(t) = e^{2\pi i t l b_F t} g(t - ka_T)$, i.e. for some function h: $h_{kl}(t) = e^{2\pi i t l b_F t} h(t - ka_T)$.
- OFDM – the functions $\{h_{k,l}\}$ are designed to be such that $g_{kl} = h_{kl}$.
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Time-frequency analysis – Schrödinger representation

- **translation** $T_x f(t) = f(t - x)$ for $x \in \mathbb{R}$, **modulation** $M_\omega f(t) = e^{2\pi i t \cdot \omega} f(t)$ for $\omega \in \widehat{\mathbb{R}}$
- **time-frequency shift** $\pi(x, \omega) f(t) = M_\omega T_x f(t)$ for $(x, \omega) \in \mathbb{R} \times \widehat{\mathbb{R}}$

$$M_\omega T_x = e^{2\pi i x \cdot \omega} T_x M_\omega$$
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Modulation spaces:

Suppose g is a Schwartz function. Then $f \in \mathcal{S}'(\mathbb{R})$ is in the **modulation space** $M_s^{p,q}(\mathbb{R})$ if

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Feichtinger's algebra, Shubin class:

$M^1_s(\mathbb{R})$ is the space of all $f \in L^2(\mathbb{R})$:

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Resolution of Identity

Matrix coefficients of the Schrödinger representation satisfy an orthogonality relation:

Moyal's Identity

For f, g, h, k are in $L^2(\mathbb{R})$ we have

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Let $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z}) = \{\pi(\alpha k, \beta l)g : k, l \in \mathbb{Z}\}$ be a **Gabor system**.

- **analysis operator**: $C_g f = (\langle f, \pi(\alpha k, \beta l)g \rangle)_{k,l\in\mathbb{Z}}$
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- **frame operator**:
  $S_{g,\Lambda} f = \sum_{k,l\in\mathbb{Z}} \langle f, \pi(\alpha k, \beta l)g \rangle \pi(\alpha k, \beta la)g$
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Discrete reconstruction

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\begin{aligned}
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Poisson summation formula:

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Suppose $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a frame for $L^2(\mathbb{R})$.

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Left action of $\mathcal{A}_s^1(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$ on $M_s^1(\mathbb{R}^d)$ by

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Projective modules over noncommutative tori

Theorem:

$M^1_s(\mathbb{R})$ is a finitely generated projective right $\mathcal{A}^1_s(\alpha^{-1}\mathbb{Z}, \beta^{-1}\mathbb{Z}, \overline{c})$.

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A Gabor frame $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ ia a line bundle over the twisted group algebra $\mathcal{A}^1(\alpha^{-1}\mathbb{Z}, \beta^{-1}\mathbb{Z}, \overline{c})$.

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Frames for Hilbert C^*-modules

- Let $\mathcal{A}$ be a unital C^*-algebra. A sequence $\{g_j : j = 1, ..., n\}$ in a (left) Hilbert $\mathcal{A}$-module ${}_{\mathcal{A}}V$ is called a **standard module frame** if there are positive reals C, D such that

$$C\,{}_{\mathcal{A}}\langle f, f\rangle \leq \sum_{j=1}^{n} {}_{\mathcal{A}}\langle f, g_j\rangle {}_{\mathcal{A}}\langle g_j, f\rangle \leq D\,{}_{\mathcal{A}}\langle f, f\rangle$$

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Existence of multi-window Gabor frames

Theorem:

Then there exist $g_1, \ldots, g_n$ in $M^1_s(\mathbb{R})$ such that for all f in $L^2(\mathbb{R})$

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$\mathcal{G}(g_1, \ldots, g_n, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a **multi-window Gabor frame** for $L^2(\mathbb{R})$).

By a result of Feichtinger and Gröchenig this implies that $\mathcal{G}(g_1, \ldots, g_n, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a multi-window Gabor frame for the class of modulation spaces $M^{p,q}_m(\mathbb{R})$.

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- $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ is a Gabor frame if and only if $M^1_s(\mathbb{R})$ is a singly-generated projective right $\mathcal{A}^1_s(\Lambda^\circ, \overline{c})$-module.

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In the discussion of time-frequency channels we considered operators of the form

$$A = \iint_{\mathbb{R}^2} a(z)\pi(z)dz$$

for $a \in L^1(\mathbb{R}^2)$, i.e. integrated representation of the Schrödinger representation.

