The Larson-Sweedler theorem and the operator algebra approach to quantum groups

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Outline of the talk

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This talk is related to joint work (in progress) with Byung-Jay Kahng (Buffalo - USA) on the Larson-Sweedler theorem for weak multiplier Hopf algebras (quantum groupoids).

Introduction

- The operator algebra approach to quantum groups finds its origin in the attempts to generalize Pontryagin's duality for abelian locally compact groups to the case of all locally compact groups. This generalization started with the work of Tannaka (1938) and Krein (1949).
- On the other hand, the theory of quantum groups, as developed by e.g. Drinfel'd, follows a different line of research, in a purely algebraic context, starting with the development of the notion of a Hopf algebra.

A new boost in both research fields came about simultaneously, in the late 80's. In the first case, it was the work of Woronowicz on the quantum $SU_q(2)$. In the second case, it was the work of Drinfel'd and Jimbo obtaining the quantizations of universal enveloping algebras. Strange enough, still there was little interaction between the two.

Hopf algebras

Definition

A Hopf algebra is a pair (A, Δ) of an algebra with identity and a coproduct Δ. The coproduct is a unital homomorphism $\Delta : A \to A \otimes A$ satisfying coassociativity $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$. It is assumed that there is a counit ε. This is an algebra homomorphism $\varepsilon : A \to \mathbb{C}$ satisfying

$$(\varepsilon \otimes \iota)\Delta(a) = a \qquad (\iota \otimes \varepsilon)\Delta(a) = a$$

for all a. Also the existence of an antipode is assumed. It is an anti-homomorphism $S : A \to A$ satisfying

$$m(S \otimes \iota)\Delta(a) = \varepsilon(a)1 \qquad m(\iota \otimes S)\Delta(a) = \varepsilon(a)1$$

for all $a \in A$.

The counit and the antipode are unique if they exist.

The basic examples

The two basic examples associated with any finite group G:

Example

Consider the algebra $K(G)$ of all complex functions on G. Define $\Delta(f)(p,q) = f(pq)$ for $f \in K(G)$ and $p,q \in G$. This is a coproduct. Define $\varepsilon(f) = f(e)$ where e is the identity in G. This is the counit. Finally, let $S(f)(p) = f(p^{-1})$ for $p \in G$, then S is the antipode.

If G is no longer finite, one can define a multiplier Hopf algebra.

Example

Consider the group algebra $\mathbb{C}G$ of G and use $p \mapsto \lambda_p$ for the imbedding of G in $\mathbb{C}G$. A coproduct is defined by $\Delta(\lambda_p) = \lambda_p \otimes \lambda_p$. The counit satisfies $\varepsilon(\lambda_p) = 1$ and the antipode $S(\lambda_p) = \lambda_{p^{-1}}$ for all p.

Integrals on Hopf algebras

Definition

A left integral on a Hopf algebra (A, Δ) is a non-zero linear functional $\varphi : A \to \mathbb{C}$ satisfying left invariance $(\iota \otimes \varphi)\Delta(a) = \varphi(a)1$ for all a. A right integral is a non-zero linear functional ψ on A satisfying $(\psi \otimes \iota)\Delta(a) = \psi(a)1$ for all a.

Integrals are unique (up to a scalar) if they exist. On a finite-dimensional Hopf algebra, integrals always exist.

Example

If $A = K(G)$ for a finite group G, the left and right integrals are the same and given by $\varphi(f) = \sum_{p \in G} f(p)$. Also for $\mathbb{C}G$ the integrals coincide and are given by $\varphi(\lambda_p) = 1$ if $p = e$ and $\varphi(\lambda_p) = 0$ otherwise.

Locally compact quantum groups

Definition (Kustermans & Vaes)

A locally compact quantum group is a pair (M, Δ) of a von Neumann algebra M and a coproduct Δ on M such that there exists a left and a right Haar weight.

- The coproduct is a unital normal *-homomorphism $\Delta : M \to M \otimes M$ satisfying coassociativity $(\Delta \otimes \iota)\Delta = (\iota \otimes \Delta)\Delta$.
- A left Haar weight is a faithful normal semi-finite weight φ on M satisfying left invariance:

$$\varphi((\omega \otimes \iota)\Delta(x)) = \omega(1)\varphi(x)$$

  for all positive $\omega \in M_*$ and all positive elements $x \in M$ satisfying $\varphi(x) < \infty$.
- Similarly for a right Haar weight ψ.

