Motives and algebraic cycles on twisted flag varieties

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1 Root datum

We introduce a combinatorial language which will be used to classify split semisimple linear algebraic groups.

Root datum. Following [SGA3, Vol.3, Exp. XXI, §1.1] we define a *root datum* to be a nonempty finite subset Σ of a free finitely generated abelian group Λ together with an embedding

$$\Sigma \hookrightarrow \Lambda^\vee, \quad \alpha \mapsto \alpha^\vee$$

into the dual $\Lambda^\vee = Hom(\Lambda, \mathbb{Z})$ such that

1. $\Sigma \cap 2\Sigma = \emptyset$,
2. $\alpha^\vee(\alpha) = 2$ for all $\alpha \in \Sigma$, and
3. $\beta - \alpha^\vee(\beta)\alpha \in \Sigma$ and $\beta^\vee - \beta^\vee(\alpha)\alpha^\vee \in \Sigma^\vee$ for all $\alpha, \beta \in \Sigma$,

where $\Sigma^\vee$ denotes the image of Σ in $\Lambda^\vee$. The elements of Σ (resp. $\Sigma^\vee$) are called roots (resp. coroots).

Root and weight lattices. The subgroup of Λ generated by Σ is called the *root lattice* and is denoted by Λ_r. A root datum is called semisimple if $\Lambda \otimes_{\mathbb{Z}} \mathbb{Q} = \Lambda_r \otimes_{\mathbb{Z}} \mathbb{Q}$. From now on by a root datum we will always mean a semisimple one.

The subgroup of $\Lambda_{\mathbb{Q}} = \Lambda \otimes_{\mathbb{Z}} \mathbb{Q}$ generated by all $\omega \in \Lambda_{\mathbb{Q}}$ such that $\alpha^\vee(\omega) \in \mathbb{Z}$ for all $\alpha \in \Sigma$ is called the *weight lattice* and is denoted by Λ_w. Observe that we have

$$\Lambda_r \subseteq \Lambda \subseteq \Lambda_w \quad \text{and} \quad \Lambda_r \otimes_{\mathbb{Z}} \mathbb{Q} = \Lambda_{\mathbb{Q}} = \Lambda_w \otimes_Z \mathbb{Q}.$$

The $\mathbb{Q}$-rank of $\Lambda_{\mathbb{Q}}$ is called the *rank* of the root datum.

Simple roots and fundamental weights. The root lattice Λ_r admits a $\mathbb{Z}$-basis $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ such that each $\alpha \in \Sigma$ is a linear combination of α_i's with either all positive or all negative coefficients. So the set Σ splits into two disjoint subsets $\Sigma = \Sigma_+ \amalg \Sigma_-$, where Σ_+ (resp. Σ_-) is called the set of positive (resp. negative) roots. The roots α_i are called *simple roots*. Observe that n here is the rank of the root datum.

Given the set Π we define the set of *fundamental weights* $\{\omega_1, \ldots, \omega_n\} \subset \Lambda_w$ as $\alpha_i^\vee(\omega_j) = \delta_{ij}$, where δ_{ij} is the Kronecker symbol. Fundamental weights form a basis of the weight lattice Λ_w.

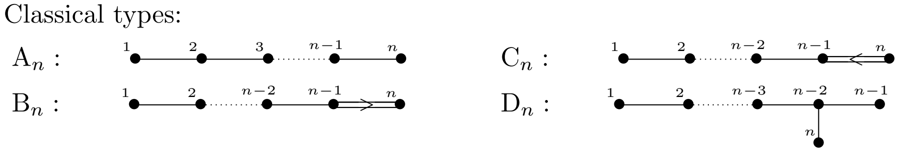
We denote $c_{ij} = \alpha_j^\vee(\alpha_i)$, $i, j = 1 \ldots n$. The matrix $C = (c_{ij})$ is called the *Cartan matrix* of the root datum. By definition we have $\alpha_i = \sum_{j=1}^n c_{ij}\omega_j$, i.e. the Cartan matrix expresses simple roots in terms of fundamental weights.

The Dynkin diagram. A root datum is called *irreducible* if it can not be represented as a direct sum of root data, i.e. Λ can not be written as $\Lambda = \Lambda_1 \oplus \Lambda_2$, where $\Sigma_1 \subset \Lambda_1$, $\Sigma_2 \subset \Lambda_2$ are the root data. To any irreducible root datum we associate a graph called the *Dynkin diagram*. Its vertices are in 1-1 correspondence with the set of simple roots $\{\alpha_1, \ldots, \alpha_n\}$ and the number of edges connecting two different vertices α_i, α_j is given by $c_{ij} \cdot c_{ji}$. Moreover, if there are more than one edge connecting α_i and α_j we orient these edges towards α_j if $c_{ij} < c_{ji}$. It can be shown that an irreducible root datum is determined uniquely by its Dynkin diagram and the intermediate subgroup $\Lambda_r \subseteq \Lambda \subseteq \Lambda_w$.

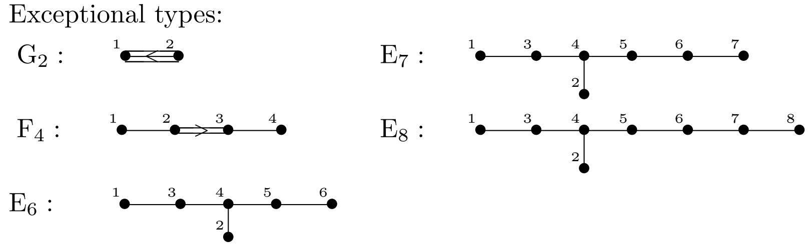
If $\Lambda = \Lambda_w$ (resp. $\Lambda = \Lambda_r$), then the root datum is called simply connected (resp. adjoint) and will be denoted by $\mathcal{D}_n^{sc}$ (resp. $\mathcal{D}_n^{ad}$), where $\mathcal{D} = \mathrm{A, B, C, D, E, F, G}$ is one of the Dynkin diagrams and n is its rank.

Our enumeration of vertices of Dynkin diagrams follows Bourbaki and looks as follows

Classical types:



Exceptional types:



The Weyl group. A $\mathbb{Z}$-linear map $s_\alpha \colon \Lambda_w \to \Lambda_w$, $\alpha \in \Sigma$, defined by

$$s_\alpha(\lambda) = \lambda - \alpha^\vee(\lambda)\alpha, \quad \lambda \in \Lambda_w,$$

is called the *reflection* corresponding to the root α. Observe that by definition we have $s_\alpha \circ s_\alpha = id$. The group W generated by all reflections s_α is called the *Weyl group* of the root datum. One can show that

$$W = \langle s_1, \ldots, s_n \mid (s_i s_j)^{m_{ij}} = 1 \rangle$$

where $s_i = s_{\alpha_i}$ is the reflection corresponding to the simple root α_i, $m_{ii} = 2$ and $m_{ij} = 2c_{ij}c_{ji}$ if $i \neq j$. The group W is a finite group that acts by permutations on Σ. It provides an example of the so called Coxeter group.

