What is the minimal critical exponent of quasiperiodic words?

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Quasiperiodicity?

A notion introduced by Apostolico, Ehrenfeucht (1990, 1993).

Definition

w is q-quasiperiodic if $w \neq q$ (finite case) and
w can be obtained by concatenations and overlaps of q

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- $$abaababaabaabababa$$
- $$abacaba\,abac\,aba\,cab\,abacaba$$

Critical exponent?

Fractional power

$x^{\frac{p}{q}} = x^n y$ with $n = \lfloor \frac{p}{q} \rfloor$, $q = |x|$ and y prefix of x of length $p - nq$

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Critical exponent of w

$$E(w) = sup\{k \in \mathbf{Q} \mid w \text{ contains a } k\text{th power}\}$$

$$E(\text{Thue-Morse}) = 2$$

$$E(\text{Fibonacci}) = 2 + \phi$$

Reformulation of the question

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Observation

$$w \text{ quasiperiodic} \Rightarrow E(w) > 2.$$

Indeed w contains an overlap of q or q^2.

On at least three-letter alphabets

Result to be verified

For all $\epsilon > 0$, over a 3-letter alphabet, there exists an infinite word with critical exponent less than $2 + \epsilon$

So the question holds only on binary alphabets:

Is the smallest exponent $\frac{7}{3}$? $\frac{5}{2}$? $\frac{8}{3}$? other ?

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Recent idea (friday)

to use Karhumäki, Shallit 1994 and their 21-uniform morphism:
$\Rightarrow \frac{7}{3}$

Ideas for the 7-letter alphabet

Step 1

$$f \begin{cases} a \mapsto xyxzxyx \\ b \mapsto xyxzxy \\ c \mapsto xyxz \end{cases}$$

for all infinite word w, $f(w)$ is $xyxzxyx$-quasiperiodic

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Step 2

Choose:

- w, y and z square-free
- x letter, $x \notin \text{alph}(yz)$, $\text{alph}(y) \cap \text{alph}(z) = \emptyset$

Maximal runs of exponent > 2 are:

$$xyxyx$$
$$f(ba) = xyxzxyxyxzxyx$$

Ideas for the 7-letter alphabet (continue)

Consequence of Step 2

$$E(w) = \max(2 + \frac{1}{1 + |y|}, 2 + \frac{1}{1 + \frac{|z|}{2+|y|}})$$

Final step

y and z can be chosen on disjoint 3-letter alphabets such that

$$E(w) \leq 2 + \epsilon$$

from 7-letter alphabet to 3-letter alphabet

Use following square-free Brandenburg's morphism (1983) twice:

$$\begin{cases} a_1 \mapsto aba\ cab\ cac\ bab\ cba\ cbc \\ a_2 \mapsto aba\ cab\ cac\ bac\ aba\ cbc \\ a_3 \mapsto aba\ cab\ cac\ bca\ bcb\ abc \\ a_4 \mapsto aba\ cab\ cba\ cab\ acb\ abc \\ a_5 \mapsto aba\ cab\ cba\ cbc\ acb\ abc \end{cases}$$

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with following extensions for the first time:

$a_6 \mapsto dbd\ cdb\ cdc\ bdb\ cbd\ cbc$,

$a_7 \mapsto ebe\ ceb\ cec\ beb\ cbe\ cbc$

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If w has a run of period p and exponent $2 + \epsilon$ with $\epsilon > 0$, then $f(w)$ has a run of exponent $2 + \epsilon + 17/p$

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(In the construction on 7 letter alphabt, we can prove periods of repetitions of exponent at least 2 are $> |xyz|$.)

Recent idea to go from 3-letter alphabet to 2-letter alphabet

Use paper by Karhumäki and Shallit in 1994 and their morphism:

$$\begin{cases} a \mapsto 0110100110010011101001 \\ b \mapsto 1001011001001100010110 \\ c \mapsto 1001011001101100010110 \\ d \mapsto 0110100110110011101001 \end{cases}$$

KS1994: If w is square-free:

- $f(w)$ contains no square yy with $|y| > 13$;
- $f(w)$ contains no $\frac{7}{3}^+$-powers.

It seems that taking suitable w quasiperiodic over $\{a, b, c\}$ with exponent $2 < E(w) < \frac{7}{3}$, we can get $\mathrm{E(f(w))} = \frac{7}{3}$.

Lower bound on two-letter alphabet

Theorem (Karhumäki, Shallit 1994)

Let x be a word avoiding α-powers, with $2 < \alpha \leq \frac{7}{3}$.
Let μ be the Thue–Morse morphism.
Then there exist u, v with $u, v \in \{\varepsilon, 01, 00, 11\}$ and a word y avoiding α-powers, such that $x = u\mu(y)v$.

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Consequence:
for w infinite avoiding such α-powers, $n \geq a$, $w = u\mu^n(w')$ with w'.
w q-quasiperiodic + n such that $3|q| \leq |\mu^n(a)|$: contradiction.

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$E(w) \geq \frac{7}{3}$

Another problem

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What about bounds on $|u|$ and $|v|$?