KK-lifting Problem and Order Structures on K-groups

Li Zhiqiang

University of Toronto

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joint with George A. Elliott

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- To determine which KK-elements can be realized by a $*$-homomorphism makes sense at its own in KK-theory.
- Such a lifting problem is closely related to the classification of C*-algebras: when the approximate (asymptotic) unitary equivalence classes of homomorphisms are determined by their induced KK-classes, for the corresponding existence theorem, we need to lift a KK-class to a homomorphism.

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- To look for criterion for KK-lifting.
- For classification, try to connect the criterion to an invariant of C*-algebras, i.e., an order structure on the K-groups.

Cuntz's picture of KK-groups:

Definition

For two C*-algebras A and B, define $KK(A, B)$ to be the homotopy classes of quasi-homomorphisms from A to B, where a quasi-homomorphism is a pair of homomorphisms $\phi_\pm : A \to M(B \otimes \mathcal{K})$ with $\phi_+(a) - \phi_-(a) \in B \otimes \mathcal{K}$.

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Theorem

Let A be a C^*-algebra in the Bootstrap class, and B be a separable C^*-algebra, then the following sequence is exact:

$$0 \to Ext^1_{\mathbb{Z}}(K_*(A), K_{*+1}(B)) \to KK(A, B) \to Hom(K_*(A), K_*(B)) \to 0.$$

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  from UCT, the KK-group of two such algebras A and B is just $\mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(B))$. The order structure is the usual one induced by projections.

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- Finite dimensional C*-algebras, Interval algebras: from UCT, the KK-group of two such algebras A and B is just $\mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(B))$. The order structure is the usual one induced by projections.
- Circle algebras: from UCT, the KK-group of two such algebras A and B is $\mathrm{Hom}(\mathrm{K}_0(A), \mathrm{K}_0(B)) \oplus \mathrm{Hom}(\mathrm{K}_1(A), \mathrm{K}_1(B))$. Elliott introduced an order structure on $\mathrm{K}_* = \mathrm{K}_0 \oplus \mathrm{K}_1$. There is an alternative picture due to Dadarlat and Nemethi: $\mathrm{K}_*^+(A) := \{([\varphi(1)], [\varphi(e^{2\pi it})]) \mid \varphi$ is a homomorphism from $\mathrm{C}(S^1)$ to $\mathrm{M}_k(A)$ for some $k\}$.

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Dimension drop algebra I_n and $\widetilde{I_n}$,

$$I_n = \{f \in \mathrm{C}([0,1], \mathrm{M}_n) \mid f(0) = 0, f(1) = \lambda 1, \lambda \in \mathbb{C}\}.$$

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For UCT, this time, we have a nontrivial Ext. part, so situation is not as same as before.
Indeed,

$$\mathrm{K}_*^+(\widetilde{I_n}) = \{(a, \bar{b}) \mid a \geq 1\} \cup \{(0,0)\}.$$

Then any multiple (not exceeding n) of the KK-element $[\delta_1] - [\delta_0]$ preserves this order, but can not be lifted to a homomorphism.

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Definition

$K_0(A;\mathbb{Z}/p\mathbb{Z}) := K_1(A \otimes P) = KK(P, A)$ for any C*-algebra P in the Bootstrap class such that $K_0(P) = 0$ and $K_1(P) = \mathbb{Z}/p\mathbb{Z}$. $K_0(A;\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}) := KK(\widetilde{P}, A)$. We can choose P to be I_p.

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Definition

The order structure is defined as follows:

$$K_0^+(A;\mathbb{Z}\oplus\mathbb{Z}/p\mathbb{Z}) := \{([\varphi(1)],[\varphi|_{I_p}])\,|\,\varphi\in Hom(\widetilde{I_p},M_k(A))\}.$$

Lemma

There is a natural short exact sequence of groups:

$$K_0(A) \stackrel{\times p}{\rightarrow} K_0(A) \stackrel{\mu_{A;p}}{\rightarrow} K_0(A;\mathbb{Z}_p) \stackrel{\nu_{A;p}}{\rightarrow} K_1(A) \stackrel{\times p}{\rightarrow} K_1(A).$$

where $p \geq 2$, $\mu_{A;p}, \nu_{A;p}$ are the Bockstein operations defined by the Kasparov product with the element of $KK(I_p, \mathbb{C})$ given by the evaluation $\delta_1 : I_p \to \mathbb{C}$ and the element of $KK^1(\mathbb{C}, I_p)$ given by the inclusion $i : SM_p \to I_p$ respectively.

