Using Optimal Control of Parabolic PDEs to Investigate Population Questions

Suzanne Lenhart

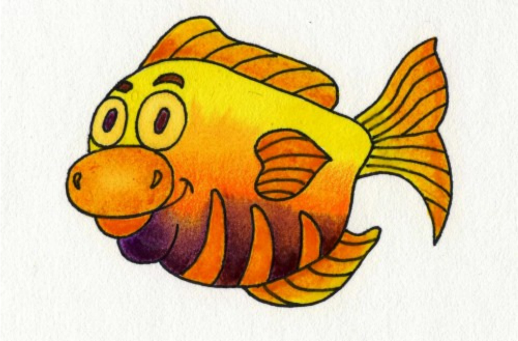
University of Tennessee - Knoxville

OUTLINE

- Motivation
- Fish Harvesting Examples
  (collaborators: Neubert, Herrera, Joshi)
- Example with Controls on Resources
  (collaborators: Bintz, Finotti, Y. Lou, Ding, Ye)
- Example with Controls on Advection Direction
  (collaborators: Phan, Finotti)
- Conclusions and Discussion.

Acknowledge support from the National Science Foundation

Harvesting



Resource Allocation

Choosing Movement Direction

Motivation

- No-take marine reserves may be a part of optimal harvest strategy designed to maximize yield.
- Marine reserves can protect habitat and defend endangered stock from overexploitation.
- Marine reserves as a part of fishery management plan are controversial.

Motivating Work

Neubert (Ecology Letters, 2003)

- rescaled equation $u'' + u(1-u) - h(x)u = 0$
- $u = 0$ at the boundary $x = 0$ and $x = L$
- max yield $\int_0^L h(x)u(x)dx$
- u STATE and h CONTROL
- TOOL: Pontryagin's Maximum Principle
- Depending on length of domain, marine reserves are part of optimal harvesting strategy.
- For large length, there are many intervals of no harvest(reserve), leading to 'chattering'.
  For small length, there is one reserve in the middle.

Idea in 1D

Rough Idea in 1 dimensional domain:

Optimal control and Pontryagin's Maximum Principle

Pontryagin and his collaborators developed optimal control theory for ordinary differential equations about 1950.

Pontryagin's KEY idea was the introduction of the adjoint variables to attach the differential equations to the objective functional (like a Lagrange multiplier attaching a constraint to a pointwise optimization of a function).

Converted problem of finding an optimal control to maximize the objective functional subject to dynamic equations (with initial conditions) to maximizing the Hamiltonian pointwise.

Extensions

To consider extensions... consider non-steady state, include time and more than 1 space variable. Parabolic PDE

There is no complete generalization of Pontryagin's Maximum Principle to PDEs.

After setting up a PDE with a control in a specifed set and an objective functional, proving existence of an optimal control is a first step.

Necessary Conditions

To derive the necessary conditions, we need to differentiate the map

control $\rightarrow$ objective functional

Note that the state contributes to the objective functional, so we also must differentiate the map

control $\rightarrow$ state

The "sensitivity" is the derivative of the control-to-state map. The sensitivity solves a PDE, which is linearized version of the state PDE.

How to find and use the adjoint function

The formal adjoint of the operator in the sensitivity PDE is found.

Transversality Condition: final time condition $\lambda = 0$ at $t = T$

nonhomogeneous term

$$\frac{\partial \text{integrand of J}}{\partial \text{state}}$$

Differentiate the objective functional $J(control)$ with respect to the control.

Use the adjoint problem and the sensitivity problem to simplify and obtain the explicit characterization of an optimal control.

Parabolic Fishery Model

Our fishery model in domain $Q = \Omega \times (0, T)$ with $\Omega \subset \mathbb{R}^n$ is :

$$u_t = \Delta u + u(1-u) - hu \qquad \text{in } Q \tag{1}$$

with initial and boundary conditions:

$$\begin{aligned} u(x,0) &= u_0(x) \qquad \text{for } x \in \Omega \\ u(x,t) &= 0 \qquad \text{on } \partial\Omega \times (0, T) \end{aligned}$$

- u represents fish population STATE
- h represents harvest rate CONTROL

We have completed the analysis for general semilinear parabolic PDE in a multidimensional domain but here we present a simpler case

Goal

We seek to maximize the objective functional over $h \in U$:

$$J(h) = \int_0^T \int_\Omega e^{-\delta t} hu \, dx \, dt \qquad (2)$$

where $U = \{h \in L^\infty(Q) : 0 \leq h(x,t) \leq M \leq 1\}$ is class of admissible controls and $e^{-\delta t}$ represents a discount factor with interest rate δ.

