A reconstruction theorem for noncommutative G-manifolds

Branimir Ćaćić

Department of Mathematics
Texas A&M University

Canadian Operator Symposium
Fields Institute, Toronto, ON
June 27, 2014

Spectral triples

Definition

A *spectral triple* $(\mathcal{A}, H, D)$ consists of

- a unital $*$-algebra $\mathcal{A}$ faithfully $*$-represented on
- H a Hilbert space, together with
- D self-adjoint on H with $(D^2+1)^{-1/2} \in \mathcal{K}(H)$, $[D,a] \in B(H)$ for all $a \in \mathcal{A}$.

If, in addition $\mathcal{A} + [D, \mathcal{A}] \subset \cap_k \operatorname{Dom} \delta^k$ for $\delta(T) := [|D|, T]$, then $(\mathcal{A}, H, D)$ is called *regular*.

Example

If X is a compact oriented Riemannian p-manifold, then

$$(C^\infty(X), L^2(X, \wedge T^*_{\mathbb{C}} X), d + d^*)$$

is a p-dimensional *commutative* spectral triple.

The reconstruction theorem for commutative spectral triples

Theorem (Connes 1996, 2013, cf. Rennie–Várilly 2006)

A spectral triple $(\mathcal{A}, H, D)$ *is commutative with dimension* $p \in \mathbb{N}$ *iff* $(\mathcal{A}, H, D) \cong (C^\infty(X), L^2(X,E), D)$ *for*

- X *a p-dimensional compact oriented Riemannian manifold,*
- $E \to X$ *a Hermitian vector bundle,*
- D *a symmetric Dirac-type operator on E, i.e.,*

$$D^2 = -g^{ij}\partial_i\partial_j + \text{lower order terms}.$$

Equivariant spectral triples

Definition

Let G be a compact abelian Lie group, e.g., $G = \mathbb{T}^N$. A *G-equivariant regular spectral triple* is

- a regular spectral triple $(\mathcal{A}, H, D)$, together with
- a strongly smooth, isometric action $\alpha : G \to \mathrm{Aut}(\mathcal{A})$ and
- a strongly continuous unitary action $U : G \to U(H)$,

such that

- for all $t \in G$, $U_t L(\cdot) U_t^* = L \circ \alpha_t$,
- for all $t \in G$, $U_t D U_t^* = D$.

Observation

The G-action on $(\mathcal{A}, H, D)$, by Fourier analysis, yields a $\widehat{G}$-grading of $(\mathcal{A}, H, D)$, where D has degree $\mathbf{0} \in \widehat{G}$.

Group cohomology

Theorem (Kleppner 1965)

Let Γ be a discrete abelian group, e.g., $\Gamma = \widehat{G}$ for G a compact abelian Lie group.

1. *Every $U(1)$-valued 2-cocycle is cohomologous to a $U(1)$-valued bicharacter, which in turn is cohomologically trivial iff it is symmetric. Hence,*

$$H^2(\Gamma, \mathbb{T}) \cong \mathrm{Hom}(\Gamma^{\otimes 2}, U(1)) / \mathrm{Hom}(S^2\Gamma, U(1)).$$

2. *We have a canonical injection* $\lambda : H^2(\Gamma, \mathbb{T}) \hookrightarrow \mathrm{Hom}(\wedge^2\Gamma, U(1))$, *defined for* $\theta \in H^2(\widehat{G}, \mathbb{T})$ *by*

$$\lambda_\theta(\mathbf{x}, \mathbf{y}) := \sigma(\mathbf{x}, \mathbf{y})^{-1}\sigma(\mathbf{y}, \mathbf{x}), \quad \sigma \in \theta \subset \mathrm{Hom}(\Gamma^{\otimes 2}, U(1)).$$

Connes–Landi deformation

Theorem (Yamashita 2010, after Connes–Landi 2001, Connes–Dubois-Violette 2002, cf. Rieffel 1993)

Let $(\mathcal{A}, H, D)$ be a G-equivariant regular spectral triple and let $\sigma \in \theta \in H^2(\widehat{G}, \mathbb{T})$. Then $(\mathcal{A}_\theta, H, D)$ is a G-equivariant regular spectral triple, where:

