Elementary amenable groups are quasidiagonal

Joint work with N. Ozawa and M. Rørdam

20, June, 2014. Toronto

Quasi Diagonal C^*-algebras

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A is **quasidiagonal**
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- **J. Rosenberg proved that if the reduced group C^*-algebra is Q.D. then the given group is amenable.**

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Conjecture (J. Rosenberg)

For any amenable group G, is the group C^*-algebra QD ??

Examples of QD group

- **residually finite groups.**
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- **the lamplighter group $\mathbb{Z}/2\mathbb{Z} \wr \mathbb{Z}$. (2013. J. Carrion, M. Dadarlat, C. Eckhardt)**
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- **Abel's group (2013. J. Carrion, M. Dadarlat, C. Eckhardt)**
- **The full group C^*-algebra $C^*(F_n)$ is residually finite dimensional (then Q.D.) for the free groups F_n, $n \in \mathbb{N}$. (M. Choi)**

Main Theorem

Theorem (2014. N.Ozawa, M.Rørdam, Y.S.)

Any elementary amenable group G (not necessary countable) is QD, i.e., the group C^*-algebra $C^*(G)$ is QD.

- **1956, M. Day. The class of elementary amenable group EG is defined as the smallest class of groups satisfying the following conditions: EG contains all abelian groups and all finite groups, EG is closed under the following elementary operations**
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  (i) subgroups, (ii) quotients, (iii) inductive limits, (iv) extensions.
- **The class of amenable groups AG is closed under (i),(ii),(iii), and (iv).**

Elementary amenable Groups, EG

- **1985, Grigorchuk showed that EG $\neq$ AG.**
- **H. Abel gave a counter example of EG $\neq$ AG as a simple (then non residually finite) group.**

Classification theorem for neclear C^*-algebras

Theorem (2013. H. Matui-Y.S. and H.Lin-Z.Niu, W.Winter.)

Let A, B be unital separable simple C^*-algebras with a unique tracial state (Basic conditions).
Assume that A, B are Strict-comparison, QD, UCT, and Amenable, (SQUAB).

Then $A \cong B$ if and only if
$(K_0(A), K_0(A)_+, [1_A]_0, K_1(A)) \cong (K_0(B), K_0(B)_+, [1_B]_0, K_1(B))$.

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Sketch of the proof

Proof of the main theorem

To show EG $\Longrightarrow$ QD, we have to consider (i) subgroups, (ii) quotients, (iii) inductive limits, (iv) extensions.

However the main obstacle is (iv) for QD.

Sketch of the proof

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Here $\bigotimes_G M_2 \rtimes H$ is SQUAB. Then by the classification theorem $(\bigotimes_G M_2 \rtimes H) \otimes \mathcal{U}$ is AT-algebra (inductive limit of $C(\mathbb{T}) \otimes M_N$). Therefore $((\bigotimes_G M_2 \rtimes H) \otimes \mathcal{U}) \rtimes_{\alpha \otimes \mathrm{id}} \mathbb{Z}$ becomes an AH-algebra,
(then $C^*(G)$ is QD).