$(1+\delta)/2 < M$

This problem is linear in the control.

Existence of an Optimal Control

Solution space u in $V = L^2(O, T, H_0^1(\Omega))$ with
u_t in $L^2(0, T; H^{-1}(\Omega))$
Note $u > 0$ in Q.

Theorem

There exists an optimal control h^* maximizing the functional $J(h)$ over U.

Proof.

- Choose a maximizing sequence $\{h^n\}$ in U.
- Use apriori estimates.
- Use weak convergence results.

□

Derivation of Optimality System

Theorem

The mapping $h \to u = u(h)$ is differentiable in the following sense:

$$\frac{u(h+\epsilon l) - u(h)}{\epsilon} \rightharpoonup \psi$$

weakly in V as $\epsilon \to 0$ for any $h \in U$ and $l \in L^{\infty}(Q)$ s.t. $(h+\epsilon l) \in U$ for ϵ small. The sensitivity ψ satisfies:

$$\begin{aligned} &\psi_t = \Delta\psi + \psi - 2u\psi - h\psi - lu && \text{in } Q \\ &\psi(x,0) = 0 && \text{for } x \in \Omega \\ &\psi(x,t) = 0 && \text{on } \partial\Omega \times (0,T) \end{aligned} \qquad (3)$$

Characterization of Opt Control

Theorem

Given an optimal control h^* and corresponding solution $u^* = u(h^*)$ there exists a weak solution $\lambda \in V$ with $\lambda_t \in L^2(0, T; H^{-1}(\Omega))$ satisfying the adjoint equation:

$$\begin{aligned} -\lambda_t - \Delta\lambda - \lambda + 2u^*\lambda + h^*\lambda + \delta\lambda = h^* \quad in\ Q \\ \lambda(x,t) = 0 \quad on \quad \partial\Omega \times (0,T). \end{aligned} \tag{4}$$

and transversality condition
$\lambda(x, T) = 0$ *for* $x \in \Omega$.

Adjoint PDE, backwards with FINAL time condition

Characterization Continued

And furthermore the characterization of an OC:

$$h^*(x,t) = \begin{cases} 0 & \text{if} \quad \lambda(x,t) > 1 \\ \frac{1+\delta}{2} & \text{if} \quad \lambda(x,t) = 1 \\ M & \text{if} \quad \lambda(x,t) < 1 \end{cases} \tag{5}$$

Note that possible bang-bang or singular cases.

Solve numerically the state and adjoint equations coupled with this optimal control characterization.

Optimal Control for Unexploited Stock Initial Condition

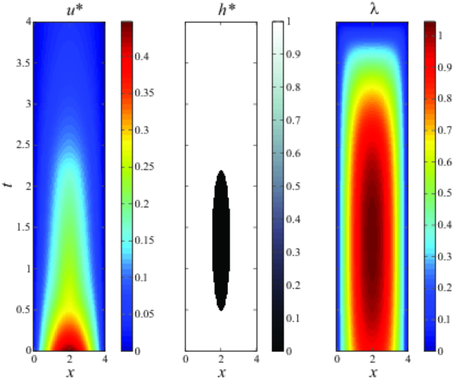
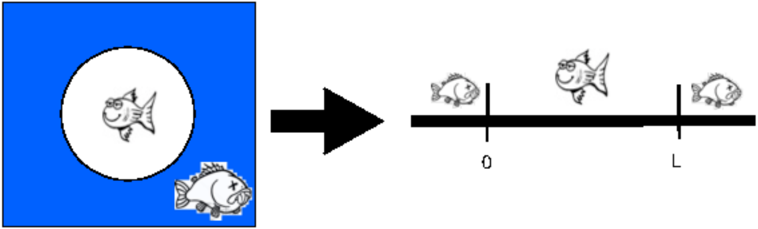


