The influence of an interacting substrate on Turing instability conditions

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5th November 2013

Can diffusion cause instability?

What is Turing instability?

Can diffusion cause instability?

Diffusion driven instability (Turing instability):

A steady state of the system is
- asymptotically stable without diffusion,
- unstable with diffusion.

I. Lengyel and I. R. Epstein:

- $u \ldots$ concentration of unbound chemical 1,
- $v \ldots$ concentration of unbound chemical 2,
- $w \ldots$ concentration of bound chemical 1,

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\begin{aligned}
u_t &= d_u \Delta u + f_u u + f_v v - h_u u - h_w w, \\
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\end{aligned}
\tag{RDES}
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\end{aligned} \qquad \text{(RDES)}$$

We assume $h_u > 0$ and $h_w < 0$.
For $|h_w|$ and h_u large one can approximate the system RDES by

$$\begin{aligned}
(1 + K)u_t &= d_u \Delta u + f_u u + f_v v, \\
v_t &= d_v \Delta v + g_u u + g_v v, \qquad \text{(RDER)}
\end{aligned}$$

where $K = -h_u/h_w > 0$.

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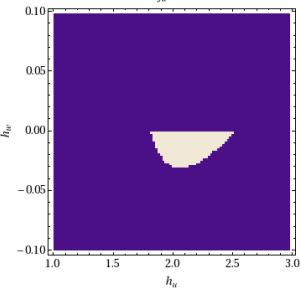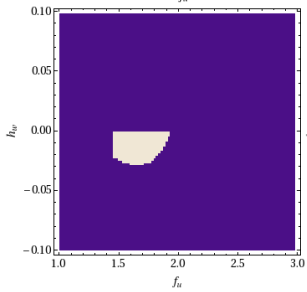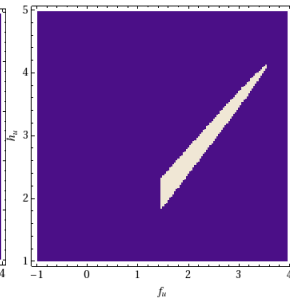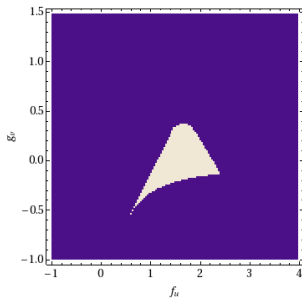
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- SI-SI type reaction does not permit TI.
- If $d_u = d_v$ then the necessary and sufficient conditions for TI do not depend on d_u and d_v.
- Hopf bifurcation cannot occur (as without binding).
- Instability condition does not depend on the parameters of binding !

Thank you.