Locally compact quantum groups
The basic examples

Let G be any locally compact group.

Theorem

Let $M = L^\infty(G)$. Define Δ on M by $\Delta(f)(p,q) = f(pq)$ whenever $p, q \in G$. Then (M, Δ) is a locally compact quantum group. The left and right Haar weights are the integrals with respect to the left and the right Haar measure respectively.

For $f \in L^\infty(G)$ we have

$$((\iota \otimes \varphi)\Delta(f))(p) = \int f(pq)dq = \int f(q)dq.$$

In the C*-framework, one takes $A = C_0(G)$ and defines $\Delta : A \to M(A \otimes A)$ as above. The Haar weights are now faithful lower semi-continuous, semi-finite weights.

Locally compact quantum groups
The basic examples - continued

Theorem

Let $M = VN(G)$, the von Neumann algebra generated by the left translations λ_p on $L^2(G)$. There is a coproduct Δ satisfying $\Delta(\lambda_p) = \lambda_p \otimes \lambda_p$ for all p. The pair (M, Δ) is also a locally compact quantum group. The left Haar weight satisfies

$$\varphi\left(\int f(p)\lambda_p dp\right) = f(e)$$

when $f \in C_c(G)$. Now the left and right Haar weights coincide.

In the C*-framework, one takes $A = C_r^*(G)$, the reduced C*-algebra of G. Also here the coproduct is a non-degenerate *-homomorphism from $A \to M(A \otimes A)$ defined as above. The left and right Haar weights are still the same and they are given by the same formula as in the von Neumann case.

Locally compact quantum groups

More history and other comments

- Pontryagin duality is a symmetric theory: the dual object $\widehat{G}$ is again a locally compact group.
- This is no longer the case with Tannaka-Krein duality and its generalizations.
- The first self-dual theory comes with the Kac algebras: Kac and Vainerman (1973), Enock and Schwartz (1973).
- The new developments (Woronowicz' $SU_q(2)$ (1987) and the Drinfel'd-Jimbo examples (1985)), show that the theory is too restrictive (the antipode S is assumed to be a *-map).
- New research by various people finally led to the theory of locally compact quantum groups as we know it now: Baaj and Skandalis (1993), Masuda, Nakagami and Woronowicz (1995), Kustermans and Vaes(1999)... and others.

Locally compact quantum groups

More comments

Consider again the definition.

Definition

A locally compact quantum group is a pair (M, Δ) of a von Neumann algebra M and a coproduct Δ on M such that there exists a left and a right Haar weight.

- It is nice and simple (if compared e.g. with the definition of a Kac algebra - and it is more general).
- There is no counit, nor an antipode from the start - the existence is proven (at least of the antipode).
- This is in contrast with (1) the definition of a (locally compact) group and with (2) the notion of a Hopf algebra.
- It is assumed that the Haar weights exist whereas in the theory of locally compact groups, the existence is proven from the axioms.

The Larson-Sweedler theorem for Hopf algebras

Theorem (Larson & Sweedler)

Assume that A is an algebra with identity and a full coproduct Δ. If there is a faithful left integral and a faithful right integral, then (A, Δ) is a Hopf algebra.

- A coproduct on a unital algebra is called full if elements of the form
$$(\iota \otimes \omega)\Delta(a) \qquad\qquad (\omega \otimes \iota)\Delta(a)$$
where $a \in A$ and ω is a linear functional on A each span A.
- In the original formulation, it is assumed that there is a counit. That is a stronger condition.
- Compare fullness of the coproduct with the density conditions we need in the C*-algebraic formulation.
- All this is precisely as in the definition of a locally compact quantum group in the operator algebraic framework.

The proof: the underlying ideas

The following results motivate the proof.

Proposition

Let G be a finite group and f a complex function on G. Write

$$f(p) = f(pqq^{-1}) = \sum g_i(pq)h_i(q).$$

Then $\sum g_i(q)h_i(pq) = f(q.(pq)^{-1}) = f(p^{-1})$.

The result is easily generalized to Hopf algebras.

Proposition

Let (A, Δ) be a Hopf algebra and $a \in A$. Write

$$a \otimes 1 = \sum \Delta(b_i)(1 \otimes c_i).$$

Then $\sum(1 \otimes b_i)\Delta(c_i) = S(a) \otimes 1$. *Also* $\varepsilon(a)1 = \sum b_i c_i$.

