Avoiding k-abelian cube/square

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k-abelian equivalence

Two words u and v are k-abelian equivalent if:

- for every w with $|w| \leq k$, $|u|_w = |v|_w$.

Equivalently:

- for every w with $|w| = k$, $|u|_w = |v|_w$
  and $u[1 : k-1] = v[1 : k-1]$
  and $u[|u| - k + 2 : |u|] = v[|v| - k + 2 : |v|]$.

pqr is a k-abelian cube if p, q, r are pairwise k-abelian equivalent.

Problems # 16

Problem

Are 2-abelian cube avoidable on a binary alphabet ?

Problem

For which k are k-abelian squares avoidable on a ternary alphabet ?

Problems # 16

Problem

Are 2-abelian cube avoidable on a binary alphabet ?

$\Rightarrow$ Yes

Problem

For which k are k-abelian squares avoidable on a ternary alphabet ?

$\Rightarrow k = 3$

Let $h : \Sigma_3^* \to \Sigma_2^*$ be the following 47-uniform morphism.

$$h : \begin{cases} 0 \to 00100101001011001001010010011001001100101101011 \\ 1 \to 00100110010011001101100110110010011001101101011 \\ 2 \to 00110110101101001011010110100101001001101101011 \end{cases}$$

Theorem

For every abelian-cube-free word $w \in \Sigma_3^*$, $h(w)$ is 2-abelian-cube-free.

Sketch of proof (1)

- $\psi(w) = (|w|_0, |w|_1, |w|_2)^T$.
- $\psi'(w) = (|w|_{00}, |w|_{01}, |w|_{11})^T$.
- Let M s.t. $M\psi(x) = \psi'(h(x)0)$ for every $x \in \Sigma_3$.

$$M = \begin{pmatrix} 10 & 9 & 5 \\ 16 & 14 & 17 \\ 5 & 10 & 8 \end{pmatrix}.$$

- We have $\psi'(h(w)0) = M\psi(w)$ for every word $w \in \Sigma_3^*$.
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$$M^{-1} = \begin{pmatrix} \frac{58}{517} & \frac{2}{47} & \frac{-83}{517} \\ \frac{43}{517} & \frac{-5}{47} & \frac{90}{517} \\ \frac{-90}{517} & \frac{5}{47} & \frac{4}{517} \end{pmatrix}.$$

Sketch of proof (2)

- $w = ap'bq'cr'd, \quad p = p_1p_2p_3, \quad q = q_1q_2q_3, \quad r = r_1r_2r_3,$
  $h(a) = up_1, \quad h(p') = p_2, \quad h(b) = p_3q_1, \quad h(q') = q_2,$
  $h(c) = q_3r_1, \quad h(r') = r_2, \quad h(d) = r_3v.$
- General case: u p_1, p_3, q_1, q_3, r_1, r_3 and v are not empty. (Other cases are treated similarly.)
- Since pqr is a 2-abelian-cube, the first letter of p_1, q_1, r_1 has the same first letter, and p_3, q_3, r_3 has the same last letter.
- We have $\psi'(p) = \psi'(p_1p_3) + M\psi(p')$,
  $\psi'(q) = \psi'(q_1q_3) + M\psi(q')$ and $\psi'(r) = \psi'(r_1r_3) + M\psi(r')$
- The condition $\psi'(p) = \psi'(q) = \psi'(r)$ become:
  $M^{-1}\psi'(p_1p_3) + \psi(p') = M^{-1}\psi'(q_1q_3) + \psi(q') =$
  $M^{-1}\psi'(r_1r_3) + \psi(r')$.
- Note: this condition cannot be fulfilled if $M^{-1}(\psi'(p_1p_3) - \psi'(q_1q_3))$, or $M^{-1}(\psi'(p_1p_3) - \psi'(r_1r_3))$ are not integer vectors.

Sketch of proof (3)

By computer, we check that for every $a, b, c, d \in \Sigma$, and for every non-empty $u, p_1, p_3, q_1, q_3, r_1, r_3, v \in \Sigma^*$ such that:

- $up_1 = h(a)$, $p_3 q_1 = h(b)$, $q_3 r_1 = h(c)$, $r_3 v = h(d)$,
- p_1, q_1, r_1 have the same first letter,
- p_3, q_3, r_3 have the same last letter,
- $M^{-1}(\psi'(p_1 p_3) - \psi'(q_1 q_3))$ and $M^{-1}(\psi'(p_1 p_3) - \psi'(r_1 r_3))$ are integer vectors,

then one of the following condition holds:

- $\psi(ap') = \psi(bq') = \psi(cr')$, that is:

$$M^{-1}\psi'(p_1 p_3) - \psi(a) = M^{-1}\psi'(q_1 q_3) - \psi(b) = M^{-1}\psi'(r_1 r_3) - \psi(c),$$

- $\psi(p'b) = \psi(q'c) = \psi(r'd)$,
- $\psi(ap') = \psi(bq') = \psi(cr'd)$,
- $\psi(ap') = \psi(bq'c) = \psi(r'd)$.

In all cases, w has an abelian cube. Contradiction.

Let $h : \Sigma_3^* \to \Sigma_2^*$ be the following morphism.

$$h : \begin{cases} 0 \to 00100101001100100101001001100100110011011 \\ 1 \to 010110110011011001100100110011011 \\ 2 \to 0101101001010010110011011 \end{cases}$$

Theorem

For every abelian-cube-free word $w \in \Sigma_3^*$, $h(w)$ is 2-abelian-cube-free.

3-abelian-square-free ternary word

Let $h : \Sigma_4^* \to \Sigma_3^*$ be the following 25-uniform morphism.

$$h : \begin{cases} 0 \to 0102012021012010201210212 \\ 1 \to 0102101201021201210120212 \\ 2 \to 0102101210212021020120212 \\ 3 \to 0121020120210201210120212 \end{cases}$$

Theorem

For every abelian-square-free word $w \in \Sigma_4^*$, $h(w)$ is 3-abelian-square-free.