- *$\mathcal{A}_\theta$ is $\mathcal{A}$ with the multiplication, $*$-operation*

$$a_{\mathbf{x}} \star_\theta b_{\mathbf{y}} := \sigma(\mathbf{x}, \mathbf{y}) a_{\mathbf{x}} b_{\mathbf{y}}, \quad (a_{\mathbf{x}})^{*_\theta} := \sigma(\mathbf{x}, \mathbf{x})\, (a_{\mathbf{x}})^* ;$$

- *$L_\sigma := \pi_\sigma \circ L : \mathcal{A}_\theta \to B(H)$ is defined by*

$$L_\sigma(a_{\mathbf{x}})\xi_{\mathbf{y}} := \pi_\sigma(L(a_{\mathbf{x}}))\xi_{\mathbf{y}} := \sigma(\mathbf{x}, \mathbf{y}) L(a_{\mathbf{x}})\xi_{\mathbf{y}}.$$

Observation (cf. Venselaar 2013)

Up to G-equivariant unitary equivalence, the construction above only depends on the *cohomology class* $\theta \in H^2(\widehat{G}, \mathbb{T})$.

Examples

Example

Noncommutative 2-tori $(C^2(\mathbb{T}^2_\theta), L^2(\mathbb{T}^2_\theta)^{\oplus 2}, \not{D})$, where $\theta \in \mathbb{T} \cong H^2(\mathbb{Z}^2, \mathbb{T})$.

Example (Connes–Landi 2001; Connes–Dubois-Violette 2002; Landi, Van Suijlekom et al.)

More generally, $(C^\infty(X)_\theta, L^2(X, E), D)$ for

- X a p-dimensional compact oriented Riemannian G-manifold,
- $E \to X$ a G-equivariant Hermitian vector bundle,
- D a G-invariant symmetric Dirac-type operator on E,
- $\theta \in H^2(\widehat{G}, \mathbb{T})$.

These are *concrete noncommutative G-manifolds*, termed *toric noncommutative manifolds* when $G = \mathbb{T}^N$ by Van Suijlekom.

The main definition

Definition (Ć. 2014, cf. Connes 1996, 2013)

We call a G-equivariant regular spectral triple $(\mathcal{A}, H, D)$ a *noncommutative G-manifold* with *metric dimension* $p \in \mathbb{N}$ and *deformation parameter* $\theta \in H^2(\widehat{G}, \mathbb{T})$ if the following conditions hold:

Order zero

$R := \pi_{\lambda_\theta} \circ L : \mathcal{A}^{\mathrm{op}} \to B(H)$ makes H into a $\mathcal{A}$-*bimodule*.

Implications of order zero

1. $\mathcal{A}$ satisfies the commutation relations $b_{\mathbf{y}} a_{\mathbf{x}} = \lambda_\theta(\mathbf{x}, \mathbf{y}) a_{\mathbf{x}} b_{\mathbf{y}}$.
2. $\mathcal{H}_\infty$ is an $\mathcal{A}$-bimodule.

The main definition continued

Order one

For all, $a, b \in \mathcal{A}$, $[[D, L(a)], R(b)] = 0$.

Metric dimension

$\lambda_k((D^2+1)^{-1/2}) = O(k^{-1/p})$ as $k \to +\infty$.

Finiteness and absolute continuity

$\mathcal{H}_\infty := \cap_k \operatorname{Dom} |D|^k$ defines a G-equivariant finitely generated projective right $\mathcal{A}$-module, admitting a G-equivariant Hermitian metric $(\cdot,\cdot)_{\mathcal{A}}$, such that for all $\xi, \eta \in \mathcal{H}_\infty$,

$$\langle \xi, \eta \rangle = \operatorname{Tr}_\omega \left((\xi, \eta)_{\mathcal{A}} (D^2+1)^{-p/2} \right).$$