2 Linear algebraic groups over arbitrary fields

We recall the notion of an affine group scheme/linear algebraic group over an arbitary field k, e.g. over a number field or a function field of an algebraic variety.

Affine group schemes. An *affine group scheme* over an arbitrary field k is an affine algebraic variety G over k endowed with a group structure, i.e. endowed with an identity element and two algebraic morphisms over k

$$\mathrm{mult}\colon G \times G \longrightarrow G \quad \text{and} \quad \mathrm{inv}\colon G \longrightarrow G$$

defining the multiplication and the inverse respectively and satisfying the usual group axioms. Alternatively, using the language of *functors of points* an affine group scheme G can be identified with a *functor*

$$\mathcal{G}\colon \mathcal{A}lg_k \longrightarrow \mathcal{G}roups$$

from the category of commutative associative k-algebras to the category of groups *represented* by a commutative finitely generated *Hopf algebra* H over k. This identification is given by $G = \operatorname{Spec} H$ and $\mathcal{G}(-) = \operatorname{Hom}_k(-, G)$.

Examples. Here are basic examples of affine group schemes

- The additive group $\mathbb{G}_a = \operatorname{Spec} k[t]$;
- The multiplicative group $\mathbb{G}_m = \operatorname{Spec} k[t, t^{-1}]$.
- The group of n-th roots of unity $\mu_n = \operatorname{Spec} k[t]/(t^n - 1)$. This is a closed subgroup of $\mathbb{G}_m$.
- The general linear group $\mathrm{GL}_n = \operatorname{Spec} k[t_{ij}, \det^{-1}]$, where $1 \leq i, j \leq n$ and det is the determinant of the $n \times n$ matrix (t_{ij}).

One can show that G is an affine group scheme over k if and only if it is a closed subgroup of a general linear group GL_n over k. To stress this fact we will call G by a *linear algebraic group* over k.

Smoothness. If the base field k has characteristic 0 any linear algebraic group G is *smooth* as a variety over k. In the language of functors of points it means that for any commutative k-algebra R and any nilpotent ideal I of R the induced map $\mathcal{G}(R) \to \mathcal{G}(R/I)$ is surjective or, equivalently, the Hopf algebra representing $\mathcal{G}$ is reduced. For instance, the group $G = \mu_n$ is smooth if and only if n is invertible in k.

Connected groups. A linear algebraic group G over k is called *connected*, if it is irreducible as an algebraic variety over k. For example, the orthogonal group of isometry classes of the quadratic form $x_1^2 + \ldots + x_n^2$ is not connected. It consists of two connected components, where the connected component of the identity is isomorphic to the special orthogonal group.

In the present lectures we will deal with smooth connected linear algebraic groups only.

Simple, semisimple groups. A connected non-trivial linear algebraic group G is called *simple* (resp. *semisimple*) if it does not have any non-trivial closed (resp. solvable) connected normal subgroups over the algebraic closure k_a of k. For instance, the product of two copies of $\mathbb{G}_m$ is not semisimple, since it contains a copy of $\mathbb{G}_m$ as the diagonal. The group SL_n, $n > 1$ provides an example of a simple linear algebraic group. A product of simple groups provides an example of a semisimple group.

Tori. A linear algebraic group $\mathbb{T}$ over k is called a *torus*, if over the algebraic closure k_a it becomes isomorphic to a product of several copies of $\mathbb{G}_m$, i.e.

$$\mathbb{T}_{k_a} \simeq \mathbb{G}_{m,k_a} \times \ldots \times \mathbb{G}_{m,k_a}.$$

If this isomorphisms is defined already over k, then $\mathbb{T}$ is called a split torus.

Example. Let k'/k be a finite separable field extension. Consider the functor

$$\mathcal{G} \colon R \longrightarrow (R \otimes_k k')^\times$$

which maps a commutative k-algebra R to the group of units of its base change. One can show that $\mathcal{G}$ is representable by a Hopf algebra and, therefore, defines a linear algebraic group over k denoted by $\mathrm{R}_{k'/k}(\mathbb{G}_m)$ and called the *Weil restriction*. The usual norm map induces a morphism of group schemes $\mathrm{R}_{k'/k}(\mathbb{G}_m) \to \mathbb{G}_m$ over k. Its kernel provides an example of a non-split torus over k.

Root datum of a linear algebraic group. Let G be a semisimple linear algebraic group over an algebraically closed field k. We associate a root datum to G as follows:

Choose a maximal torus $\mathbb{T}$ inside G. We denote the character group of $\mathbb{T}$ by $\Lambda = \mathrm{Hom}(\mathbb{T}, k^\times)$. The abelian group Λ is a free abelian group of rank equals to the rank of G. Consider the adjoint representation $\mathrm{Ad}\colon \mathbb{T} \to GL(\mathrm{L})$ of $\mathbb{T}$ on the associated Lie algebra L. A character $\alpha \in \Lambda$ is called a *weight* of the adjoint representation if the respective weight subspace L_α defined by

$$\mathrm{L}_\alpha \;:=\; \{\, v \in \mathrm{L} \mid \mathrm{Ad}(t)v = \alpha(t)v \;\; \forall t \in \mathbb{T}\,\}$$

is non-trivial. Note that all non-trivial weight subspaces are one dimensional and there is a direct sum decomposition $\mathrm{L} = \bigoplus_{\text{weights } \alpha} \mathrm{L}_\alpha$. We take Σ to be the set of weights of the adjoint representation.

The set of coroots $\Sigma^\vee$ can then be identified with a subset of the cocharacter group $\Lambda^\vee = Hom(k^\times, \mathbb{T})$ in such a way that $\alpha^\vee(\beta) = \langle \beta, \alpha^\vee\rangle$, where

$$\langle \cdot, \cdot \rangle \;:\; \Lambda \times \Lambda^\vee \;\longrightarrow\; \mathbb{Z} = \mathrm{Hom}(k^\times, k^\times)\,,$$

is the perfect pairing defined on points by

$$\lambda(\rho(x)) = x^{\langle \lambda, \rho\rangle} \text{ for all } \lambda \in \Lambda,\; \rho \in \Lambda^\vee \text{ and } x \in k^\times.$$

We define the *Weyl group* W of G to be the quotient

$$W(G) = N_G(\mathbb{T})/\mathbb{T},$$

where $N_G(\mathbb{T})$ is the normalizer of $\mathbb{T}$ in G. It is a finite group which acts on Λ by

$$w(\lambda)(t) \;:=\; \lambda(w^{-1}(t)), \quad \lambda \in \Lambda.$$

It can be shown that Σ does not depend on the choice of $\mathbb{T}$.

Classification of split simple groups. Given a root datum (Λ, Σ) one can associate to it a group scheme $\mathbb{G}$ defined over $\mathrm{Spec}\,\mathbb{Z}$ called a *Chevalley group* such that the base change $\mathbb{G} \times_{\mathrm{Spec}\,\mathbb{Z}} k$, where k is algebraically closed, is a semisimple linear algebraic group over k with the root datum (Λ, Σ).