Proof of the Larson-Sweedler theorem

Assume that we have an unital algebra A with a full coproduct Δ. Assume that φ is a faithful left integral and that ψ is a faithful right integral on A. Consider $p, q \in A$ and write

$$(\iota \otimes \iota \otimes \varphi)(\Delta_{13}(p)\Delta_{23}(q)) = \sum b_i \otimes c_i .$$

Then

$$\sum \Delta(b_i)(1 \otimes c_i) = (\iota \otimes \iota \otimes \varphi)((\iota \otimes \Delta)(\Delta(p)(1 \otimes q))) = a \otimes 1$$

where $a = (\iota \otimes \varphi)(\Delta(p)(1 \otimes q))$ by the left invariance of φ. Similarly we find

$$\sum (1 \otimes b_i)\Delta(c_i) = (\iota \otimes \iota \otimes \varphi)((\iota \otimes \Delta)((1 \otimes p)\Delta(q))) = b \otimes 1$$

where $b = (\iota \otimes \varphi)((1 \otimes p)\Delta(q))$.

Proof of the Larson-Sweedler theorem - continued

Now we would like to define $S : A \to A$ by $S(a) = b$. In order to do that, we need to verify two things:

- A is spanned by elements of the form $(\iota \otimes \varphi)(\Delta(p)(1 \otimes q))$.
- If $\sum \Delta(b_i)(1 \otimes c_i) = 0$, then $\sum b_i \otimes c_i = 0$.

For the first result, we need that φ is faithful and that Δ is full. For the second, we need the right integral ψ. Indeed, multiply with $\Delta(p)$ from the left and apply $\psi \otimes \iota$ to get $\sum \psi(pb_i)c_i = 0$. As this is true for all $p \in A$ and because ψ is assumed to be faithful, it follows that $\sum b_i \otimes c_i = 0$.

The counit is defined in a similar way by $\varepsilon(a) = \varphi(pq)$ if $a = (\iota \otimes \varphi)((\Delta(p)(1 \otimes q))$.

The proof is completed by showing that ε is the counit and that S is the antipode. This is straightforward.

The antipode in the operator algebra framework

Consider the previous arguments and introduce the involution.

Proposition

Let (A, Δ) be a Hopf *-algebra and let $a \in A$.

If $a \otimes 1 = \sum \Delta(p_i)(1 \otimes q_i^*)$ then $S(a)^* \otimes 1 = \sum \Delta(q_i)(1 \otimes p_i^*)$.

What we do now is:

- Start with a locally compact quantum group (M, Δ) and a right Haar weight ψ.
- Consider the map $V : p \otimes q \mapsto \Delta(p)(1 \otimes q^*)$ on the Hilbert space level $\mathcal{H}_\psi \otimes \mathcal{H}$.
- The right invariance will imply that V is unitary. It is considered as the right regular representation.
- This will yield a closed, densely defined, conjugate linear map on $\mathcal{H}_\psi$

The right regular representation

Let (M, Δ) be a locally compact quantum group with a right Haar weight ψ. Let $\mathcal{H}$ be the underlying Hilbert space of M. Let $\mathcal{H}_\psi$ be the GNS space of ψ. Denote by Λ_ψ the canonical imbedding of the left ideal $\mathfrak{N}_\psi$ in $\mathcal{H}_\psi$ where $\mathfrak{N}_\psi$ is the left ideal of M of elements satisfying $\psi(x^*x) < \infty$. Let M act directly on $\mathcal{H}_\psi$.

Proposition

There is a unitary operator V on $\mathcal{H}_\psi \otimes \mathcal{H}$ satisfying (formally)

$$V(\Lambda_\psi(x) \otimes \xi) = \sum \Lambda_\psi(x_i) \otimes y_i\xi$$

where $x \in \mathfrak{N}_\psi$, where $\xi \in \mathcal{H}$ and where $\sum x_i \otimes y_i$ stands for $\Delta(x) \in M \otimes M$.

To prove that V is well-defined and isometric, one uses right invariance of ψ. To show that it is unitary is more tricky.

