Universal Tomographic Measurements

Lane P. Hughston

Department of Mathematics
University College London
London WC1E 6BT, United Kingdom

—

20 February 2013

—

Workshop on Quantum Tomography
Fields Institute, Toronto

Collaborator: Dorje C. Brody (Brunel University, London)

Introduction

The goal is to construct certain classes of measurement operations in quantum theory that only involve a minimal amount of structure.

To the extent that the structure involved is minimal, to the same extent the class of measurement operations associated with it is maximal—or "universal"—that is to say, applicable to any system.

We shall look at some explicit examples of UQMs (universal quantum measurements).

Some of these are tomographic, in the sense that the statistics of the measurement outcomes allow one to determine the state of the system.

States

We consider a quantum system represented by a Hilbert space $\mathcal{H}^\alpha$ of finite dimension n.

Write ξ^α $(\alpha = 1, \ldots, n)$ for a typical element of $\mathcal{H}^\alpha$, and η_α for a typical element of the dual space $\mathcal{H}_\alpha$.

Complex conjugation defines a map $\mathcal{H}^\alpha \to \mathcal{H}_\alpha$ given by $\xi^\alpha \in \mathcal{H}^\alpha \to \bar{\xi}_\alpha \in \mathcal{H}_\alpha$.

The state of the system is a positive operator w^α_β. Thus $w^\alpha_\beta \xi^\beta \bar{\xi}_\alpha \geq 0$ for all ξ^β.

We do not insist that w^α_β should have trace unity—physical expressions will involve ratios.

States of the form $w^\alpha_\beta = Z^\alpha \bar{Z}_\beta \,/ Z^\gamma \bar{Z}_\gamma$ are in one to one correspondence with points of the complex projective space $\mathbb{CP}^{n-1}$.

There is a natural Riemannian metric on the space of pure states, called the Fubini-Study metric, which is invariant under the action of the unitary group.

Experiments

For a given quantum system, we need to specify the space of possible results for an experiment on it, and to say how the state changes as a consequence of the experiment having been made and a certain result having been obtained.

Each experiment is described by a measurable space $(\Omega, \mathcal{F})$ endowed with some structure that relates it to $\mathcal{H}^{\alpha}$ and the state w^{α}_{β}.

Here the set Ω represents the possible outcomes of chance, and $\mathcal{F}$ is a collection subsets of Ω forming a σ-algebra.

In a given experiment, if $\omega \in \Omega$ is the outcome of chance, then the result of the experiment is the smallest element $A \in \mathcal{F}$ such that $\omega \in A$.

It may be that the smallest subset A of Ω that contains ω is the set $\{\omega\} \in \mathcal{F}$ that only contains ω.

This happens for example in the case of "refined" experiments where the result of the experiment is sufficient to determine the outcome of chance.

We distinguish between outcomes of chance (which are elements of Ω) and results of experiments (which are elements of $\mathcal{F}$).

If $\mathcal{E}$ and $\mathcal{F}$ are σ-algebras on Ω, and if $\mathcal{E}$ is a sub-σ-algebra of $\mathcal{F}$, then we say that the experiment $\mathcal{F}$ is a refinement of the experiment $\mathcal{E}$.

Transformations

Another ingredient that we require for the specification of an experiment is a system of state transformations $\mathbb{T} = \{T(A),\ A \in \mathcal{F}\}$ satisfying the following conditions:

1. For each $A \in \mathcal{F}$, $T(A)$ is given by a completely positive map
$$T:\ w^{\alpha}_{\beta} \longrightarrow T^{\alpha\beta'}_{\beta\alpha'}(A) w^{\alpha'}_{\beta'}\ ; \tag{1}$$
2. $\mathbb{T} = \{T(A),\ A \in \mathcal{F}\}$ is countably additive. Thus, $T^{\alpha\beta'}_{\beta\alpha'}(\emptyset) = 0$, and if the sets $\{A_n :\ n \in \mathbb{N}\}$ are disjoint, and such that $A = \cup_n A_n$, then
$$T^{\alpha\beta'}_{\beta\alpha'}(A) = \sum_n T^{\alpha\beta'}_{\beta\alpha'}(A_n)\ ; \tag{2}$$

