Avoiding Circular Repetitions

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Challenges in Combinatorics on Words
Fields Institute

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Avoidability

- Is *ABCBABC avoidable*?
- Is it 3-*avoidable*?
- Smallest n for which P is n-avoidable?
- Decidability
- Over *circular words* (necklaces)
- **Smallest avoidable exponent (repetition threshold)**

Repetitions are popular patterns

- *k-power*: x^k
- x^2 is 3-avoidable but not 2-avoidable (Thue).
- *α-power*: $y = x^{\lfloor \alpha \rfloor} x'$ such that $\frac{|y|}{|x|} = \alpha$. We then write

$$y = x^{\alpha}.$$

Examples

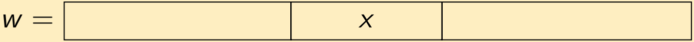
- $\texttt{hotshots} = (\texttt{hots})^2$
- $\texttt{alfalfa} = (\texttt{alf})^2 a = (\texttt{alf})^{\frac{7}{3}}$

Definition

- w is *α-power-free* if none of its factors is a β-power for any $\beta \geq \alpha$.
- w is *α^+-power-free* if none of its factors is a β-power for any $\beta > \alpha$.

Observation

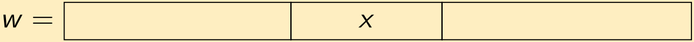
w is α-power-free if for every factor x



x is α-power-free

Definition

w is *circularly α-power-free* if for every factor x



x and all its conjugates are α-power-free.

Circular repetition: example

Example

$$w = \texttt{dividing}$$
$$x = \texttt{dividi}$$

a conjugate of x is

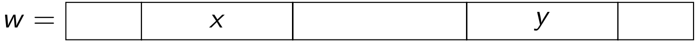
$$\texttt{vididi}$$

which has a $\frac{5}{2}$-power: $\texttt{ididi} = (\texttt{id})^{\frac{5}{2}}$

- So w is not circularly $\frac{5}{2}$-power-free.
- In fact, w is circularly $(\frac{5}{2})^{+}$-power-free.

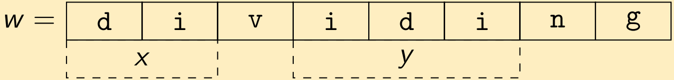
Circular repetition

- w is circularly α-power-free if for every pair of factors x and y



  yx is α-power-free.
- (x, y) is a circular α-power if yx is α-power.

Example



$$yx = \texttt{ididi} = (\texttt{id})^{\frac{5}{2}}$$

Hence (x, y) is a circular $\frac{5}{2}$-power.

Repetition threshold

Definition

The *repetition threshold*, RT(n), is the smallest α for which there exists an infinite α^+-power-free word over Σ_n.

Dejean's conjecture

Thue, Dejean, Pansiot, Moulin Ollagnier, Carpi, Currie, Mohammad-Noori, Rampersad, and Rao:

$$\mathrm{RT}(n) = \begin{cases} \frac{7}{4}, & \text{if } n = 3; \\ \frac{7}{5}, & \text{if } n = 4; \\ \frac{n}{n-1}, & \text{if } n \neq 3, 4. \end{cases}$$

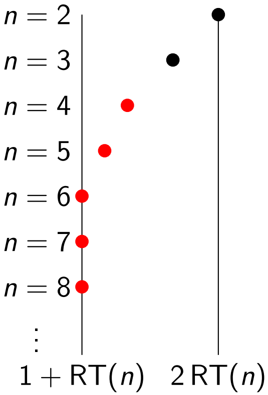
Repetition threshold for circular factors

Definition

The *repetition threshold for circular factors*, RTC(n), is the smallest α for which there exists an infinite circularly α^+-power-free word over Σ_n.

n	RT(n)	RTC(n)
2	2	4
3	$\frac{7}{4}$	$\frac{13}{4}$
4	$\frac{7}{5}$	$\frac{5}{2}$
5	$\frac{5}{4}$	$\frac{105}{46}$
6	$\frac{6}{5}$	$1 + \frac{6}{5} = \frac{11}{6}$
$\vdots$	$\vdots$	$\vdots$
k	$\frac{k}{k-1}$	$1 + \mathrm{RT}(k) = \frac{2k-1}{k-1}$

Bounds on RTC(n)



Theorem

$1 + \mathrm{RT}(n) \leq \mathrm{RTC}(n) \leq 2\,\mathrm{RT}(n)$

Thue-Morse word and RTC(2)

- Thue morphism

$$h(0) = 01$$
$$h(1) = 10.$$

- The *Thue-Morse word*

$$\mathbf{t} = h^{\omega}(0) = 01101001\cdots$$

is 2^+-power-free.