Moreover, let k be an arbitrary field and let G be a semisimple linear algebraic group G containing a *split* maximal torus $\mathbb{T}$; such a group will be called *split*. Then G can be identified with the base change $\mathbb{G} \times_{\mathrm{Spec}\,\mathbb{Z}} \mathrm{Spec}\ k$, where $\mathbb{G}$ is the Chevalley group corresponding to the root datum of $G \times_{\mathrm{Spec}\ k} \mathrm{Spec}\ k_a$ over the algebraic closure k_a of k.

In other words, there is a 1-1 correspondence between root data and split semisimple linear algebraic groups over k (and semisimple linear algebraic groups over an algebraically closed field).

We now provide a complete list of split simple (corresponding to irreducible root data) groups of classical types:

A_n**:** Here $\Lambda_w/\Lambda_r \simeq \mathbb{Z}/(n+1)\mathbb{Z}$ and $G = \mathrm{SL}_{n+1}/\mu_l$, where l divides $n+1$. For $l = n+1$ we obtain the projective linear group $G^{ad} = \mathrm{PGL}_{n+1}$ (adjoint), and for $l = 1$ the special linear group $G^{sc} = \mathrm{SL}_{n+1}$ (simply connected).

B_n**:** In this case $\Lambda_w/\Lambda_r \simeq \mathbb{Z}/2\mathbb{Z}$ has only two subgroups. The trivial subgroup corresponds to the special orthogonal group O^+_{2n+1} (adjoint) while the whole group corresponds to the spinor group Spin_{2n+1} (simply connected).

C_n**:** As in the previous case $\Lambda_w/\Lambda_r \simeq \mathbb{Z}/2\mathbb{Z}$ and, therefore, there are two isogenies: the projective symplectic group PGSp_{2n} (adjoint) and the symplectic group Sp_{2n} (simply connected).

D_n**:** In this case Λ_w/Λ_r depends on n.

(a) If n is odd, then $\Lambda_w/\Lambda_r \simeq \mathbb{Z}/4\mathbb{Z}$ and the respective isogenies are given by the projective orthogonal group PGO^+_{2n} (adjoint), the special orthogonal group O^+_{2n} (corresponds to the subgroup $\mathbb{Z}/2\mathbb{Z}$) and the spinor group Spin_{2n} (simply connected).

(b) If n is even, then $\Lambda_w/\Lambda_r \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ and the respective isogenies are given by PGO^+_{2n} (adjoint), two halfspin (or semispinor) groups $\mathrm{Spin}^{\pm}_{2n}$ (corresponding to the subgroups $\mathbb{Z}/2\mathbb{Z} \times 1$ and $1 \times \mathbb{Z}/2\mathbb{Z}$ respectively), the special orthogonal group O^+_{2n} (the diagonal subgroup) and the spinor group Spin_{2n} (simply connected). Note, however, that Spin^+_{2n} and Spin^-_{2n} are isomorphic by means of the outer automorphism of Σ.

In the present lectures we will mainly study *non-split* semisimple groups. Here are the typical examples

1. Let A be a central division algebra over k. Its automorphism group $\mathrm{Aut}(A)$ is a non-split simple linear algebraic group over k (denoted PGL_A). In particular, let $\mathbb{H}_{\mathbb{R}}$ be the algebra of real quaternions. Then $\mathrm{Aut}(\mathbb{H}_{\mathbb{R}})$ is a non-split group over $k = \mathbb{R}$ (a form of $\mathrm{PGL}_2(\mathbb{R})$).

2. Let $k = \mathbb{Q}(t_1, \ldots, t_n)$ be a purely transcendental field extension of $\mathbb{Q}$. Consider a quadratic form $q(x_1, \ldots, x_n) = t_1x_1^2 + t_2x_2^2 + \ldots t_nx_n^2$ of rank n over k. The *isometry group* $\mathrm{O}^+(q)$ of all linear automorphisms f of k^n of determinant 1 preserving q, i.e. $q \circ f = q$, is a non-split simple linear algebraic group.

3 Projective homogeneous varieties (PHVs) in the split case

Homogeneous varieties. We start with the following general definition:

Let G be an arbitrary group scheme over a field k. A variety X over k is called a *homogeneous* G-variety if there is a morphism $\rho\colon G \times X \to X$ of varieties over k such that

$$\rho(g \cdot f, x) = g(f(x)) \text{ for all } f, g \in G(k_a),\ x \in X(k_a),$$

and the action of $G(k_a)$ on $X(k_a)$ is *transitive*, i.e. for every x, $y \in X(k_a)$ there exists $g \in G(k_a)$ such that $g \cdot x = y$, where $g \cdot x$ denotes $\rho(g, x)$.

Now let G be a split linear algebraic group over k. We choose a split maximal torus $\mathbb{T}$ in G and the root system Σ. We choose a Borel (maximal connected solvable) subgroup B of G containing T, hence, a set of simple root Π together with a set of positive roots Σ_+.

Consider a smooth *projective* G-homogeneous variety X. A k-point x of X corresponds to the stabilizer subgroup G_x at x and for any two k-points $x, y \in X$ the respective stabilizer subgroups are conjugate, i.e. we have $G_y = gG_xg^{-1}$, where $g \in G(k)$ is such that $g \cdot x = y$. In other words, the conjugacy class of G_x describes $X(k)$.

Combinatorial description. The set of conjugacy classes of stabilizer subgroups of a split G is in 1-1 correspondence with the set of subsets of Π.

Namely, given a subset $\Theta \subset \Pi$ we define the representative P_Θ of a conjugacy class as follows: We define P_Θ to be the subgroup of G generated by $\mathbb{T}$ and all unipotent subgroups corresponding to all positive roots and to all roots in the linear span of Θ with no Θ-terms.

Here the unipotent subgroup corresponding to a root α is the image of $u_\alpha \colon \mathbb{G}_a \hookrightarrow G$ such that $tu_\alpha(x)t^{-1} = u_\alpha(\alpha(t)x)$ for all $t \in \mathbb{T}$, $x \in k^\times$.

The representative P_Θ is called the *standard parabolic subgroup* of G. For any $\Theta \subset \Pi$ we have $B \subseteq P_\Theta \subseteq G$. In particular, $P_\emptyset = B$ and $P_\Pi = G$.

By $P_{i_1,\ldots,i_r}$ we denote the standard parabolic subgroup of the complementary subset $\Theta \setminus \{\alpha_{i_1}, \ldots, \alpha_{i_r}\}$. The subgroup P_i is called a *maximal parabolic subgroup* corresponding to the vertex i.

Classification. Observe that if we choose another Borel subgroup $\mathbb{T} \subset B'$ containing $\mathbb{T}$, then the respective representatives P_Θ and P'_Θ will be conjugate to each other. If we choose another $\mathbb{T}$ inside B this leads to a possible permutation of Π. Hence, the correspondence above doesn't depend on the choices of $\mathbb{T}$ and B up to a permutation.