The map $x \mapsto S(x)^*$ on the Hilbert space level

Proposition

There exists a closed, densely defined, conjugate linear operator K on $\mathcal{H}_\psi$ so that the domain of K consists of vectors $\xi \in \mathcal{H}_\psi$ such that there is a vector $\xi_1 \in \mathcal{H}_\psi$ with the property that for all $\varepsilon > 0$ and all vectors $\eta_1, \eta_2, \dots, \eta_n$ in $\mathcal{H}_\psi$, there exist elements $p_1, p_2, \dots, p_m, q_1, q_2, \dots, q_m$ in $\mathcal{N}_\psi$ such that

$$\|\xi \otimes \eta_k - V(\sum \Lambda_\psi(p_j) \otimes q_j^* \eta_k)\| < \varepsilon \quad (1)$$

$$\|\xi_1 \otimes \eta_k - V(\sum \Lambda_\psi(q_j) \otimes p_j^* \eta_k)\| < \varepsilon \quad (2)$$

for all k. If ξ_1 exists, it is unique and $K\xi = \xi_1$.

- First one has to show that ξ_1 is unique, if it exists (i.e. $\xi_1 = 0$ if $\xi = 0$).
- Then one must show that the domain of K is dense.

Two aspects of the proof - 1

Proposition

The operator K is well-defined.

Proof.

Assume that

$$\sum \Lambda_\psi(p_j) \otimes q_j^* \eta \to V^*(\xi \otimes \eta) \quad \text{and} \quad \sum \Lambda_\psi(q_j) \otimes p_j^* \zeta \to 0.$$

Take the scalar product of the first expression with a vector $\pi'(\zeta')^* \zeta \otimes \eta'$ where ζ' and η' are right bounded. Then

$$\sum \langle \Lambda_\psi(p_j) \otimes q_j^* \eta, \pi'(\zeta')^* \zeta \otimes \eta' \rangle = \sum \langle \zeta' \otimes \pi'(\eta')^* \eta, p_j^* \zeta \otimes \Lambda_\psi(q_j) \rangle$$

This proves that $\langle V^*(\xi \otimes \eta), \pi'(\zeta')^* \zeta \otimes \eta' \rangle = 0$ and hence $V^*(\xi \otimes \eta) = 0$. $\square$

Two aspects of the proof - 2

To show that K is densely defined, we need to use the left Haar weight φ and the associated left regular representation W, defined on $\mathcal{H}_\psi \otimes \mathcal{H}_\varphi$.

Proposition

Let $c, d \in \mathcal{N}_\psi$ *and* $\omega \in \mathcal{B}(\mathcal{H}_\varphi)_*$ *and define*

$$\xi = \Lambda_\psi((\iota \otimes \omega(c \,\cdot\, d^*))W).$$

Then $\xi \in \mathcal{D}(K)$ *and* $K\xi = \Lambda_\psi((\iota \otimes \overline{\omega}(d \,\cdot\, c^*))W)$.

Proof.

We take $\omega = \langle \cdot \xi', \eta' \rangle$, an orthonormal basis (ξ_j) and

$$p_j = (\iota \otimes \langle \cdot \xi_j, c^* \eta' \rangle)W \qquad \text{and} \qquad q_j = (\iota \otimes \langle \cdot \xi_j, d^* \xi' \rangle)W.$$

Then $p_j, q_j \in \mathcal{N}_\psi$ and they will give the required elements. $\square$

Further steps to develop the theory

- The closed operator K has a polar decomposition $IL^{\frac{1}{2}}$.
- It implements the antipode S in the sense that $S(x)^* = KxK$ and gives rise to the polar decomposition of the antipode with the unitary antipode $x \mapsto I(x^*)I$ and the scaling group, implemented by L^{it}.
- The antipode is characterized by $(S \otimes \iota)W = W^*$.
- We have $(K \otimes T)W(K \otimes T) = W^*$ where T is the closure of the map $\Lambda_\varphi(x) \mapsto \Lambda_\varphi(x^*)$ where $x \in \mathfrak{N}_\varphi \cap \mathfrak{N}_\varphi^*$.
- This gives rise to the properties of the scaling group, the modular automorphism groups, ...
- Relative modular theory is used to prove uniqueness of the Haar weights.
- This gives relative invariance of the Haar weights w.r.t. the scaling group.
- Etc. ...

Conclusions

- We talked about two 'interacting' fields of research: (1) The operator algebra approach to quantum groups and (2) the 'Hopf algebra' approach to quantum groups.
- We discussed the problems that make such an interaction rather difficult and not so obvious.
- In this talk we gave one clear example to illustrate the intimate link between the two topics. It is the Larson-Sweedler theorem, as known in Hopf algebra theory, that is implicit in the definition of e.g. a quantum group in the operator algebra setting.
- I choose this topic for this talk, not only because of the above, but also because I am presently working on a generalization along these lines within the theory of (locally quantum) groupoids. This is joint work with B.-J. Kanhg.

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