Theorem

$\mathbf{t}$ *is circularly* 4^+*-power-free.*

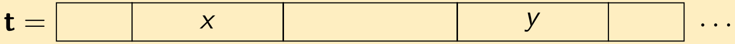
Thue-Morse word and RTC(2)

Theorem

$\mathbf{t}$ *is circularly* 4^+*-power-free.*

Proof.

- Suppose (x, y) is a circular 4^+-power of $\mathbf{t}$, i.e.,



  and yx is a 4^+-power.
- Then either y or x is a 2^+-power, a contradiction.

□

$\mathrm{RTC}(2) = 4$

Theorem

$\mathrm{RTC}(2) = 4$.

Proof.

- Since $\mathbf{t}$ is circularly 4^+-power-free, we have

$$\mathrm{RTC}(2) \leq 4.$$

- No binary word of length 12 is circularly 4-power-free, so

$$\mathrm{RTC}(2) \geq 4.$$

□

RTC(3) $= \frac{13}{4}$

Overview of the proof:

- A finite search shows that RTC(3) $\geq \frac{13}{4}$.
- So to prove RTC(3) $= \frac{13}{4}$, we just need to construct an infinite word that is circularly $(\frac{13}{4})^+$-power-free.
- We give a pair of morphisms:

$$\psi : \Sigma_6^* \to \Sigma_6^*$$

$$\mu : \Sigma_6^* \to \Sigma_3^*$$

- We prove $\mu(\psi^\omega(0))$ is circularly $(\frac{13}{4})^+$-power-free.

Pair of morphisms

$$\psi(0) = 0435$$
$$\psi(1) = 2341$$
$$\psi(2) = 3542$$
$$\psi(3) = 3540$$
$$\psi(4) = 4134$$
$$\psi(5) = 4105.$$

$$\mu(0) = 012102120102012$$
$$\mu(1) = 201020121012021$$
$$\mu(2) = 012102010212010$$
$$\mu(3) = 201210212021012$$
$$\mu(4) = 102120121012021$$
$$\mu(5) = 102010212021012.$$

$\mu(\psi^\omega(0))$

Theorem

$\mu(\psi^\omega(0))$ *is circularly* $(\frac{13}{4})^+$*-power-free.*

Proof idea

The proof has two parts

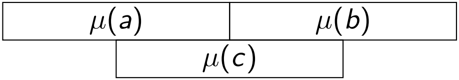
1. $\mathbf{r} = \psi^\omega(0)$ is circularly cubefree.
2. $\mathbf{s} = \mu(\mathbf{r})$ is circularly $(\frac{13}{4})^+$-power-free.
    1. $\mathbf{s}$ has no short circular $(\frac{13}{4})^+$-power. (This is checked by computer)
    2. $\mathbf{s}$ has no long circular $(\frac{13}{4})^+$-power.

μ is well-behaved!

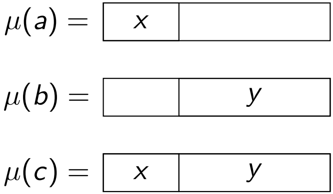
- $\mu : \Sigma_6^* \to \Sigma_3^*$ is 15-*uniform*

$$|\mu(a)| = 15 \text{ for all } a \in \Sigma_6.$$

- μ is *synchronizing*, i.e., for no $a, b, c \in \Sigma_6$



- μ is *strongly synchronizing*, i.e., for all $a, b, c \in \Sigma_6$ and $x, y \in \Sigma_3^*$ if

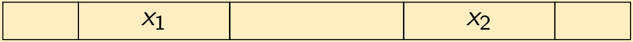


either $c = a$ or $c = b$.

Main lemma

Lemma

- *Let ϕ be a strongly synchronizing q-uniform morphism.*
- *Let w be a circularly cubefree word.*
- *If (x_1, x_2) is a circular $(\frac{13}{4})^+$-power in $\phi(w)$, i.e.,*

$\phi(w) =$ 

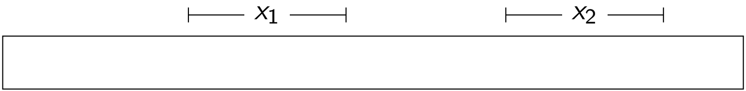
$x_2 x_1$ is $(\frac{13}{4})^+$-power, then

$$|x_2 x_1| < 22q.$$

Main lemma: proof

- Proof is by contradiction.
- Suppose (x_1, x_2) is a circular $(\frac{13}{4})^+$-power of $\phi(w)$, and $|x_2 x_1| \geq 22q$.