Moreover, since any two isogeneous split groups have the same Π, it doesn't depend on the isogeny class of G. In particular, when dealing with projective homogeneous G-varieties we may always assume that G is adjoint.

Summarizing the above discussion, all smooth projective homogeneous G-varieties (where G is split simple) are classified (up to an isomorphism) by subsets Θ of the set of vertices of the Dynkin diagram of G. An isomorphism class of a projective homogeneous G-variety corresponding to a subset Θ will be denoted by $\mathcal{D}/P_\Theta$, where $\mathcal{D}$ is the type of the root datum of G.

Examples. We now provide basic examples of projective homogeneous G-varieties, where G is a split simple linear algebraic group over k (enumeration of roots follows Bourbaki):

A_n**:** We have $\mathrm{A}_n/P_1 \simeq \mathrm{A}_n/P_n \simeq \mathbb{P}^n$ and, more generally,

$$\mathrm{A}_n/P_i \simeq \mathrm{A}_n/P_{n-i} \simeq \mathrm{Gr}(i, n+1),$$

where $\mathrm{Gr}(i, n+1)$ is the *Grassmannian* of i-dimensional linear subspaces in $\mathbb{A}^{n+1}$. The variety $\mathrm{A}_n/P_{1,n}$ is called the *incidence variety*. A point on this variety is given by the pair (l, H), where l is a line and H is a hyperplane in $\mathbb{A}^{n+1}$ such that $l \subset H$. In geometric terms, it is given by the equation $\sum_{i=0}^{n} x_i y_i = 0$ in $\mathbb{P}^n \times \mathbb{P}^n$, where x_i (resp. y_i) are the projective coordinates of the first (resp. second) factor.

Finally, $\mathrm{A}_n/P_\emptyset$ is isomorphic to the *variety of complete flags*: points are given by n-tuples of linear subspaces $(l_1, l_2, \ldots, l_n)$ in $\mathbb{A}^{n+1}$ such that $\dim l_i = i$ and $l_1 \subset l_2 \subset \ldots \subset l_n$.

B_n**,** D_n**:** The variety B_n/P_1 (resp. D_n/P_1) is isomorphic to a smooth projective *quadric* Q of dimension $2n-1$ (resp. $2n-2$) given by the equation $x_1^2 - x_2^2 + x_3^2 - \ldots = 0$, where $x_1, \ldots, x_{2n+1}$ (resp. $2n+1$ is replaced by $2n$) are the projective coordinates. The variety B_n/P_n (resp. D_n/P_n or D_n/P_{n-1}) is a (resp. a connected component of) *maximal orthogonal Grassmannian* that is a variety of maximal totally-isotropic linear subspaces in the quadratic space of rank $2n+1$ (resp. $2n$). The variety B_n/P_i, $i < n$ (resp D_n/P_i, $i < n-1$) is a Grassmannian of isotropic linear subspaces of dimension i.

G_2**,** F_4**,** E_6**:** The variety G_2/P_2 is isomorphic to a 5-dimensional smooth projective quadric and the variety G_2/P_1 is a 5-dimensional Fano variety. The variety E_6/P_6 is isomorphic to the so called *Cayley plane* $\mathbb{OP}^2$ that is the octonionic projective plane of dimension 16. The variety F_4/P_4 can be identified with a hyperplane section of E_6/P_6.

4 PHVs in the general (non-split) case

Twisted forms. We recall the notion of a twisted form:

A smooth group scheme G over an arbitrary field k is called a *twisted form* of a linear algebraic group G' if there is an isomorphism of group schemes $G \times_k k_a \simeq G' \times_k k_a$ over the algebraic closure k_a. The descent theory provides a bijection of pointed sets

$$\begin{array}{ccc} H^1(k, \mathrm{Aut}_k(G')) & \simeq & \mathrm{Twist}_k(G') \\ \xi \in Z^1(k, \mathrm{Aut}_k(G')) & \mapsto & G = {}_\xi G' \end{array}$$

between the set of cohomology classes of the first étale cohomology group of the automorphism group scheme $\mathrm{Aut}_k(G')$ and the set of twisted forms of G'.

Since any semisimple linear algebraic group G becomes split over k_a, it can be viewed as a twisted form of the respective Chevalley group $\mathbb{G}$, i.e. as an element of $H^1(k, \mathrm{Aut}_k(\mathbb{G}))$.

Inner and outer forms. Consider an action of the adjoint group $\mathbb{G}^{ad}$ on $\mathbb{G}$ by means of inner automorphisms $\mathbb{G}^{ad} \to \mathrm{Aut}_k(\mathbb{G})$. It induces the map

$$inn\colon\ H^1(k, \mathbb{G}^{ad}) \longrightarrow H^1(k, \mathrm{Aut}_F(\mathbb{G})).$$

A linear algebraic group G is called an *inner form* of $\mathbb{G}$ if it is a twisted form of $\mathbb{G}$ such that the corresponding cohomology class lies in the image of this map.

The isogeny $\mathbb{G}^{sc} \to \mathbb{G}^{ad}$, where $\mathbb{G}^{sc}$ is the simply connected cover of the adjoint form $\mathbb{G}^{ad}$ of $\mathbb{G}$, induces a map

$$sc\colon\ H^1(k, \mathbb{G}^{sc}) \longrightarrow H^1(k, \mathbb{G}^{ad}).$$

Twisted forms that correspond to cohomology classes in the image of the composition $inn \circ sc$ are called *strongly inner forms* of $\mathbb{G}$.

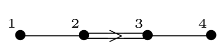
A twisted form of $\mathbb{G}$ is called an *outer form*, if the respective cohomology class is not in the image of the map *inn*. Such cohomology classes correspond to nontrivial automorphisms of the root datum of $\mathbb{G}$ [SGA, Exp. XXIV, Thm.1.3]. In particular, if the Dynkin diagram of $\mathbb{G}$ has no nontrivial graph-automorphisms, then the map *inn* is surjective. This implies that outer forms exist only for groups of types A_n, D_n and E_6.

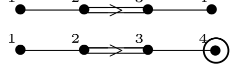
We will write the order of the respective graph-automorphism of the Dynkin diagram of $\mathbb{G}$ as an upper-left index. For example, the notation ${}^1\mathrm{A}_n$ means an inner form of a group of type A_n, and ${}^6\mathrm{D}_4$ means an outer form of a group of type D_4 corresponding to an automorphism of order 6.

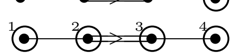
The Tits index. In general, a twisted form G of a Chevalley group $\mathbb{G}$ is not necessary a split group. To measure how far is G from being split one uses the Tits index of G.

Let G be an inner form of $\mathbb{G}$. The *Tits index* of G is a set of Dynkin digrams of $\mathbb{G}$ where certain vertices are circled. Each circled vertex corresponds to the copy of a split one dimensional torus $\mathbb{G}_m$ sitting inside G. For instance, if all vertices are circled then the group G is split. If non of them are circled, then the group G is called *anisotropic*. If at least one of the vertices is circled, then the group G is called *isotropic*. For each type of $\mathbb{G}$ the list of all possible Tits indexes can be found in the table.