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Stone-von Neumann:

$\mathcal{A}^\infty(\mathbb{R}^2, c)$ is Morita equivalent to the complex numbers $\mathbb{C}$.

Define a Hermitian structure on $\mathcal{A}^\infty(\mathbb{R}^2, c)$ via

$$_{\mathbb{R}^2}\langle f, g\rangle = \iint_{\mathbb{R}^2} \langle f, \pi(z)g\rangle \pi(z) dz$$

and as left action:

$$_{\mathbb{R}^2}\langle A \cdot f, g\rangle = \iint_{\mathbb{R}^2} (a\natural V_g f)(z)\pi(z) dz$$

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Derivations ∂_1 and ∂_2 on $\mathcal{A}^\infty(\mathbb{R}^2, c)$:

$$\partial_1 A = 2\pi i \iint_{\mathbb{R}^2} x a(x,\omega)\pi(x,\omega)dxd\omega$$

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compatibility condition

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For example,

$$2\pi i\omega V_g f(x,\omega) = V_g f'(x,\omega) + V_{g'} f(x,\omega)$$

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derivations on the noncommutative torus $C^*(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$:

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$$\nabla_1^\circ g(t) = 2\pi i \beta^{-1} t g(t) \quad \text{and} \quad \nabla_2^\circ g(t) = \alpha^{-1} g'(t)$$

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Let's consider the case $A = {}_{\Lambda}\langle f, g\rangle$ in more detail:

- $$\nabla_i({}_{\Lambda}\langle f, g\rangle \cdot h) = \delta_i({}_{\Lambda}\langle f, g\rangle) \cdot h + {}_{\Lambda}\langle f, g\rangle \cdot \nabla_i h$$
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  shows that the analysis and synthesis steps need to have the same regularity.

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  shows that the analysis and synthesis steps need to have the same regularity.
- Suppose $\langle g, h\rangle_{\Lambda^\circ} = I$, i.e. $\mathcal{G}(g, \Lambda)$ is a frame.
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Let $\mathcal{G}(g, \alpha\mathbb{Z} \times \beta\mathbb{Z})$ be a Riesz basis for its closed span $\mathcal{H}$ in $L^2(\mathbb{R})$. Then $\nabla_i g$ or $\nabla_i h$ is not in $\mathcal{H}$, where h denotes the canonical dual Gabor atom $h = S^{-1}_{g,g}$.

Proof is based on an observation of G. Battle, which uses the left Leibniz property for $A = \pi(\alpha k, \beta l)$ implies:

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$$\begin{aligned}
\langle \nabla_1 g, \nabla_2 h\rangle &= \langle \sum_{k,l} \langle \nabla_1 g, \pi(\alpha k, \beta l)h\rangle \pi(\alpha k, \beta l) g, \nabla_2 h\rangle \\
&= \sum_{k,l} \langle \pi(-\alpha k, -\beta l) g, \nabla_1 h\rangle \langle \nabla_2 g, \pi(-\alpha k, -\beta l) h\rangle \\
&= \langle \nabla_2 g, \sum_{k,l} \langle \nabla_1 h, \pi(\alpha k, \beta l) g\rangle \pi(\alpha k, \beta l) h\rangle \\
&= \langle \nabla_2 g, \nabla_1 h\rangle
\end{aligned}$$

However, $\nabla_1\nabla_2 - \nabla_2\nabla_1 = 2\pi i I$, canonical commutation relations, gives

$$1 = \langle g, h\rangle = \langle \nabla_2 g, \nabla_1 h\rangle - \langle \nabla_1 g, \nabla_2 h\rangle = 0.$$

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Let g be in ${}_\Lambda V_{\Lambda^\circ}$. Then $P_g := {}_\Lambda\langle g, g\rangle$ is a projection in $C^*(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$ if and only if $g\langle g, g\rangle_{\Lambda^\circ} = g$. If $g \in M^1_s(\mathbb{R})$ or $\mathscr{S}(\mathbb{R})$, then P_g gives a projection in $\mathcal{A}^1_s(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$ or $\mathcal{A}^\infty(\alpha\mathbb{Z} \times \beta\mathbb{Z}, c)$, respectively.

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