$\phi(w) =$



Main lemma: proof

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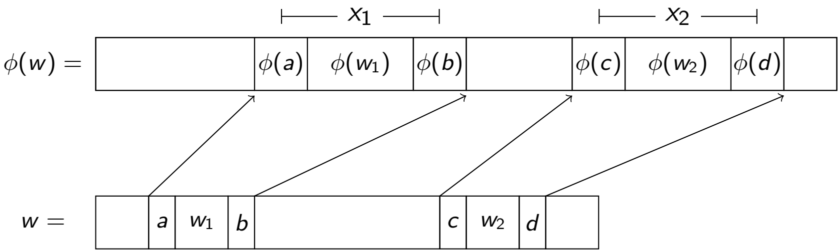
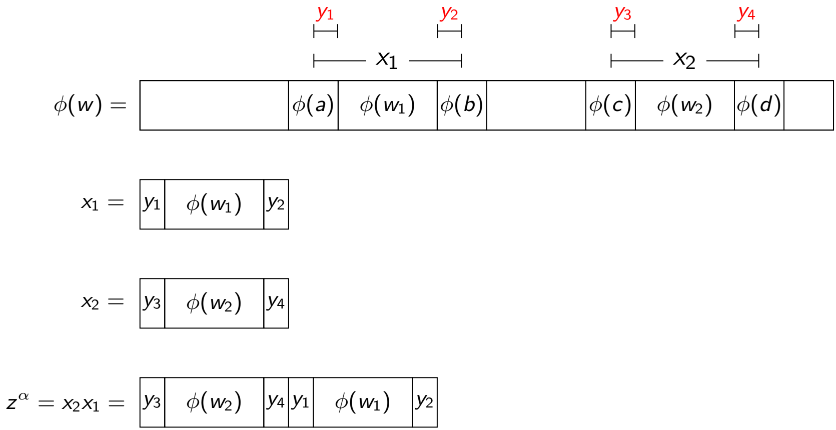


Main lemma: proof



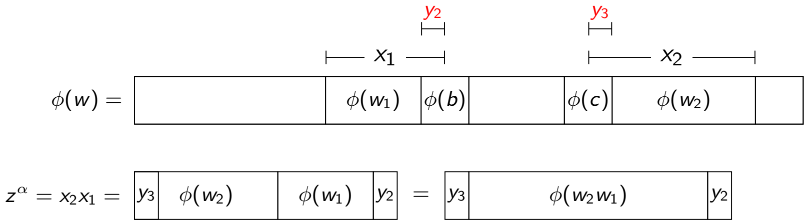
Here z is a word and $\alpha > \frac{13}{4}$. There are two cases to consider:

1. $y_4 y_1 = \epsilon$
2. $y_4 y_1 \neq \epsilon$

Main lemma: case 1

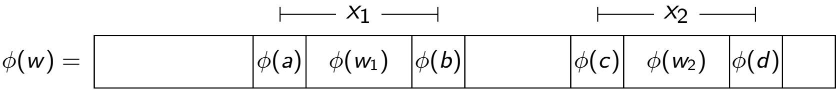
Case 1 is $y_4 y_1 = \epsilon$. Therefore we have

$$x_1 = \phi(w_1) y_2,$$
$$x_2 = y_3 \phi(w_2).$$

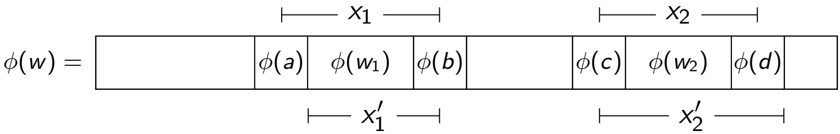


- Note that $\alpha > \frac{13}{4} = 3.25$.
- We get that $\phi(w_2 w_1)$ contains a cube.
- $w_2 w_1$ contains a cube. (synchronizing)
- w contains a circular cube, a contradiction.

Main lemma: case 2



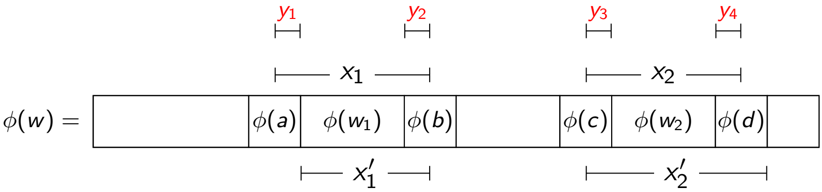
- We would like to show that by shrinking x_1 and enlarging x_2 we can get another circular $(\frac{13}{4})^+$-power of the same length:



- Then clearly (x_1', x_2') falls into case 1.