Figure : Final time 4, length of domain 4, discount .2

Dirichlet Boundary Condition



Effect of Boundary Conditions

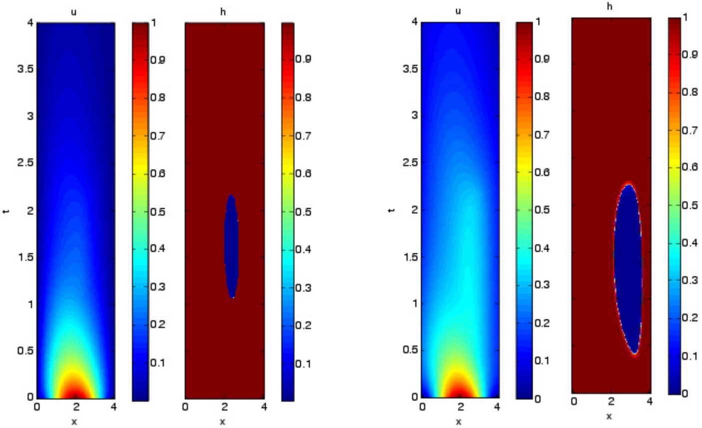
NOTE that the Dirichlet boundary condition gives a type of heterogeneity to the problem.

If you use Neumann BC for this problem, the optimal control is constant and the optimal state is also constant.

What about Robin BC?

My student, Mike Kelly, is working on this case.

Comparison: Dirichlet BC vs. Robin BC with Advection Coefficient ($b = 0.25$)



Work continues

These models are quite simple and only begin to investigate these issues.

Mike Neubert and Holly Moeller are investigating this when including habitat damage.

Resource Allocation

Motivation: Controls on Resources

- How resource allocation affects the population dynamics of species remains an important issue in conservation biology.
- Given a fixed amount of resources, how can we determine the optimal spatial arrangement of the favorable and unfavorable parts of the habitat for species to survive?
- This question was first addressed by Cantrell and Cosner

$$u_t = \lambda \Delta u + m(x)u - u^2 \qquad \text{in } \Omega,$$

  subject to Dirichlet, Robin, or Neumann BC
  $u(x,t)$ is the density of the species
- $m(x)$ represents the intrinsic growth rate of the species and measures the availability of the resources.

A related question?

How does resource allocation affects population size of the species?

Population abundance is clearly a good measurement of conservation effort.

Control problem about Population Size

Given $0 < \delta < |\Omega|$, define the control set

$$U = \{m \in L^\infty(\Omega) \mid 0 \le m(x) \le 1,\ \int_\Omega m(x)\ dx = \delta\}.$$

We seek to find $m^* \in U$, such that

$$J(m^*) = \max_m J(m),$$

with objective functional

$$J(m) = \int_\Omega \left[u - (Bm^2)\right] dx, \qquad (6)$$

$$\begin{cases} -\lambda \Delta u = mu - u^2, & x \in \Omega, \\ \dfrac{\partial u}{\partial n} = 0, & x \in \partial\Omega, \end{cases} \qquad (7)$$

paper with Ding, Finotti, Y. Lou, Y. Ye, in Nonlinear Analysis: Real World Applications 2010.

Numerical Illustrations

Used an iterative scheme to solve state and adjoint system with control characterization

Take *measure*$(\Omega) = 1$ and $\delta = .5$

1-D case

For $\lambda = .1$, $B > 1$ implies optimal control is constant.

Next we show $B = .5$ cases

See non-uniqueness and lack of symmetry

1-D case

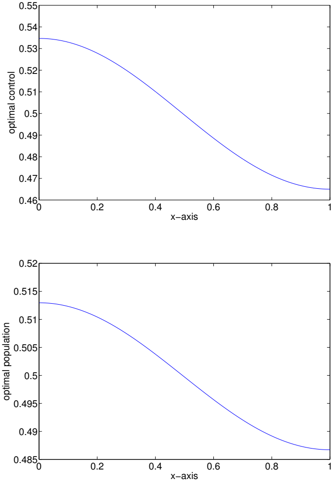


Figure : An OC and Corresponding State in 1D for $\lambda = 0.1$, $B = 0.5$

Another solution to same case

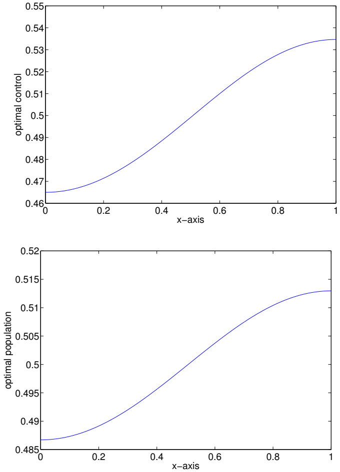


Figure : Another Optimal Control and Corresponding State in 1D for $\lambda = 0.1,\ B = 0.5$