Lemma

Given a KK-element $\alpha \in KK(A, B)$, it induces the following commutative diagram:

$$\begin{array}{ccccccccc} K_0(A) & \xrightarrow{\times p} & K_0(A) & \xrightarrow{\mu_{A;p}} & K_0(A;\mathbb{Z}_p) & \xrightarrow{\nu_{A;p}} & K_1(A) & \xrightarrow{\times p} & K_1(A) \\ & & \Big\downarrow{K_0(\alpha)} & & \Big\downarrow{K_0(\alpha;\mathbb{Z}_p)} & & \Big\downarrow{K_1(\alpha)} & & \\ K_0(B) & \xrightarrow{\times p} & K_0(B) & \xrightarrow{\mu_{B;p}} & K_0(B;\mathbb{Z}_p) & \xrightarrow{\nu_{B;p}} & K_1(B) & \xrightarrow{\times p} & K_1(B) \end{array}.$$

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Theorem

Given natural numbers n, m, and p, with n dividing p, if $\alpha \in KK(\widetilde{I_n}, \widetilde{I_m})$ induces a positive homomorphism on the K-groups with coefficients above, then α can be lifted to a homomorphism.

Generalized dimension drop interval algebras

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$$\mathbf{I}[\mathrm{m}_0, \mathrm{m}, \mathrm{m}_1] = \{f \in \mathrm{C}([0,1], \mathrm{M}_m) : f(0) = a_0 \otimes \mathrm{id}_{\frac{\mathrm{m}}{\mathrm{m}_0}}, f(1) = \mathrm{id}_{\frac{\mathrm{m}}{\mathrm{m}_1}} \otimes a_1\},$$

where a_0 and a_1 belong to $\mathrm{M}_{\mathrm{m}_0}(\mathbb{C})$ and $\mathrm{M}_{\mathrm{m}_1}(\mathbb{C})$ respectively, and m_0, m_1 divide m.

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Answer: No.

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The natural map $\eta : KK(A, B) \to Hom(K^0(B), K^0(A))$ is an isomorphism, where η is the Kasparov product with K-homology elements.

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A KK-element α *can be lifted to a homomorphism if and only if* $\eta(\alpha)$ *is positive on the K-homology groups, where the positive cone is the subset of all K-homology classes of finite dimensional representations.*

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However, K-homology is not a direct invariant for classification of C*-algebras.

Let us assume $(m_0, m_1) = 1$, we look at the K-theory with coefficients for $A_m = \mathbf{I}[m_0, m, m_1]$.

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Theorem

For a natural number $p \geq 2$ with m dividing p, denote $\mathbb{Z} \oplus \mathbb{Z}_p$ by G_p, then we have $K_0(A_m; G_p) = \mathbb{Z} \oplus Z(m, p)$, where $Z(m, p) = \{(\bar{b}, \bar{c}) \in \mathbb{Z}_p \oplus \mathbb{Z}_p \mid \frac{m}{m_1}c - \frac{m}{m_0}b \in p\mathbb{Z}\}$.

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$$K_0^+(A_m; G_p) = \{(a, \bar{b}, \bar{c}) \in \mathbb{Z} \oplus Z(m,p) \mid a m_0 \geq \bar{b}, a m_1 \geq \bar{c}\}.$$

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$$K_0^+(A_m; G_p) = \{(a, \bar{b}, \bar{c}) \in \mathbb{Z} \oplus Z(m,p) \mid am_0 \geq \bar{b}, am_1 \geq \bar{c}\}.$$

The Bockstein operations are given by

$$\mu_{A_m;p} = \begin{pmatrix} m_0 \\ m_1 \end{pmatrix}, \text{ and } \nu_{A_m;p} = (-\frac{m}{pm_0}, \frac{m}{pm_1}).$$

Moreover, these K-theoretic data have the following generators δ_0, δ_1, id, and $\overline{id}$. We even have:

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Lemma

Given a generalized dimension drop interval algebra A_m, and a positive integer p with $m|p$, then each element of the Dadarlat-Loring positive cone of $K_0(A_m; \mathbb{Z} \oplus \mathbb{Z}_p)$ can be written as a linear combination of $[\delta_0], [\delta_1], [id]$, and $[\overline{id}]$ with non-negative integer coefficients.