Example. A group G of type F_4 can have three possible Tits indexes:

(i) 

(ii) 

(iii) 

where the index (i) corresponds to the anisotropic case, (ii) to the non-split isotropic case and (iii) to the split case.

Description of PHVs in general. Consider a projective homogeneous G-variety X over k. It becomes isomorphic over k_a to a projective homogeneous $\mathbb{G}$-variety $\mathbb{G}/P_\Theta$ corresponding to a standard parabolic subgroup P_Θ of $\mathbb{G}$ and, therefore, to a subset Θ of the Dynkin diagram of $\mathbb{G}$. In other words, X is a twisted form of the respective PHV corresponding to the split group. Moreover, it can be shown that X is, indeed, a projective G-homogeneous variety, where $G = {}_\xi\mathbb{G}$ is a twisted form of $\mathbb{G}$, that is $X \simeq {}_\xi\mathcal{D}/P_\Theta$, where $\mathcal{D}$ is the Dynkin type of $\mathbb{G}$.

Observe that the projective homogeneous G-variety $X = {}_\xi\mathcal{D}/P_\Theta$ has a k-rational point if and only if all non-circled vertices of the Tits index of G belong to the subset Θ. A projective homogeneous variety is rational if and only if it has a rational point.

Examples. We now provide a list of examples of projective homogeneous G-varieties over k, where G is a twisted form of an adjoint split group $\mathbb{G}^{ad}$.

A_n**:** Let $\mathbb{G} = \mathrm{PGL}_{n+1}$ be the projective linear group. The pointed set $H^1(F, \mathrm{PGL}_{n+1})$ is isomorphic to the set of isomorphism classes of central simple algebras A of degree $n+1$ over k. Moreover, cohomology classes in the image of

$$H^1(k, \mathrm{SL}_{n+1}/\mu_r) \longrightarrow H^1(k, \mathrm{PGL}_{n+1}), \quad r \mid n+1,$$

correspond to central simple algebras A of index r. In particular, PGL_{n+1} has no non-trivial strongly inner forms.

For an algebra A the respective inner twisted form G is given by the automorphism group $\mathrm{PGL}_A = \mathrm{Aut}_k A$ of A. The respective projective homogeneous G-variety X can be identified with the variety of flags of (right) ideals in A. For instance, if $X = {}_A(\mathrm{A}_n/P_i)$, then $X \simeq \mathrm{SB}_i(A)$ is the generalized *Severi-Brauer variety* of ideals of reduced dimension i in A.

The set of outer forms of PGL_{n+1} is in one-to-one correspondence with the set of isomorphism classes of central simple algebras with unitary involutions. In this case any projective homogeneous varieties becomes isomorphic over k_a to the variety $\mathrm{A}_n/P_{i_1,\ldots,i_m}$, where $i_j = n+1-i_{m-j+1}$ for all $j = 1\ldots m$.

$\mathrm{B}_n, \mathrm{D}_n$**:** Let $\mathbb{G}$ be the projective orthogonal group. Depending on the type $\mathbb{G}$ is either O^+_{2n+1} or PGO^+_{2n}. We assume $char(k) \neq 2$.

In the first case (B_n) the pointed set $H^1(k, \mathrm{O}^+_{2n+1})$ is isomorphic to the set of isometry classes of quadratic forms q of rank $2n+1$. The respective inner form G is given by the special orthogonal group $\mathrm{O}^+(q)$ of q and the respective projective homogeneous G-variety $X = {}_q(\mathrm{B}_n/P_1)$ is the *projective quadric* given by the equation $q = 0$. Observe that for O^+_{2n+1} all twisted forms are inner.

In the second case (D_n) the pointed set $H^1(k, \mathrm{PGO}^+_{2n})$ is parametrised by the set of isomorphism classes of certain central simple algebras A with orthogonal involutions σ. For an inner form $G = {}_{(A,\sigma)}\mathrm{PGO}^+_{2n}$ the projective homogeneous G-variety $X = {}_{(A,\sigma)}\mathrm{D}_n/P_1$ is a *projective quadric* if the algebra A is split and is an *involution variety* if A is non-split. More precisely, if A is split, then the variety X is given by the equation $q = 0$, where q is the quadratic form of rank $2n$ associated to the involution σ. All outer forms of PGO_{2n}^+ correspond to algebras with involutions which have non-trivial discriminant.

Pfister case: An important example of a twisted inner form of the projective orthogonal group $\mathrm{PGO}_{2^n}^+$ (resp. $\mathrm{O}_{2^n+1}^+$) is given by the so called *Pfister* quadratic form ϕ (resp. its maximal neighbor). By an n-fold Pfister form ϕ we call the tensor product of n binary quadratic forms $\phi = \bigotimes_{i=1}^n \langle 1, -a_i\rangle$, $a_i \in k^\times$, where the notation $\langle 1, -a_i\rangle$ stands for the form $x^2 - a_i y^2$.

5 Equivariant algebraic oriented cohomology theories

We now recall several facts concerning algebraic oriented cohomology theories following [Levine-Morel].

An algebraic oriented cohomology theory is a contravariant functor $\mathtt{h}$ from the category of smooth projective varieties over a field k to the category of commutative unital rings which satisfies certain properties (e.g. localization, homotopy invariance).

Pull-backs and push-forwards. Given a morphism $f\colon X \to Y$ the functorial map $\mathtt{h}(f)$ will be denoted by f^* and called the *pull-back*. One of the characterizing properties of $\mathtt{h}$ says that for any proper map $f\colon X \to Y$ there is an induced map $f_*\colon \mathtt{h}(X) \to \mathtt{h}(Y)$ of $\mathtt{h}(Y)$-modules called the *push-forward* (here $\mathtt{h}(X)$ is an $\mathtt{h}(Y)$-module via f^*).

Characteristic classes. Another characterizing property of an algebraic oriented cohomology theory is the existence of characteristic classes. The latter is a collection of maps $c_i\colon K_0(X) \to \mathtt{h}(X)$, $i \geq 1$ that satisfy the following properties:

Let $c(x) = 1 + c_1(x)t + c_2(x)t^2 + \ldots \in \mathtt{h}(X)[[t]]$ denote the *total characteristic class*. Then

- $c(E) = 1$ for a trivial bundle E over X,
- $c_i(E) = 0$ for a bundle E with $i > rk(E)$,
- $c(E \oplus E') = c(E) \cdot c(E')$ for any two bundles E and E' over X.

Given two line bundles L_1 and L_2 over X, we have

$$c_1(L_1 \otimes L_2) = c_1(L_1) +_F c_1(L_2), \tag{1}$$

where F is a one-dimensional commutative *formal group law* over the coefficient ring $R = \mathtt{h}(Spec(k))$ associated to $\mathtt{h}$.