2-D, optimal control concentrated at boundary

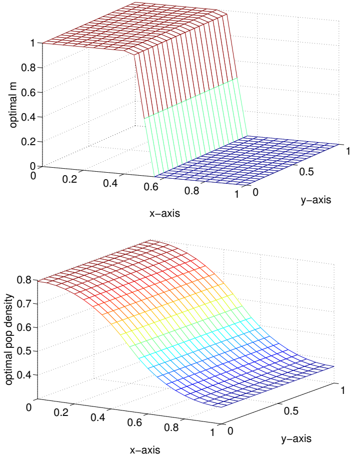


Figure : An OC and State in 2D for $\lambda = 0.1$, $B = 0.1$, $\delta = 0.5$

2-D, optimal control concentrated at corner

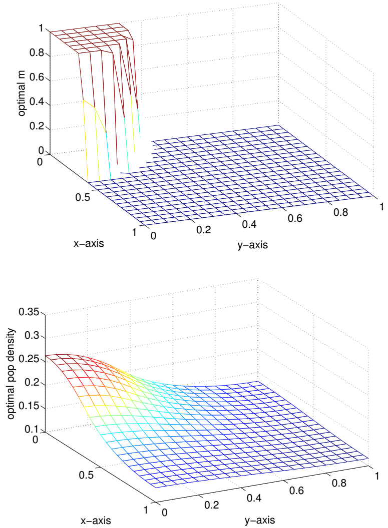


Figure : An OC and State in 2D for $\lambda = 0.1$, $B = 0.1$, $\delta = 0.1$

Conclusions from Some Numerical Results

(i) For 1-D habitat, the characterization of OC depends on the choice of the diffusion rate λ. For small λ the OC seems to be symmetric, and so may be unique.
This is in contrast to the case when λ is suitably larger, where OC is not unique and non-symmetric.

(ii) For rectangular domains, the shape of OC depends on the choice of the amount of total resources, δ. When the amount is small, the OC is concentrated at one of the corners of the rectangle.

This is different from the situation where the amount of total resources is suitably large, for which the OC concentrates at a boundary edge of the rectangle.

Further investigation on relationships of λ, B and δ

Dynamic Model: Parabolic Case with Bintz and Finotti

- $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $Q = \Omega \times [0, T]$ where $T \in (0, \infty)$.
- Model with $u(x, t)$, population density

$$\begin{cases} u_t - \mu \Delta u & = & u(m - u), & Q, \\ u(\cdot, 0) & = & u_0 \geq 0, & \Omega \times \{t = 0\}, \\ \frac{\partial u}{\partial \nu} & = & 0, & \partial\Omega \times [0, T]. \end{cases}$$

- $m = m(x, t)$ in $L^\infty(Q)$ measures the availability of resources.
- $\mu > 0$ is fixed (diffusion coefficient).
- $u_0 \in L^\infty(\Omega \times \{t = 0\})$ is sufficiently smooth.
- Compare with $u = 0$ boundary condition

Problem Formulation

- Seek $m(x,t)$ that maximizes the total population size while minimizing the resource "cost".
- Find $m^* \in U$ such that

$$J(m^*) = \max_U J(m), \quad \text{where} \quad J(m) = \int_Q [u(x,t) - Bm^2]\, dxdt.$$

- $U = \{m \in L^\infty(Q) : 0 \leq m \leq M\}$.
- B is a positive weight constant on the cost term.
- $u = u(m)$ denotes the dependence of the state on the control.

Theorem

There exists an optimal control $m^* \in U$ such that

$$J(m^*) = \max_{m \in U} \int_Q [u(x,t) - Bm^2] \, dxdt.$$

The proof uses a maximizing sequence of controls and corresponding estimates to obtain the desired convergence.