Given $A_m = \mathbf{I}[m_0, m, m_1]$ and $B_n = \mathbf{I}[m_0, n, m_1]$, realize the KK-data on K-groups with coefficients, we have:

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Theorem

Given positive integers n, m and p with $m|p$, then the canonical map

$$\Gamma : KK(A_m, B_n) \to \mathbf{Hom}(\mathbf{K}(A_m; p), \mathbf{K}(B_n; p))$$

is an isomorphism.

For m_0, m_1, we always choose $\beta_0 \geq 0$, and $\beta_1 \leq 0$, such that $\beta_0 m_0 + \beta_1 m_1 = 1$.

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Theorem (Structure of $\mathbf{Hom}(\mathbf{K}(A_m; p), \mathbf{K}(B_n; p))$)

Each element $\Phi = (x, \rho, y)$ in $\mathbf{Hom}(\mathbf{K}(A_m; p), \mathbf{K}(B_n; p))$ with K_0-multiplicity x and K_1-multiplicity y is of the following form:

$$\Phi = (x, \sigma, y) + d(0, \begin{pmatrix} -m_1 m_0 & m_0 m_0 \\ -m_1 m_1 & m_0 m_1 \end{pmatrix}, 0). \qquad (\star)$$

where $\sigma = \begin{pmatrix} x m_0 \beta_0 + \frac{m y m_1 \beta_1}{n} & x m_0 \beta_1 - \frac{m y m_0 \beta_1}{n} \\ x m_1 \beta_0 - \frac{m y m_1 \beta_0}{n} & x m_1 \beta_1 + \frac{m y m_0 \beta_0}{n} \end{pmatrix}$, *and d is an integer with* $0 \leq d < \frac{m}{m_0 m_1}$.

From the two theorems above, if we assume the K_1-multiplicity of given KK-element α is zero, then α must have the following form:

$$\alpha = (\beta_0 x - dm_1)\delta_0 + (\beta_1 x + dm_0)\delta_1.$$

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Theorem

Given a KK-element α as above with $0 \leq d < \dfrac{m}{m_0 m_1 m_0}$ or $0 \leq d < \dfrac{m}{m_0 m_1 m_1}$, if $\Gamma(\alpha; p)$ preserves the Dadarlat-Loring order structure, then $\beta_0 x - dm_1 \geq 0$.

Theorem

Given a KK-element $\alpha \in KK(A_m, B_n)$ with K_1-multiplicity zero, if the K_0-multiplicity $x \geq m$, then α can be lifted to a homomorphism between the algebras.

To obtain the exact conditions under which α preserves the Dadarlat-Loring order, we simplify further by assuming $d = 0$, then we have:

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Theorem

Given $\alpha = \beta_0 x \delta_0 + \beta_1 x \delta_1$, let R be the remainder of $\beta_0 m_0 m_0 x$ divided by m , and S be the remainder of $\beta_0 m_0 m_1 x$ divided by m. Then $\Gamma(\alpha; p)$ preserves the Dadarlat-Loring order structure if and only if $x = 0$ or

$$\beta_0 m_0 m_0 x \geq m, \ \beta_0 m_0 m_1 x \geq m \tag{1}$$

$$m_0 x \geq R, \ m_1 x \geq S. \tag{2}$$

Note that not only in the statement above, but also in t he proof, the whole requirements for preserving the Dadarlat-Loring order are included in the first part coefficient involving β_0; the negative number β_1 (with proper choices of K_0-multiplicity) provides flexibility for α not to be representable by a homomorphism.

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Concrete Examples: Given $m_0 = 2, m_1 = 3$, and $m = 12$, by the first inequality above, take $x = 2$, then the second inequality is also satisfied. We get $\alpha = 4\delta_0 - 2\delta_1$, which can not be representable by a $*$-homomorphism, since the minimal relation between δ_0 and δ_1 is $6\delta_0 = 4\delta_1$.

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Another example could be $x = 5$, then we get

$$\alpha = 10\delta_0 - 5\delta_1 = 4\delta_0 - \delta_1.$$

This result shows that the existence theorem with Dadarlat-Loring order structure fails at the level of building blocks. If one puts some restrictions on the inductive limit algebras, e.g., real rank zero, then the dynamical behaviour of the connecting maps can be controlled, we can still get a classification in terms of Dadarlat-Loring order.

Thank you!