Formal group laws. A (commutative one-dimensional) formal group law over a commutative unital ring R is a power series in two variables (see [Hazewinkel78])

$$F(u,v) = u + v + \sum_{i,j\geq 1} a_{ij}u^i v^j, \quad a_{ij} = a_{ji}, \quad a_{ij} \in R$$

which satisfies the following axioms:

- (associativity) $F(u, F(v,w)) = F(F(u,v),w)$,
- (commutativity) $F(u,v) = F(v,u)$
- $F(u,0) = 0$.

A morphism of formal group laws $f\colon F \to F'$ over R is a power-series $f \in R[[u]]$ such that $f(F(u,v)) = F'(f(u), f(v))$. For any formal group law F there is a unique power series $\imath_F(u) \in R[[u]]$ called the formal inverse of F which satisfies the identity $F(u, \imath_F(u)) = 0$.

We will use the following notation

$$u +_F v = F(u,v),\ -_F u = \imath_F(u) \text{ and } a \cdot_F u = \underbrace{u +_F u +_F \cdots +_F u}_{a\ times},\ a \geq 1.$$

Examples. (a) The additive formal group law is given by $F_a(u,v) = u + v$.
(b) The multiplicative formal group law is given by $F_m(u,v) = u + v - uv$.
(c) The Lorentz formal group law is given by

$$F_l(u,v) = \tfrac{u+v}{1+uv} = u + v + \sum_{i\geq 1}(-1)^i \left(u^i v^{i+1} + u^{i+1}v^i\right).$$

(d) Let E be an elliptic curve defined by

$$E: \qquad v = u^3 + a_1 uv + a_2 u^2 v + a_3 v^2 + a_4 uv^2 + a_6 v^3,\ a_i \in \mathbb{Z}.$$

The group law on E induces an elliptic formal group law

$$\begin{aligned} F_e(u,v) \quad &= \quad u + v - a_1 uv - a_2(u^2 v + v^2 u) + \\ &\qquad 2a_3(u^3 v + uv^3) + (a_1 a_2 - 3a_3)u^2 v^2 + O(5). \end{aligned}$$

Algebraic cobordism. There is a universal formal group law F_u. Its coefficient ring, called the Lazard ring $\mathbb{L}$, is generated by coefficients a_{ij} modulo the relations which come from the axioms of the formal group law. Any commutative one-dimensional formal group law over a ring R corresponds to a ring homomorphism from $\mathbb{L}$ to R (which specializes the coefficients).

There is the respective algebraic oriented cohomology theory Ω defined over a field of characteristic zero, called *algebraic cobordism*, that is universal in the following sense: Given any algebraic oriented cohomology theory $\mathtt{h}$ there is a unique morphism $\Omega \to \mathtt{h}$ of algebraic oriented theories The formal group law associated to Ω is exactly the universal formal group law F_u.

To a formal group law F over R one can associate an algebraic oriented cohomology theory $\mathtt{h}_F$ via

$$\mathtt{h}_F(-) = \Omega(-) \otimes_{\mathbb{L}} R$$

So there is a '1-1' correspondence (the cohomology theory correspoding to F will be universal among all the theories with the formal group law F):

$$\boxed{\text{formal group laws}} \overset{1-1}{\longleftrightarrow} \boxed{\text{Algebraic oriented cohomology theories}}.$$

Under this correspondence the addiitve formal group law F_a corresponds to Chow theory and the multiplicative F_m to Grothendieck's K_0.

Observe that for each formal group law there always exists a corresponding algebraic oriented theory but not necessary a topological one (possible example is Lorentz).

Equivariant pretheories. We follow now [Gille-Zainoulline]. Let G be an algebraic group over k. Consider a contravariant functor $\mathtt{h}_G$ from the category of smooth G-varieties over k to the category of commutative rings. Given G-varieties X and Y and a G-equivariant map $f\colon X \to Y$ the induced map $\mathtt{h}_G(Y) \to \mathtt{h}_G(X)$ is called a pull-back and is denoted by f_G^* .

The functor $\mathtt{h}_G$ is called a G-equivariant pretheory over k if it satisfies the following two axioms:

- (homotopy invariance) For a G-equivariant map $p\colon \mathbb{A}_k^n \to pt = Speck$ (where G acts trivially on pt) the induced pull-back $p_G^*\colon \mathtt{h}_G(pt) \to \mathtt{h}_G(\mathbb{A}_k^n)$ is an isomorphism.
- (localization) For a smooth G-variety X and a G-equivariant open embedding $i\colon U \to X$ the induced pull-back $i_G^*\colon \mathtt{h}_G(X) \to \mathtt{h}_G(U)$ is surjective.

We then extend this definition to an equivariant pretheory over the category of fields over k by introducing the restriction map

$$res_{l/k}^X\colon \mathtt{h}_G(X) \to \mathtt{h}_G(X_l) = \mathtt{h}_{G_l}(X_l)$$

which commutes with pull-backs and is compatible with taking towers of field extension.

We define $\bar{\mathtt{h}}_G(X) := \lim_{l/k} \mathtt{h}_G(X_l)$ where the colimit is taken over all field extensions. We denote the canonical morphism $\mathtt{h}_G(X) \to \bar{\mathtt{h}}_G(X)$ by $r\bar{e}s_k$.

Equivariant oriented B-theories. Let $\mathtt{h}$ be an algebraic oriented cohomology theory. Let G be a split linear algebraic group. Let $T \subset B \subset G$ be a maximal torus containing a Borel subgroup. Let $\mathtt{h}_B$ be a B-equivariant pretheory such that

- $\mathtt{h}_B(E) = \mathtt{h}(E/B)$ for every G-torsor E;
- $\bar{\mathtt{h}}_B(B) = \mathtt{h}(pt)$ and $\bar{\mathtt{h}}_B(G) = \mathtt{h}(G/B)$

Then we call $\mathtt{h}_B$ a B-equivariant oriented theory over k.

Examples. Equivariant K-theory [Thomason], equivariant Chow groups [Totaro] and equivariant cobordism [Deshpande, Krishna, Heller, Malagon-Lopez] provide examples of such theories. An equivariant cycle homology (using Rost cycle modules) [Gille-Zainoulline].

6 $\mathtt{h}$-invariants of torsors

Let G be a split linear algebraic group over k. Let $\phi\colon G \to Spec\, k$ be the structure map. Let E be a (left) G-variety over k and let $\eta\colon Spec\, K \to E$ be its generic point. Consider the G-equivariant map

$$\psi\colon G_K = G \times_k K \to G \times_k E \to E$$

which takes the identity of G to the generic point of E.