Sensitivity

Lemma

The map $m \longmapsto u(m)$ *is differentiable in the following sense: for each* m, ℓ *in* U *such that* $m + \epsilon\ell \in U$ *for all* ϵ *sufficiently small, there exists* $\psi = \psi(m, \ell) \in L^2((0, T), H^1(\Omega))$, *such that*

$$\frac{u(m + \epsilon\ell) - u(m)}{\epsilon} \rightharpoonup \psi \text{ weakly in } L^2((0, T), H^1(\Omega)) \text{ as } \epsilon \to 0,$$

and the sensitivity ψ *satisfies*

$$\begin{cases} \psi_t - \mu\Delta\psi - (m - 2u)\psi = u\ell, & Q, \\ \psi(\cdot, 0) = 0, & \Omega \times \{t = 0\}, \\ \frac{\partial\psi}{\partial\nu} = 0, & \partial\Omega \times (0, T). \end{cases}$$

Necessary Conditions

Theorem

Given an optimal control m^* and corresponding state $u^* = u(m^*)$, there exists an adjoint solution p in $L^2(0, T, H^1(\Omega))$ which satisfies $p_t \in L^2((0, T), H^1(\Omega)^*)$ and

$$\begin{cases} -p_t - \mu\Delta p - (m^* - 2u^*)p = 1, & \text{in } Q, \\ \dfrac{\partial p}{\partial \nu} = 0, & \text{in } \partial\Omega \times (0, T), \\ p = 0 & \text{in } \Omega \times \{t = T\}. \end{cases}$$

Furthermore, m^* is characterized by

$$m^* = \min\left\{1, \max\left\{\frac{u^* p}{2B}, 0\right\}\right\}$$

idea

- The operator in the adjoint PDE is the formal adjoint of the operator in the sensitivity PDE (which is linear).
- The characterization of m^* follows from the fact that

$$\lim_{\epsilon\to 0^+} \frac{J(m^*+\epsilon\ell)-J(m^*)}{\epsilon} \le 0$$

  and by making use of the equations of adjoint and sensitivity equations.
- The L^∞-bound of the solution $u(m)$ plays the key role in the analysis.

Uniqueness of Optimal Control

Theorem

There exists a positive number T_0 such that if $0 < T \leq T_0$, then there is a unique optimal control.

The small T requirement is common in systems with opposite time orientation. Here the state system is forward in time and the adjoint is backwards in time.

Numerical Illustrations

- We use a forward-backward iterative scheme with a finite difference method to solve the state and adjoint equations with optimal control characterization.
- We illustrate various scenarios with both Neumann and Dirichlet boundary conditions varying $T, M,$ and u_0.
- Baseline values,

$$\mu = 0.1, \quad , T = 1, \quad B = 0.05, \quad \text{and} \quad M = 1.$$

Single centered peak initial condition

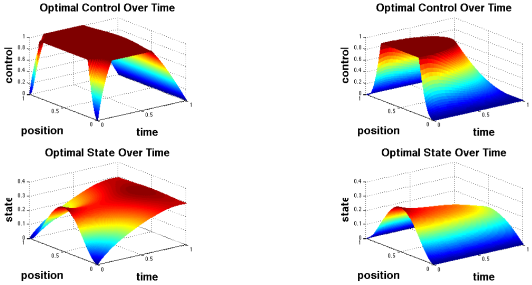


Figure : Optimal control and corresponding states for Neumann and Dirichlet boundary conditions

Vary final time $T = 1$ vs $T = 0.2$

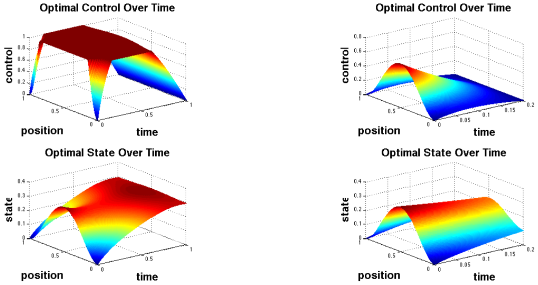


Figure : Optimal control and corresponding states for Neumann boundary conditions varying final time

Two peaks initial condition with $T = 0.2$ and $M = 2$

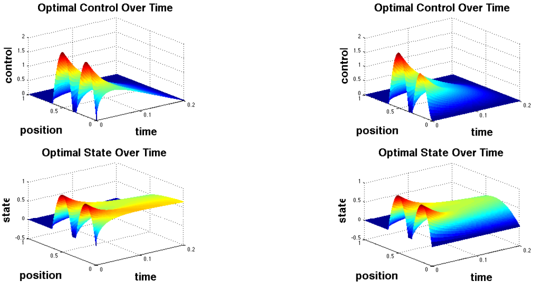


Figure : Optimal control and corresponding states for Neumann and Dirichlet boundary conditions with two peak initial condition