3. For each $A \in \mathcal{F}$, $T(A)$ is trace-reducing:

$$T^{\gamma\beta'}_{\gamma\alpha'}(A)\; w^{\alpha'}_{\beta'}/w^{\gamma}_{\gamma} \;\leq\; 1\;; \tag{3}$$

4. $T(\Omega)$ satisfies the law of total probability:

$$T^{\gamma\beta'}_{\gamma\alpha'}(\Omega)\; w^{\alpha'}_{\beta'}/w^{\gamma}_{\gamma} \;=\; 1\;\;. \tag{4}$$

Probabilities

Once we have specified the system of transformations, then in a model for an experiment $(\Omega, \mathcal{F}, \mathbb{T})$ on $\mathcal{H}$, the probability that the outcome of chance ω lies in the set $A \in \mathcal{F}$ is given by

$$\mathbb{P}(\omega \in A) = T^{\gamma\beta'}_{\gamma\alpha'}(A)\; w^{\alpha'}_{\beta'}/w^{\gamma}_{\gamma}\;. \tag{5}$$

If A_ω denotes the smallest element of $\mathcal{F}$ containing the outcome of chance ω, and if $\mathbb{P}(\omega \in A_\omega) \neq 0$, then the state transformation associated with ω is

$$w^{\alpha}_{\beta} \rightarrow \frac{T^{\alpha\beta'}_{\beta\alpha'}(A_\omega) w^{\alpha'}_{\beta'}}{T^{\gamma\beta'}_{\gamma\alpha'}(A_\omega) w^{\alpha'}_{\beta'}}\;. \tag{6}$$

The usual projective measurements (with or without selection) in quantum mechanics take this form, and so do discrete POVM measurements.

For example, in the "unrefined" experiment corresponding to a non-selective projective measurement we have $\mathcal{F} = \{\Omega, \emptyset\}$.

In that case, the outcome of the measurement is trivial in the sense that we have $\mathbb{P}(\omega \in \Omega) = 1$ and $\mathbb{P}(\omega \in \emptyset) = 0$. Nevertheless, the state transformation will in general be non-trivial.

Continuous measurements

In the continuous case, the probability of any particular outcome of chance is zero. In that situation we model the state transformations as follows.

We suppose that there exists a measure $\mu(\mathrm{d}\omega)$ on Ω and a transformation density $t^{\alpha\beta'}_{\beta\alpha'}(\omega)$ with the property that for any $A \in \mathcal{F}$ we have

$$T^{\alpha\beta'}_{\beta\alpha'}(A) = \int_{\Omega} \mathbb{1}\{\omega \in A\} t^{\alpha\beta'}_{\beta\alpha'}(\omega)\ \mu(\mathrm{d}\omega)\ .$$

Then the probability distribution for the outcome of chance is

$$\mathbb{P}(\omega \in \mathrm{d}\omega) = t^{\gamma\beta'}_{\gamma\alpha'}(\omega)\ w^{\alpha'}_{\beta'}/w^{\gamma}_{\gamma}\ \mu(\mathrm{d}\omega)\ ,$$

and the state transformation is given by

$$w^{\alpha}_{\beta} \to \frac{t^{\alpha\beta'}_{\beta\alpha'}(\omega) w^{\alpha'}_{\beta'}}{t^{\gamma\beta'}_{\gamma\alpha'}(\omega) w^{\alpha'}_{\beta'}}\ .$$

In what follows, we shall be concerned with the continuous situation, where we have a finite dimensional Hilbert space and the outcome space Ω has the structure of a manifold on which a natural candidate for the measure $\mu(\mathrm{d}\omega)$ on Ω is available.

In particular, we consider how to model the associated transformation tensor density function $t^{\alpha\beta'}_{\beta\alpha'}(\omega)$.

UQMs

We consider the case where the measurable space Ω representing the possible outcomes of chance is the manifold $\mathbb{CP}^{n-1}$, the space of pure states associated with the given Hilbert space $\mathcal{H}^{\alpha}$. This is a rather natural choice to consider since it does not involve introducing any additional structure on the quantum system.

Let $\mathit{\Gamma} := \mathbb{CP}^{n-1}$, let x denote a typical point in $\mathit{\Gamma}$, and let $Z^{\alpha}(x)$ denote a representative vector in $\mathcal{H}$ lying on the fibre above the point $x \in \Gamma$.