Characteristic vs restriction. Let $\mathtt{h}_B$ be a B-equivariant oriented theory, where B is a Borel subgroup of G. Then one of the main results of [Gille-Zainoulline] says that

Theorem. We have in $\bar{\mathtt{h}}_B(G) = \mathtt{h}(G/B)$

$$im(r\bar{e}s_k \circ \phi_B^*) \subseteq im(r\bar{e}s_k \circ \psi_B^*).$$

In particular, for a G-torsor E this implies that the image of the restriction map

$$res = r\bar{e}s_k \circ \psi_B^*\colon \mathtt{h}(E/B) = \mathtt{h}_B(E) \to \mathtt{h}_B(G) = \mathtt{h}(G/B)$$

always contains the image of the characteristic map

$$c = r\bar{e}s_k \circ \phi_B^*\colon \mathtt{h}_B(pt) \to \mathtt{h}_B(G) = \mathtt{h}(G/B)$$

and, moreover, there exists a torsor E for which the images coincide. Such a torsor will be called generic.

In the case $\mathtt{h} = \mathrm{CH}$ (the Chow group) this is a result by [Karpenko-Merkurjev].

Cohomology of a group. Now consider the action of T on the affine space $\mathbb{A}_k^n$ with weights $\omega_1, \ldots, \omega_n$ together with an action of T on G by left multiplication. Then T embeds into $\mathbb{A}_k^n$ as the complement of the coordinate hyperplanes Z_i, $i = 1 \ldots n$. Let $V = \mathbb{A}_k^n \times^T G$ be the associated vector bundle over G/T. By definition $V = \mathcal{L}_{G/T}(\omega_1) \oplus \ldots \oplus \mathcal{L}_{G/T}(\omega_n)$ and $G = T \times^T G$ embeds into V as the complement of the union of zero-sections [Brion]

$$V_j = \bigoplus_{j \neq i} \mathcal{L}_{G/T}(\omega_i) = Z_j \times^T G.$$

This allows to prove (using the homotopy invariance and localization) that

$$\mathtt{h}(G) \simeq \mathtt{h}(G/B)/\big(c_1(\mathcal{L}_{G/B}(\omega_1)), \ldots, c_1(\mathcal{L}_{G/B}(\omega_n))\big).$$

The characteristic map $c\colon \mathtt{h}_B(pt) \to \mathtt{h}(G/B)$ can be also defined by taking the characteristic classes of line bundles associated to characters of a split maximal torus T of G (we will discuss this map in details later).

In view of this fact we can identify the ideal generated by the first characteristic classes by the ideal generated by augmented (non-constant) elements of the image of c. In other words, there is an exact sequence (in the augmented sense)

$$\mathtt{h}_B(pt) \xrightarrow{c} \mathtt{h}(G/B) \to \mathtt{h}(G) \to 0.$$

where the second map is the pull-back induced by $G \to G/B$.

$\mathtt{h}$-invariant. The latter says that the cokernel (in the augmented sense)

$$\mathtt{h}(G/B)/(im\, res_{K/k+})$$

of the restriction map $res_{K/k}$ is a quotient of $\mathtt{h}(G)$. Moreover, for generic torsors it coincides with $\mathtt{h}(G)$ and for trivial torsors it is trivial. We call this cokernel by the $\mathtt{h}$-invariant of a G-torsor E.

Examples. Take $\mathtt{h} = \mathrm{CH}$ with $\mathbb{F}_p$-coefficients. $\mathrm{CH}(G; \mathbb{F}_p)$ is a cocommutative Hopf algebra over a finite field so it has to be of the form

$$\bigotimes_{i=1}^{r} \mathbb{F}_p[x_i]/(x_i^{p^{k_i}})$$

for some integers k_i, where the element x_i has degree d_i. All these parameters depend only on the root datum of G and were computed by [V. Kac].

It follows that the CH-invariant of E (as a quotient of $\mathrm{CH}(G; \mathbb{F}_p)$) must have similiar form (as comodule)

$$\bigotimes_{i=1}^{r} \mathbb{F}_p[x_i]/(x_i^{p^{j_i}})$$

for some $j_i \leq k_i$, $i = 1 \dots r$. The r-tuple of positive integers $(j_1, \dots, j_r)$ is precisely the J-invariant of E as was defined in the motivic context in [Petrov-Semenov-Zainoulline].

Question 1: What about other cohomology theories $\mathtt{h}$, say Lorentz, elliptic, Lubin-Tate ? Does the $\mathtt{h}$-invariant give an interesting (discrete) invariant of a torsor ?

Question 2: How to compute $\mathtt{h}(G)$ using the exact sequence above ?

Question 3: Does the theorem extend to outer forms (say, to quasi-split groups) or to other G-varieties ?

7 Motives of generically split PHV's

We explain now how the CH-invariant (the J-invariant) gives rise to a motivic decomposition of a generically split projective homogeneous variety. We follow [Petrov-Semenov-Zainoulline].

Characteristic decomposition of $\mathrm{CH}(G/B)$**.** Let G be a split semisimple linear algebraic group over k. We fix $T \subset B \subset G$ a split maximal torus together with a Borel subgroup of G. Let $A = \mathrm{CH}(G/B; \mathbb{F}_p)$ denote the Chow ring of G/B with $\mathbb{F}_p$-coefficients and let $C = im\, c$ denote the characteristic subring of A.

Consider the pull-back $A = \mathrm{CH}(G/B; \mathbb{F}_p) \to \mathrm{CH}(G; \mathbb{F}_p)$. Pick up the generators $\{x_1, \dots, x_r\}$ of $\mathrm{CH}(G, \mathbb{F}_p)$. Let $\{e_1, \dots, e_r\}$ be their preimages in A. Observe that the ring A is a free C-module with basis consisting of all products of e_i (proven by comparing the Poincare polynomials). Given an r-tuple of non-negative integers $M = (m_1, \dots, m_r)$ we denote by e^M the product $e_1^{m_1} \cdot \ldots \cdot e_r^{m_r}$. Let $N = (p^{k_1} - 1, \dots, p^{k_r} - 1)$ (so that e^N is the maximal element with respect to the induced lexicographic order on the basis elements).

Using the grading on CH and the dimension axiom ($\mathrm{CH}^i(X) = 0$ if $i > \dim X$) we can show that

Lemma. (Shifted Duality) The pairing

$$C \times C \to \mathbb{F}_p, \qquad (x, y) \mapsto \deg(e^N xy)$$

is non-degenerated, i.e. for any non-zero element $x \in C$ there exists $y \in C$ such that $\deg(e^N xy) \neq 0$.

We fix a (homogeneous) basis $\{x_i\}_{i \in I}$ of C and the dual basis $\{x_i^\vee\}_{i \in I}$ (with respect to the pairing). We denote

$$\rho_i = e_i \otimes 1 + 1 \otimes e_i \in \mathrm{CH}(G/B \times G/B; \mathbb{F}_p) \simeq Hom_{\mathbb{F}_p}(A, \mathbb{F}_p) \otimes A = End_{\mathbb{F}_p}(A).$$

Observe that ρ_i's correspond to primitive elements in $\mathrm{CH}(G; \mathbb{F}_p) \otimes \mathrm{CH}(G; \mathbb{F}_p)$. As before, we denote $\rho^M = \rho_1^{m_1} \cdot \ldots \cdot \rho_r^{m_r}$.