The main definition continued

- Define $\epsilon_\theta : \mathcal{A}^{\otimes(p+1)} \to \mathcal{A}^{\otimes(p+1)}$ by

$$a_0 \otimes a_1 \otimes \cdots \otimes a_p \mapsto \sum_{\pi \in S_p} \left(\prod_{\substack{i<j \\ \pi(i)>\pi(j)}} (-\lambda_\theta(\mathbf{x}_{\pi(i)}, \mathbf{x}_{\pi(j)})) \right) a_0 \otimes a_{\pi(1)} \otimes \cdots \otimes a_{\pi(p)}$$

  for $a_0 \otimes a_1 \otimes \cdots \otimes a_p$ with $a_k \in \mathcal{A}_{\mathbf{x}_k}$.
- Define $L_D : \mathcal{A}^{\otimes(p+1)} \to B(H)$ by

$$a_0 \otimes a_1 \otimes \cdots \otimes a_p \mapsto L(a_0)[D, L(a_1)] \cdots [D, L(a_p)].$$

Orientability

There exists $\mathbf{c} \in (\mathcal{A}^{\otimes(p+1)})^G$ with $\epsilon_\theta(\mathbf{c}) = \mathbf{c}$, such that $\chi := L_D(\mathbf{c})$ is self-adjoint and unitary, and

$$\forall a \in \mathcal{A}, \quad \chi a = a\chi, \quad \chi[D, a] = (-1)^{p+1}[D, a]\chi.$$

The main definition continued

Implications of orientability

1. Since $\epsilon_\theta(\mathbf{c}) = \mathbf{c}$, $\mathbf{c}$ is a Hochschild p-cycle.
2. One can view $\mathbf{c}$ as a G-invariant p-form in the (deformed) Kähler differential calculus on $\mathcal{A}$ with relations

$$b_{\mathbf{y}} a_{\mathbf{x}} = \lambda_\theta(\mathbf{x},\mathbf{y}) a_{\mathbf{x}} b_{\mathbf{y}}, \quad db_{\mathbf{y}} \wedge a_{\mathbf{x}} = \lambda_\theta(\mathbf{x},\mathbf{y}) a_{\mathbf{x}} \wedge db_{\mathbf{y}},$$
$$db_{\mathbf{y}} \wedge da_{\mathbf{x}} = -\lambda_\theta(\mathbf{x},\mathbf{y}) da_{\mathbf{x}} \wedge db_{\mathbf{y}}.$$

Strong regularity

$\mathrm{End}_{\mathcal{A}^{\mathrm{op}}}(\mathcal{H}_\infty) \subset \cap_k \mathrm{Dom}\, \delta^k$, where $\delta(T) := [|D|, T]$.

Overall observation

The cohomological datum θ serves as the gauge of noncommutativity. In particular, if $\theta = 0$, one gets a *commutative* spectral triple.

Stability under Connes–Landi deformation

Theorem (Ć. 2014, cf. Connes–Landi 2001, Connes–Dubois-Violette 2002)

Let

- $(\mathcal{A}, H, D)$ *be a noncommutative G-manifold with deformation parameter* $\theta \in H^2(\widehat{G}, \mathbb{T})$ *and dimension* $p \in \mathbb{N}$,
- $\theta' \in H^2(\widehat{G}, \mathbb{T})$.

Then $(\mathcal{A}_{\theta'}, H, D)$ *is a noncommutative G-manifold with deformation parameter* $\theta + \theta'$ *and dimension p.*

Remark

In particular, the construction of the orientation cycle for the deformed noncommutative G-manifold from that of the original generalises the construction of orientation cycles for noncommutative tori.

A reconstruction theorem for noncommutative G-manifolds

Theorem (Ć. 2014, cf. Connes 2013)

Let $(\mathcal{A}, H, D)$ be a noncommutative G-manifold with metric dimension $p \in \mathbb{N}$ and deformation parameter $\theta \in H^2(\widehat{G}, \mathbb{T})$. Then $(\mathcal{A}, H, D) \cong_G (C^\infty(X)_\theta, L^2(X, E), D)$ for

- *X a p-dimensional compact oriented Riemannian G-manifold,*
- *$E \to X$ a G-equivariant Hermitian vector bundle,*
- *D a G-invariant symmetric Dirac-type operator on E.*

Proof.

1. Deform by $-\theta$ to get something with deformation parameter $\theta - \theta = 0$, which is therefore *commutative*.
2. Apply Connes's reconstruction theorem for commutative spectral triples [2013]! □