Then we can construct a system of transformations $\mathbb{T}$ by setting

$$T^{\alpha\beta'}_{\beta\alpha'}(A) = n \int_{x \in \mathit{\Gamma}} \mathbb{1}\{x \in A\} \frac{Z^{\alpha}(x) Z^{\beta'}(x) \bar{Z}_{\beta}(x) \bar{Z}_{\alpha'}(x)}{(Z^{\gamma}(x)\bar{Z}_{\gamma}(x))^2} \mu(\mathrm{d}x).$$

Here

$$\mu(\mathrm{d}x) = \frac{\mathcal{D}Z(x)}{\int_{x \in \mathit{\Gamma}} \mathcal{D}Z(x)}$$

where

$$\mathcal{D}Z = \epsilon_{\alpha\beta\cdots\gamma} Z^{\alpha} \mathrm{d}Z^{\beta} \cdots \mathrm{d}Z^{\gamma} \; \epsilon^{\alpha\beta\cdots\gamma} \bar{Z}_{\alpha} \mathrm{d}\bar{Z}_{\beta} \cdots \mathrm{d}\bar{Z}_{\gamma} \; / \; [Z^{\gamma}(x)\bar{Z}_{\gamma}(x)]^n.$$

We see that $\mu(\mathrm{d}x)$ defines the uniform distribution on $\mathit{\Gamma} = \mathbb{CP}^{n-1}$.

The transformation tensor density is given by

$$t^{\alpha\beta'}_{\beta\alpha'}(\omega) = n\frac{Z^{\alpha}(x)Z^{\beta'}(x)\bar{Z}_{\beta}(x)\bar{Z}_{\alpha'}(x)}{(Z^{\gamma}(x)\bar{Z}_{\gamma}(x))^2} \ .$$

One sees that the outcome of chance in such a measurement is a pure state.

The probability that the outcome lies in a region $A \subset \mathbb{CP}^{n-1}$ is given by

$$\mathbb{P}(\omega \in A) = E^{\alpha}_{\beta}(A) w^{\beta}_{\alpha},$$

where

$$E^{\alpha}_{\beta}(A) = n\int_{x\in\Gamma} \mathbb{1}\{x \in A\}\frac{Z^{\alpha}(x)\bar{Z}_{\beta}(x)}{Z^{\gamma}(x)\bar{Z}_{\gamma}(x)}\ \mu(\mathrm{d}x).$$

Clearly we have $E^{\alpha}_{\beta}(\mathit{\Gamma}) = \delta^{\alpha}_{\beta}$.

It follows in particular that

$$\rho(x) := n\frac{Z^{\alpha}(x)\bar{Z}_{\beta}(x)w^{\beta}_{\alpha}}{Z^{\gamma}(x)\bar{Z}_{\gamma}(x)}$$

is the probability density for the outcome

$$\mathbb{P}(\omega \in \mathrm{d}\omega) = \rho(x)\ \mu(\mathrm{d}x).$$

If we make a large number of measurements, and analyze the statistics of the measurements, then we can determine $\rho(x)$, and hence determine the state w^{α}_{β} .

The ensemble of outcomes has density $\rho(x)$, and therefore the state of the ensemble representing the outcomes of a large number of measurements is

$$r^{\alpha}_{\beta} = \int_{\Gamma} \rho(x) \frac{Z^{\alpha}(x)\bar{Z}_{\beta}(x)}{Z^{\gamma}(x)\bar{Z}_{\gamma}(x)}\, \mu(\mathrm{d}x). \tag{7}$$

The integral can be worked out explicitly by use of the following formula:

$$\int_{\Gamma} \frac{Z^{\alpha}(x)Z^{\beta'}(x)\bar{Z}_{\beta}(x)\bar{Z}_{\alpha'}(x)}{(Z^{\gamma}(x)\bar{Z}_{\gamma}(x))^2}\, \mu(\mathrm{d}x) = \frac{1}{n(n+1)}\big(\delta^{\alpha}_{\beta}\delta^{\beta'}_{\alpha'} + \delta^{\alpha}_{\alpha'}\delta^{\beta}_{\beta'}\big). \tag{8}$$

A calculation then shows that

$$r^{\alpha}_{\beta} = \frac{1}{n+1}\big(\delta^{\alpha}_{\beta} + w^{\alpha}_{\beta}\big) \ . \tag{9}$$

We see that the original state w^{α}_{β} is “diluted” as a consequence of the measurement operation.