Theorem. (Direct sum decomposition) The elements

$$\{\rho^M \cdot (x_i \otimes x_i^\vee)\}_{M\leq N, i\in I}$$

form a pairwise-orthogonal system of idempotents in the ring $(End_{\mathbb{F}_p}(A),' \circ')$.

The case of a generic torsor. Now assume that E is a generic G-torsor. Consider the variety E/B. We know that $im\, res = im\, c$, where

$$res\colon \mathrm{CH}(E/B;\mathbb{F}_p) \to \mathrm{CH}((G/B)_{k_a};\mathbb{F}_p) = A,$$

is the restriction map, so the elements $x_i \otimes x_i^\vee \in im(res)$, i.e. are defined over the base field k.

We know that E/B is generically split, meaning that G becomes a split group over the function field of E/B. Using this fact we can show that each $\rho_i \in im(res)$. This immediately implies that all idempotents appearing in the theorem belong to the image of the restriction map.

General case. For arbitrary E we replace N by $(p^{j_1}-1,\ldots,p^{j_r}-1)$ and proceed as in the generic case.

Motivic decompositions. A family of pairwise orthogonal idempotents in $End_{\mathbb{F}_p}(A)$ defined over k (that belong to the image of res) gives rise (via the Rost nilpotence theorem) to a family of parwise orthgonal idempotents over k, hence, to a direct sum decomposition of the Chow motive of $M(E/B;\mathbb{F}_p)$.

In this way we obtain the main result of [Petrov-Semenov-Zainoulline]

Theorem. Given a G-torsor E the Chow-motive $M(E/B;\mathbb{F}_p)$ splits as a direct sum of indecomposable motives

$$M(E/B) \simeq \bigoplus_{i\geq 0} R_{p,E}(i)^{\oplus n_i}$$

where the Poincare polynomial (the coefficient at t^l of the Poincare polynomial equals to the number of Tate motives in codimension l) of $M(R_{p,E}\times_k k_a)$ is given by

$$P(M(R_{p,E}\times_k k_a),t) = \prod_{i=1}^r \frac{1-t^{d_i p^{j_i}}}{1-t^{d_i}}$$

and the integers c_i are the coefficients of the quotient

$$P(A,t)/P(M(R_{p,E}\times_k k_a),t).$$

The motives $R_{p,E}$ satisfy the following properties:

$$cd_p(E/B) = \dim R_{p,E} = \deg P(M(R_{p,E}\times_k k_a),t) = \sum_{i=1}^r d_i(p^{j_i}-1).$$

In particular, $R_{p,E} = \mathbb{F}_p$ is Tate iff $j_1 = \ldots = j_r = 0$ iff E/B contains a closed point of degree coprime to p.

If E is generic, then it gives the formula for the canonical p-dimension of G that is

$$cd_p(G) = \mathrm{cd}_p(E/B) = \sum_{i=1}^{r} d_i(p^{k_i} - 1).$$

Following the same approach as for Chow groups (replacing the symbol CH by $\mathtt{h}$ evrywhere in the defintion of Grothendieck-Chow motives) one can define the pseudo-abelian category of $\mathtt{h}$-motives. The result of Vishik-Yagita says that there is 1-1 correspondence between isomorphism classes of Ω-motives and CH-motives.

Question 1: Can one obtain ($\mathtt{h}$-)motivic decompositions of the $\mathtt{h}$-motive of E/B in a similar way ?

Question 2: What is the $\mathtt{h}$-analogue (say $\mathtt{h} = K_0$) of the canonical dimension of G ?

8 The characteristic map

We are now going to define and to describe the characteristic map

$$c\colon \mathtt{h}_B(pt) \to \mathtt{h}_B(G)$$

in general. The key idea is that this map should depend only on the root datum (Σ, Λ) and the formal group law F. We follow [Calmes-Petrov-Zainoulline].

Assume we are given a root datum (Λ, Σ) and a formal group law F.

Formal group algebra. We first define $R[x_\Lambda] := R[x_\lambda,\ \lambda \in \Lambda]$ to be the polynomial ring over a ring R with variables indexed by Λ. Let $\epsilon\colon R[x_\Lambda] \to R$ be the augmentation morphism which maps any x_λ to 0. Consider the $\ker(\epsilon)$-adic topology on $R[x_\Lambda]$ given by ideals $\ker(\epsilon)^i$, $i \geq 0$, which form a fundamental system of open neighborhoods of 0. We define $R[[x_\Lambda]]$ to be the $ker(\epsilon)$-adic completion of the polynomial ring $R[x_\Lambda]$. We then set

$$R[[\Lambda]]_F := R[[x_\Lambda]]/(x_0, x_{\lambda+\mu} - F(x_\lambda, x_\mu)).$$

and call it the formal group algebra. One may think of $R[[\Lambda]]_F$ as of $\mathtt{h}_B(pt)$.

Demazure operators. The Weyl group W of the root datum acts on the Λ by means of reflections. This induces an action on $R[[\Lambda]]_F$.

Assume that all x_α, $\alpha \in \Sigma$, are regular in $R[[\Lambda]]_F$, then we can define an R-linear operator on $R[[\Lambda]]_F$

$$\Delta_\alpha(u) := \frac{u - s_\alpha(u)}{x_\alpha}$$

and call it the formal Demazure operator. The operator Δ_α can be viewed as a 'twisted' differential operator as it is satisfies the 'twisted' Leibniz rule

$$\Delta_\alpha(uv) = \Delta_\alpha(u)v + s_\alpha(u)\Delta_\alpha(v).$$

The algebra of Demazure operators. Consider the R-subalgebra D_F of $End_R(R[[\Lambda]]_F)$ generated by all formal Demazure operators and left multiplications by elements of $R[[\Lambda]]_F$. Composing with the augmentation map $\epsilon\colon R[[\Lambda]]_F \to R$ we obtain the subalgebra ϵD_F of the algebra of R-linear functions (with the usual product and sum of functions).

Take its dual $\epsilon D_F^* = Hom_R(\epsilon D_F, R)$ and consider the evaluation map

$$eval\colon R[[\Lambda]]_F \to \epsilon D_F^*, \quad u \mapsto \text{ evaluation at } u.$$

Then we have

Theorem. If $\mathtt{h}$ is a weakly birational algebraic oriented cohomology theory, then

$$\mathtt{h}(G/B) \simeq \epsilon D_F^*$$

and the characteristic map c coincides with the evaluation map $eval$.

Moreover, if the torsion index of the root datum is invertible, then the kernel of c coincides with the augmented ideal of invariants $(R[[\Lambda]]_{F\,+}^W)$.

Formal Schubert Calculus. One can show that the algebra ϵD_F is a free $R[[\Lambda]]_F$-module with basis $\epsilon\Delta_{I_w}([pt])$, where $w \in W$ and I_w is the reduced decomposition of w. The dual basis in ϵD_F^* corresponds to the classes of Bott-Samelson resolutions of Schubert varieties for I_w, $w \in W$.

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