Single peak near boundary

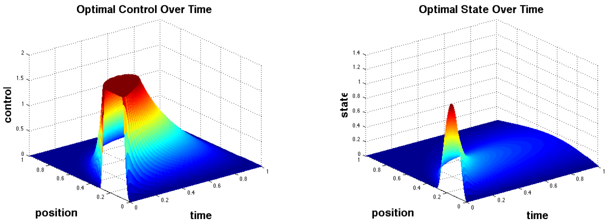


Figure : Optimal control and corresponding state for Dirichlet boundary conditions with one peak initial condition

Conclusions and Future Work

- Some numerical results demonstrate some simple cases in this preliminary investigation.
- Consider fixed amount of resources: $\int_Q m(x,t)\ dx = \delta$.
- Consider two dimensional habitat with more realistic initial conditions.
- Consider additional objective to maximize final time population size.

Choosing Movement Direction

Concentrating on Movement

- Ecological question: Given a fixed amount of resources, how does the species react to the habitat to be "beneficial"?
- Movement: Random Diffusion and Directed Advection.
- Belgacem-Cosner and Cantrell-Cosner-Lou studied the effects of the advection along an environmental resource gradient

$$u_t - \nabla \cdot [D\nabla u - \alpha u \nabla m(x)] = u[m(x) - u], \quad \Omega \times (0, \infty)$$

  with zero flux boundary condition.
- $m(x)$ represents the intrinsic growth rate and measures the availability of the resources.
- "beneficial" means the persistence of the population or the existence of a unique globally attracting steady state.

A Related Question

If a species could choose the direction for advection movement, how would such a choice be made to maximize its total population?

Would the advection be related to the spatial gradient of m, the resource function? or the spatial gradient of $\ln(m)$?

Population Dynamics Model

- $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, $Q_T = \Omega \times [0, T]$ and $S_T = \partial\Omega \times [0, T]$ for some fixed $T > 0$.
- Model with $u(x, t)$, population density

$$\begin{cases} u_t - \nabla \cdot [\mu \nabla u - u\vec{h}] &= u[m - f(x,t,u)], & Q_T, \\ \mu \dfrac{\partial u}{\partial \nu} - u\vec{h} \cdot \nu &= 0, & \mathcal{S}_T, \\ u(\cdot, 0) &= u_0 \geq 0, & \Omega. \end{cases}$$

- $\vec{h} : Q_T \to \mathbb{R}^n$ is the advection direction.
- $m = m(x, t)$ in $L^\infty(Q_T)$ measures the availability of resources.
- $f : Q_T \times \mathbb{R} \to \mathbb{R}$ is non-negative and satisfies some natural smoothness and growth conditions.
- $\mu > 0$ is fixed (diffusion coefficient).
- $u_0 \in L^\infty(\Omega)$ is sufficiently smooth.

Problem Formulation

- Seek the advection term $\vec{h}(x,t)$ control that maximizes the total population while minimizing the "cost" due to movement.
- Find $\vec{h}^* \in U$ such that

$$J(\vec{h}^*) = \max_U J(\vec{h}), \quad \text{where} \quad J(\vec{h}) = \int_{Q_T} [u(x,t) - B|\vec{h}(x,t)|^2] dxdt.$$

-

$$U = \{\vec{h} \in L^2((0,T), L^2(\Omega)^n) : |h_k| \leq M, \quad \forall k = 1, 2, \cdots, n\}.$$

- B "cost coefficient" due to the population moving along $\vec{h}$.
- Denote the dependence of the state on the control by $u = u(\vec{h})$.

Existence Solutions and Some Estimates

Theorem

Given $m \in L^\infty(Q_T)$ *and* u_0 *be non-negative, bounded and in* $H^1(\Omega)$. *Then, for each* $\vec{h} \in U$, *there is a unique weak solution* $u = u(\vec{h})$ *of*

$$\begin{cases} u_t - \nabla \cdot [\mu \nabla u - u\vec{h}] = u[m - f(x,t,u)], & Q_T, \\ \mu \dfrac{\partial u}{\partial \nu} - u\vec{h} \cdot \nu = 0, & \mathcal{S}_T, \\ u(\cdot, 0) = u_0 \geq 0, & \Omega. \end{cases}$$

Moreover, there is a finite constant $C > 0$ *such that*

$$0 \leq u(\vec{h}) \leq C, \quad \forall\, (x,t) \in Q_T,$$

and

$$\sup_{0 \leq t \leq T} \int_\Omega u(x,t)^2 dx + \int_{Q_T} |\nabla u(x,t)|^2 dxdt \leq C.$$

Steps in the proof

- Solutions $u \geq 0$ follows from Stampacchia's truncation method (the standard maximum principle is not applicable here).

