Toric Varieties and Degenerations

Fields-MITACS Undergraduate Research Program

Ana Lucía Báez Camargo

Xander Flood

Darren Gooden

Iker Martínez

Laura Walton

Supervisors: Dr. M. Harada and Dr. J. Yang

August 22, 2012

In this summer research program we studied toric varieties and toric degenerations and finally we degenerate the variety:

$$V(x, y, z) = \left\{[x : y : z] \mid y^2z - x^3 - z^3 = 0\right\} \tag{1}$$

To achieve this goal we first study what an affine variety is.

Definition 1. An *affine variety* is the vanishing locus of a set of polynomials in $\mathbb{C}[x_1, \ldots, x_n]$.

After some days working on affine varieties we were ready to begin the study on toric varieties. First we needed to know what a torus in the sense of algebraic object is:

Definition 2. An *algebraic torus* is a subset of $\mathbb{C}^m$ which is isomorphic to $(\mathbb{C}^*)^n$

There are two important things to notice. The first one is that any torus inherits a group structure from $(\mathbb{C}^*)^n$, and the second is that $\mathbb{C}^* \subset \mathbb{C}$.

Definition 3. A *toric variety* is a variety V that

1. Contains an algebraic torus as a dense open subset
2. The action of the torus on itself extends to an action of the torus on the entire variety V.

As we said before, $\mathbb{C}^* \subset \mathbb{C}$, so the first item of definition 3 is satisfied. For the second part, the group operation $\mathbb{C}^* \times \mathbb{C}^* \to \mathbb{C}^*$ can be extended to an action $\mathbb{C}^* \times \mathbb{C} \to \mathbb{C}$. Hence $\mathbb{C}$ is a toric variety.

Now we know what a toric variety is, but why is this important? In this summer we learned that toric varieties can be constructed in three different ways: algebraically, combinatorially and geometrically. This is really good because we can have a lot of information and tools from this three branches of mathematics.

Algebraically, we can construct a toric variety as follows:
Let $A = m_1, m_2, ..., m_s,\ A \subset (\mathbb{Z})^n$. Consider the matrix $M_A = [m_1 ... m_s]$ Then $Z(x^{l+} - x^{l-} \mid l \in null(M_A))$ is a toric variety.

Geometrically:
We consider the same subset A, and define:
$\phi_A : \mathbb{C}^n \to \mathbb{C}^s$
$t = (t_1, ..., t_n) \mapsto (t^{m_1}, ..., t^{m_s})$
where $t^{m_i} = t_1^{m_{i1}} \cdot t_2^{m_{i2}} \cdot ... \cdot t_n^{m_{in}}$.
The closure of the image of ϕ_A is a toric variety and is the same as the algebraic one.

The main goal of this summer was to degenerate an algebraic variety into a toric variety.

Definition 4. Let X be any algebraic variety. A *toric degeneration* of X is a map $\pi : Y \to \mathbb{C}$ such that:

$$\pi^{-1}(z) \cong \{X if z \neq 0 V if z = 0$$

for some variety Y and some toric variety V.

To construct a toric degeneration is quite complicated. Dave Anderson gives a construction for varieties with certain properties in his paper *Okounkov Bodies and Toric Degenerations*. At the end of this project we degenerate the variety:

$$X = Z(y^2 z - x^3 - z^3)$$

into:

$$V = Z(y^2 z - x^3)$$

The degeneration of X is given by:

$$Y = Z(y^2 z - x^3 - \tau^6 z^3)$$

and
$\pi : Y \to \mathbb{C}\ (x, y, z, \tau) \mapsto \tau$
As desired,

$$\pi^{-1}(c) \cong \{X = Z(y^2 z - x^3 - z^3) if c \neq 0 and V = Z(y^2 z - x^3) if c = 0$$

There is a lot of further research in this topic, like what information does a degeneration give us about the original variety. And if we can construct a toric degeneration for more general varieties?

Finally, we are so grateful for our advisors Dr. Megumi Harada, Dr. Jessie Yang, Lauren Dedieu and Vanessa Foster for their help in this project. Also we are indebted to The Fields Institute and MITACS for the great opportunity they gave us.