

Uniqueness theorem for certain 1-dimensional NCCW with non-trivial K_1 -group

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June 24, 2014

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- ▶ ϕ and ψ are **equivalent** in a suitable sense
- ▶ often one of the maps $\phi : A \rightarrow B$ is a $*$ -homomorphism of **special form**
often called "standard form" or "diagonal form"

Examples of $*$ -homomorphisms

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- ▶ $r = 0$ means unital $*$ -homomorphism

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- ▶ Application: one-step Bratteli diagram

Bratteli diagrams

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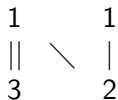
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- ▶ define the multiplicity matrix $[m_{ij}]$ the $s \times p$ matrix; the entry (i, j) is the multiplicity of $M_{k_i}(\mathbb{C})$ into $M_{n_j}(\mathbb{C})$.
- ▶ Note: $\sum_{i=1}^p m_{ij} k_j \leq n_i$, for $1 \leq i \leq s$,
with equality if the map is unital.

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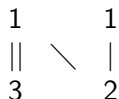
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$$\phi(\lambda, \mu) = \left[\begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} \right]$$

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- ▶ (Robert) extended this for 1-dim NCCW with K_1 -group trivial, using Cuntz semigroup

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- ▶ The index set I can be chosen to be pairs (F, ϵ) with order: $(F, \epsilon) \prec (F', \epsilon')$ if $F \subseteq F'$ and $\epsilon' \leq \epsilon$.

The building blocks

- ▶ "point-line" alg. can be described by two one step Bratteli diagram: given f.d. C and D and two $*$ -homo (left and right) from C to D with prescribed Bratteli diagrams so that C is mapped inj. into D , then alg. A is bdd. cont. from real line to D , convg. at inf. to the left/right images of $*$ -homo.

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- ▶ Note: the size of 0 in $f(0)$ is $2q$. It follows that A is non-unital, stably projectionless.

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$$KL(A, B) \cong \text{Hom}_\Lambda(\underline{K}(A), \underline{K}(B))$$

if A satisfies UCT and B is σ -unital.

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- ▶ there is $\Phi : A \rightarrow B$ iso. that induces the given maps at the invariant level

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- ▶ (i) B is a building block and $\phi, \psi : A \rightarrow B$ $*$ -homomorphisms;
- ▶ (ii) ϕ and ψ induce the same KL-class in $\text{KL}(A, B)$;

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- ▶ then the maps ψ and ϕ are approx. unit. equivalent

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- ▶ Note: the K-homology group $K^0(A) = KK(A, \mathbb{C})$ Example: $K^0(M_n) = \mathbb{Z}$.
- ▶ Claim: $K^0(A) \cong \mathbb{Z} \oplus K_1(A)$

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