Nevertheless, we can determine the original state since

$$w^{\alpha}_{\beta} = (n+1)r^{\alpha}_{\beta} - \delta^{\alpha}_{\beta}. \tag{10}$$

We call such experiments "universal tomographic measurements" (UQM) since they can be applied to any finite-dimensional quantum system, and allow for .

No additional structure (such as the specification of an observable or a measurable space) is required apart from what is already implicit in the original specification of the system $\mathcal{H}^{\alpha}$.

Direction of spin

UQMs can form elements of other measurements.

In that case, we introduce more structure on the Hilbert, but typically not involving the choice of specific observables. The idea is to keep the picture as geometrical as possible.

An example is as follows.

Consider a three-dimensional Hilbert space, for which the space of pure states has the structure of the complex projective space $\mathbb{CP}^2$ endowed with the Fubini-Study metric.

Let $\mathcal{C}$ be a real conic curve in $\mathbb{CP}^2$.

By "real" we mean the following: we require that for any point x in $\mathcal{C}$ the complex conjugate line (representing the states orthogonal to x) is tangent to $\mathcal{C}$.

Such a setup is equivalent to representing $\mathcal{H}^\alpha$ as a space of symmetric spinors $\mathcal{H}^{AB}$, with a typical element z^{AB} (where $A = 1, 2$) so $z^{AB} = z^{BA}$.

The conic is given by

$$\epsilon_{AB}\,\epsilon_{CD}\,z^{AC}z^{BD} = 0\ . \tag{11}$$

Here $\epsilon_{AB} = -\epsilon_{BA}$. Thus the conic is given by projective Hilbert space elements of the form

$$z^{AB} = \phi^A\phi^B. \tag{12}$$

The complex conjugate line consists of all states x^{AB} such that

$$\bar{\phi}_A\bar{\phi}_B x^{AB} = 0. \tag{13}$$

Thus the pure states orthogonal to the point $z^{AB} = \phi^A \phi^B$ on the conic are of the form

$$x^{AB} = \bar{\phi}^{(A} \alpha^{B)} \tag{14}$$

But any such state lies on a line tangent to $\mathcal{C}$, the tangent point being $\bar{\phi}^A \bar{\phi}^B$.

We can use $\mathcal{C}$ as the outcome space of a class of measurements.

For any spin-one initial state w^{AB}_{CD} the outcome of the measurement is a point of the conic $\mathcal{C}$, that is to say, a pure spin state with a definite direction for the axis of spin.

Thus, for the state transformation we have

$$w^{AB}_{CD} \rightarrow \phi^A \phi^B \bar{\phi}_C \bar{\phi}_D \; / \; (\phi^E \bar{\phi}_E)^2 \; . \tag{15}$$

The probability that the outcome lies in a given region $A \subset \mathcal{C}$ is given by

$$\mathbb{P}(\omega \in A) = 3 \int_A w^{AB}_{CD} \; \phi^C \phi^D \bar{\phi}_A \bar{\phi}_B (\phi^E \bar{\phi}_E)^{-2} \; \mathcal{D}\phi \; , \tag{16}$$

where $\mathcal{D}\phi$ is the volume element on the conic $\mathcal{C}$ induced by the embedding of $\mathbb{CP}^1$ in $\mathbb{CP}^2$ as a rational curve (the Veronese embedding).

Such an experiment on a spin-one system can be regarded as a "measurement of the direction of the axis of the spin" of the particle.

The result of the experiment is an answer to the question "what is the direction of the spin axis of the particle?".

The state then transforms from the original state to a pure state, which is the unique state that has that axis of spin.

Similar formulae apply in the case of higher spins, in which case the defining structure involves a rational curve of degree $2s$ in $\mathbb{CP}^{2s}$ (twisted cubic, rational quartic, etc.).

Applications: disentangling measurements

As another example we can consider the Hilbert space of a pair of qubits.

In that case the Hilbert space has dimension four, and the associated pure state space is $\mathbb{CP}^3$.

The space of disentangled pure states is a quadric surface in $\mathbb{CP}^3$.

The quadric is a doubly ruled surface, given by the product of two $\mathbb{CP}^1s$.

Each of the $\mathbb{CP}^1s$ is endowed with the Fubini-Study measure, so as a consequence the quadric has a natural measure on it, given by the product measure.