- The energy estimate

$$\sup_{0 \leq t \leq T} \int_\Omega u(x,t)^2 dx + \int_{Q_T} |\nabla u(x,t)|^2 dxdt \leq C.$$

  follows by multiplying the equation with u and using Hölder's inequality, Sobolev embeddings.

- The upper bound for u, i.e. $u \leq C$ is not trivial. It follows from de Giorgi's iteration technique.

- The existence of solution follows by standard method (Galerkin's method).

Existence of an Optimal Control

Theorem

There exists an optimal control $\vec{h}^* \in U$ *such that*

$$J(\vec{h}^*) = \max_{\vec{h} \in U} \int_{Q_T} [u(x,t,\vec{h}) - B|\vec{h}(x,t)|^2] dxdt.$$

- Careful analysis of the convergence of maximizing sequence of controls and corresponding states.
- The a-priori estimates of the solutions $u(\vec{h})$ are essential.

Necessary Conditions

Theorem

Given an optimal control $\vec{h}^*$ *and corresponding state* $u^* = u(\vec{h}^*)$, *there exists an adjoint solution* p *in* $L^2(0, T, H^1(\Omega))$ *which satisfies* $p_t \in L^2((0, T), H^1(\Omega)^*)$ *and*

$$\begin{cases} -p_t - \mu \Delta p - \vec{h}^* \cdot \nabla p - [m - g(x, t, u^*)]p = 1, & \textit{in } Q_T, \\ \dfrac{\partial p}{\partial \nu} = 0, & \textit{in } S_T, \\ p(\cdot, T) = 0 & \textit{in } \Omega. \end{cases}$$

Furthermore, $\vec{h}^*$ *is characterized by*

$$h_i^* = \max\left\{ \min\left\{ M, \frac{u^* p_{x_i}}{2B} \right\}, -M \right\}, \quad \textit{for each} \quad i \in \{1, \dots, n\}.$$

Uniqueness and Stability Result

Theorem

Let $\beta > 0$. There exist $0 < T_1$ and B_1 such that if $B > B_1$ and $0 < T < T_1$, there exists a constant $C = C_T > 0$ such that the estimate

$$||\vec{h}^*(m_1) - \vec{h}^*(m_2)||_{L^2(Q_T)} \leq C||m_1 - m_2||_{L^2(Q_T)},$$

holds for all m_1, m_2 in $L^\infty(Q_T)$ with $|m_1|, |m_2| \leq \beta$.

Numerical Illustrations

- We use a forward-backward iterative scheme with finite difference method to solve the state and adjoint equations with optimal control characterization.

- We have run several examples for different types of nonlinearity f such as

$$f(x,t,u) = u,\ f(x,t,u) = u + \frac{1}{1+u}, \quad f(x,t,u) = u + \frac{u}{1+u^2}.$$

- For each type of nonlinearity, we have run for both time-independent and time-dependent m.

- In all examples shown here,

$$\mu = 0.1, \quad , T = 0.2, \quad B = 0.05, \quad \text{and} \quad f(x,t,u) = u.$$