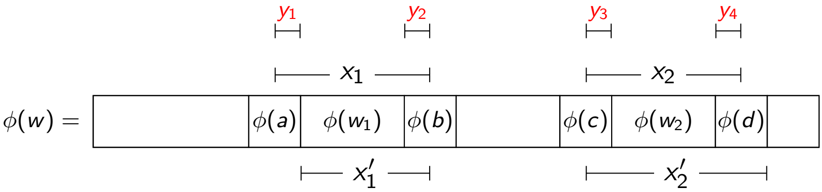
Main lemma: case 2

- Goal: $x_2' x_1' = x_2 x_1$

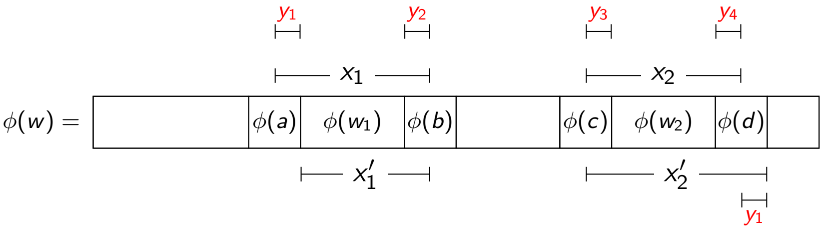


Main lemma: case 2

- Goal: $x_2'x_1' = x_2x_1$

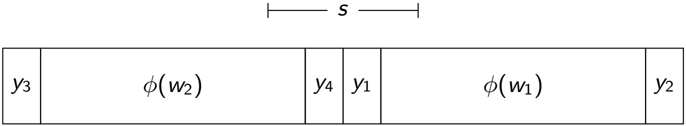


- We just need to show that $\phi(d) = y_4y_1$.



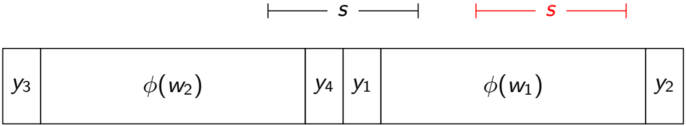
- Let $s = \phi(e)y_4y_1\phi(f)$, where e is the last letter of w_2 and f is the first letter of w_1.

$z^\alpha = x_2x_1 =$



- s also appears in $\phi(w_1)$:

$z^\alpha = x_2x_1 =$



Main lemma: case 2

- Since ϕ is synchronizing, the word y_4y_1 is the complete image of a letter:

$s =$			$\phi(e)$	y_4y_1	$\phi(f)$	
$\phi(w) =$	$\phi(w[0])$	$\cdots$	$\phi(w[i])$	$\phi(w[i+1])$	$\phi(w[i+2])$	$\cdots$

- Recall that y_4 is a prefix of $\phi(d)$ and y_1 is suffix of $\phi(a)$.

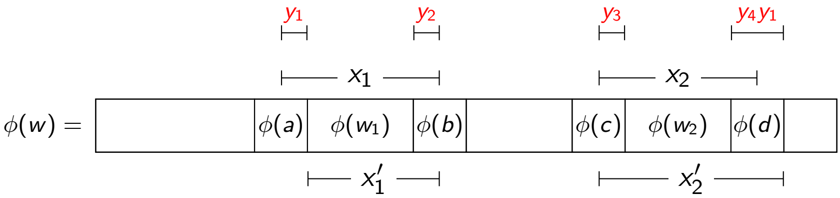
$\phi(d) =$	y_4	
$\phi(a) =$		y_1
$\phi(w[i+1]) =$	y_4	y_1

- Since ϕ is strongly synchronizing, we have either

$$y_4y_1 = \phi(d) \text{ or}$$
$$y_4y_1 = \phi(a).$$

Main lemma: case 2

- Without loss of generality, we can assume $y_4 y_1 = \phi(d)$.



- So we can write $x_2' x_1' = x_2 x_1$.
- (x_1', x_2') falls into case 1.
- So we get a contradiction.

Open problem 1

- Prove or disprove

$$\mathrm{RTC}(4) = \frac{5}{2}\ ,$$
$$\mathrm{RTC}(5) = \frac{105}{46}\ \text{, and}$$
$$\mathrm{RTC}(n) = 1 + \mathrm{RT}(n) = \frac{2n-1}{n-1} \text{ for } n \geq 6.$$

Open problem 2: generalized circular repetitions

- We can study repetition avoidance in the products of factors of words.
- Let RT_k denote the repetition threshold for this new problem, where k is the number of factors we take into consideration.
- We can easily prove
$$\mathrm{RT}_k(2) = 2k.$$
- It would be interesting to obtain more values of $\mathrm{RT}_k(n)$.
- Conjecture:
$$\mathrm{RT}_2(n) = \mathrm{RTC}(n)$$
- For large integers n, we conjecture that
$$\mathrm{RT}_k(n) = k - 1 + \mathrm{RT}(n).$$

Open problem 3: algorithmic problems

- For a finite word w, define the circular exponent, $\mathrm{cexp}(w)$, to be

$$\mathrm{cexp}(w) = \max\{\alpha : \text{ w has a circular } \alpha\text{-power}\}.$$

  Is $\mathrm{cexp}(w)$ computable in linear time?
- Given α and w, can we compute in linear time whether w avoids circular α-powers?