This gives rise to a class of UQMs that we can call "disentangling measurements". Starting with a general state of the two-qubit system, the outcome of a disentangling measurement is a point on the quadric.

The transformation density is given by the product of the transformation densities associated with the UQMs attached to each of the individual qubits.

Similar constructions apply in the case of entangled states of many particle systems, in which case the relevant outcome space is given by the Segre embedding of the product space of the pure state spaces associated with the various individual systems.

Conclusions

We are able to introduce certain rather general classes of operations in quantum mechanics that one can regard as "universal quantum measurements" (UQMs) in the sense that they are applicable to all quantum systems and involve the specification of only a minimal amount of structure on the system.

The first class of UQM that we have considered involves the Hilbert space of the system together with the specification of the state of the system—no further structure is brought into play.

We can call operations of this type "universal tomographic measurements", since given the statistics of the outcomes of such measurements it is possible to reconstruct the state of the system.

We are then able to consider the large class of UQMs that can be constructed when the universal tomographic measurements on one or more quantum systems are lifted, through appropriate embeddings, to induce certain associated operations on the embedding space.

As an example, one can make a measurement of the direction in space along which the spin of a spin-s particle is oriented $\left(s = \frac{1}{2}, 1, \ldots\right)$. In this case the additional structure involves the embedding of CP^1 as a rational curve of degree $2s$ in CP^{2s}.

As another example, we have indicated how one can construct a universal disentangling measurement, the outcome of which, when applied to a mixed state of an entangled compose system, is a disentangled product of pure constituent states.

References

[1] Adler, S. A., Brody, D. C., Brun, T. A. & Hughston, L. P. (2001) Martingale models for quantum state reduction. *J. Phys.* A**34**, 8795–8820.

[2] Arnold, V. I. (1989) *Mathematical Methods of Classical Mechanics*, second edition. Berlin: Springer-Verlag.

[3] Ashtekar, A. & Schilling, T. A. (1998) Geometrical formulation of quantum mechanics. In: *On Einstein's Path: Essays in Honor of Engelbert Schucking*, A. Harvey, ed.. Berlin: Springer-Verlag

[4] Bengtsson, I. & Zyczkowski, K. (2006) *Geometry of Quantum States: An Introduction to Quantum Entanglement* (Cambridge: Cambridge University Press).

[5] Brody, D. C. & Graefe, E. M. (2010) Coherent states and rational surfaces. *J. Phys.* A**43** 255205.

[6] Brody, D. C. & Hughston, L. P. (2001) Geometric quantum mechanics. *J. Geom. Phys.* **38** 19-53.

[7] Brody, D. C. & Hughston, L. P. 2002 Efficient simulation of quantum state reduction. *J. Math. Phys.* **43**, 5254–5261.

[8] Brody, D. C. & Hughston, L. P. 2006 Quantum noise and stochastic reduction. *J. Phys.* A**39**, 833–876.

[9] Davies, E. B. (1976) *Quantum Theory of Open Systems*. New York: Academic Press.

[10] Davies, E. B. & Lewis, J. T. (1970) An operational approach to quantum probability. *Commun. Math. Phys.* **17**, 239-260.

[11] Geroch, R. (1970) *Lecture notes on particle physics*. Department of Physics, University of Texas at Austin (unpublished).

[12] Gibbons, G. W. (1992) Typical states and density matrices. *J. Geom. Phys.* **8**, 147-162.

[13] Holevo, A. S. (1982) *Probabilistic and Statistical Aspects of Quantum Theory*. Amsterdam: North-Holland.

[14] Holevo, A. S. (2001) *Statistical Structure of Quantum Theory*. Berlin: Springer.

[15] Hughston, L. P., Jozsa, R. & Wootters, W. K. (1993) A Complete Classification of Quantum Ensembles having a Given Density Matrix, *Physics Letters A* **183**, 14-18.

[16] Hughston, L. P. (1995) Geometric aspects of quantum mechanics. In: *Twistor Theory*, Huggett, S., ed.. New York: Marcel Dekker.

[17] Hughston, L. P. (1996) Geometry of stochastic state reduction. *Proc. Roy. Soc. London* A**452**, 953-979.

[18] Kobayashi, S. & Nomizu, K. (1963, 1969) *Foundations of differential geometry*, Vols. 1, 2. New York: Wiley.