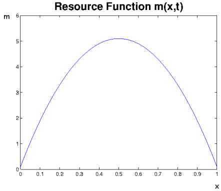


Figure : Time-Independent $m(x) = 20x(1-x) + .1$

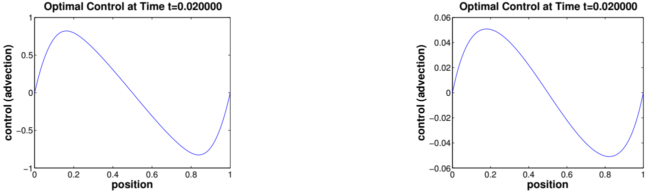


Figure : Time Slices of an Optimal Controls in 1D for $B = .05$ and $B = 1$

Second Example

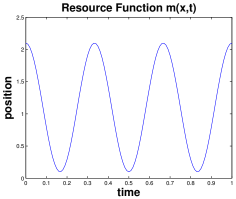


Figure : Time-Independent $m(x) = \cos(6\pi x) + 1.1$

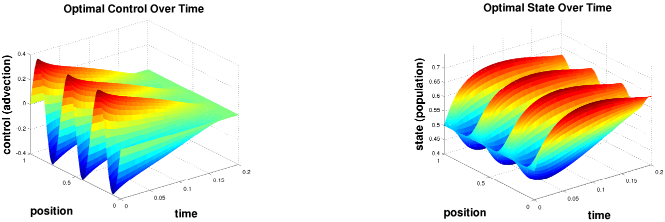


Figure : An Optimal Control and Corresponding State in 1D Over Time

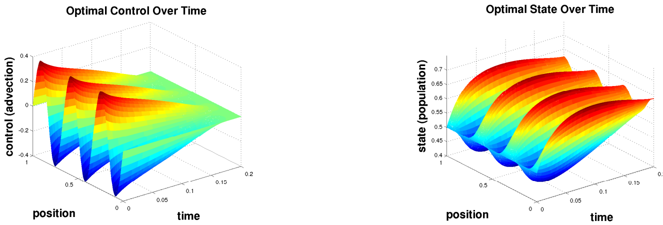


Figure : An Optimal Control and Corresponding State in 1D Over Time

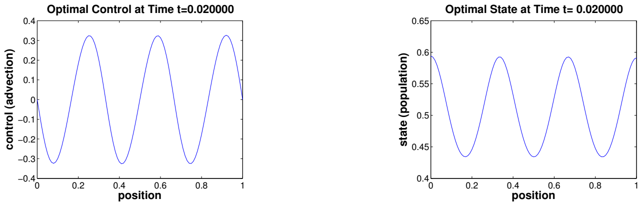


Figure : Time Slices of an Optimal Control and Corresponding State in 1D

Last Example

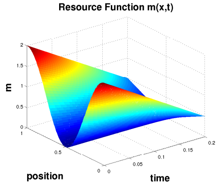


Figure : $m(x,t) = (1 - t/T)(\cos(2\pi x) + 1) + (t/T)|x - .5|$

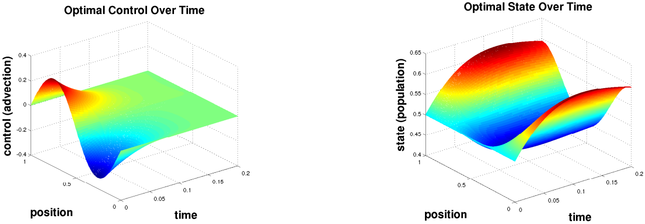


Figure : An Optimal Control and Corresponding State in 1D Over Time

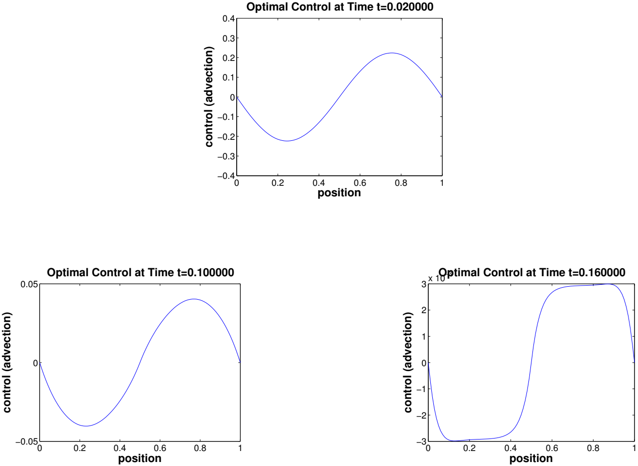


Figure : Early, Mid, and Late Time Slices of an Optimal Control and Corresponding *m* slice in 1D

Conclusions and Discussion

- The uniqueness and stability of the optimal control are obtained under some conditions on T and B.
- The numerical results indicate that the population follows the gradient of the given resource.
- Current work on elliptic case and ALSO investigate other relationships with the gradient of m

References- Papers

H. R. Joshi, G. E. Herrera, S. Lenhart, and M. Neubert, Optimal Dynamic Harvest of a Mobile Renewable Resource, Natural Resource Modeling Journal 22, 2009, 322-342.

H. Finotti, S. Lenhart, and T. Phan, Optimal control of advection director for reaction-diffusion population models, Evolution Equations and Control Theory 1 (2012), 81-107.

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