Negative Quasi-Probability, Contextuality, Quantum Magic and the Power of Quantum Computation

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Motivation: Quantum Foundations

Quantum mechanics has unfamiliar features

- Superposition, entanglement, collapse under measurement, tensor-product structure of Hilbert space, non-locality, contextuality, negative (quasi-)probability . . .

- Which of these concepts are "truly quantum" and which are "merely classical"?
- Can this cconceptual distinction help predict the unique capabilities of the quantum world?

Motivation: From Quantum Foundations to Quantum Information

The Best Information is Quantum Information

- Clear operational advantages of quantum information: CHSH games, Shor's algorithm

- Which features of quantum theory are necessary and sufficient resources for these operational advantages?

Motivation: Quantum Information

Which quantum features power quantum computation?

- *Non-locality* is the fundamental quantum resource for communication *under the LOCC restriction*
- Quantum resources (capabilities) that are necessary for power of quantum computation are less clear
    - MBQC *vs* standard circuit model *vs* adiabatic QC *vs* DQC1 model...

Both fundamental and *practical*:

- Which quantum processes/algorithms admit an efficient classical simulation?
- What experimental capabilities are needed for exponential quantum speed-up?

Background: Discrete Wigner function

Main Tool: the Wootters/Gross DWF

- A quasi-probability representation introduced by Bill Wootters (1987) and developed by David Gross (2005)
- A discrete analog of the Wigner function (DWF)

- This DWF has nice group-covariant properties relevant to quantum computation
- This DWF is well-defined only for odd-prime dimensional quantum systems:
  - qudits (for $d \neq 2$) or qupits (for $p \neq 2$)
  - . . . maybe "quopits"?
  - as only even prime, 2 is the oddest prime of them all!

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Outline of Results: Quantum Foundations

- We identify the full set of non-negative quantum *states* + *transformations* + *measurements* under this DWF
  - these define an operational *subtheory* of quantum theory
- This a large, convex subtheory of quantum theory with
  - superposition, entanglement (without non-locality), *collapse* under measurement, tensor-product structure of Hilbert space
  - quantum teleportation, the no-cloning principle and other so-called "quantum" phenomena

The non-negative DWF for this subtheory corresponds to:

- a *classical probabilistic model* for quopit systems
- a *local hidden variable model* for entangled quopits
- a *maximal* classical subtheory for quopit systems:
  - negativity of discrete Wigner function occurs if and only if the quantum state violates a contextuality inequality

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Outline of Results:
from Quantum Foundations to Quantum Information

This is all interesting but how is it *useful*?

We show that the Wootters/Gross DWF provides:

- an *efficient simulation scheme* for a class of quantum circuits – extending Gottesman-Knill to (mixed) non-stabilizer states
- a *direct link* between contextuality and the power of quantum computation:
  - a quantum state enables universal quantum computation only if it violates a *contextuality inequality*
- the quantum "Mana": the amount of negativity/contextuality is a *quantitive resource* for universal quantum computation

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Quasi-Probability Representations

The most well-known QPR is the Wigner function

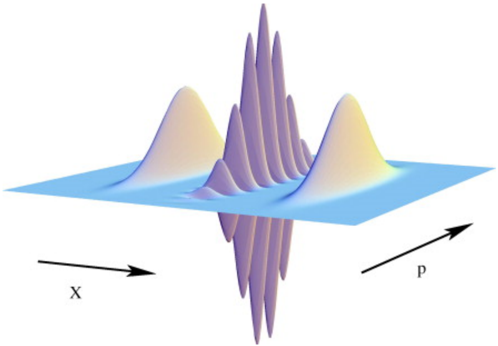
$$\mu_\rho^{\text{Wigner}}(q,p) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} d\xi d\eta \, \mathrm{Tr}\left[\rho e^{i\xi(Q-q)+i\eta(P-p)}\right]$$

- Real-valued function on classical phase space (eg, $\mathbb{R}^2$ for 1 particle in 1d).
- An *equivalent formulation* of quantum mechanics:

$$Pr(q \in \Delta) = \int_\Delta dq \int dp \mu_\rho^{\text{Wigner}}(q,p)$$

- Not unique! Other choices of QPR: P-representation, Q-representation, etc ...

Quasi-Probability Representations



- $\mu_\rho^{\mathrm{Wigner}}(q,p)$ takes on negative values for some quantum states.
- Negativity and non-classicality: negativity of given state depends on choice of QPR!
- Can even choose a QPR for which *all* states are non-negative!

Freedom in choosing QPR

The Wigner function is a non-unique choice of QPR!

- (i) Phase space can be any set Λ, e.g., $\Lambda = \mathbb{R}^2$ for Wigner function.
- (ii) Linear map taking quantum states to real-valued functions is non-unique.
- (iii) Linear map taking measurements to conditional probabilities can be non-unique.

General Class of Quasi-probability Representations

Definition: A *quasi-probability representation* of QM:

Any pair of linear (affine) maps

$$\mu_\rho : \rho \to \mu_\rho$$

$$\xi_k : E_k \to \xi_k$$

with $\mu_\rho : \Lambda \to \mathbb{R}$ and $\xi_k : \Lambda x \mathbb{K} \to \mathbb{R}$,
that reprodiuce the Born rule via the *law of total probability*

$$\Pr(k) = \mathrm{Tr}(E_k \rho) = \int_\Lambda d\lambda \xi_k(\lambda) \mu_\rho(\lambda)$$

Frames and Quasi-probability representations

The non-uniqueness of QPR is equivalent to choosing a frame and a dual frame for the Hilbert space of linear operators

- A frame of operators $\{F(\lambda)\}$ is just a spanning set*, viz. an overcomplete basis, indexed by $\lambda \in \Lambda$.
- A Hermitian frame $\{F(\lambda)\}$ and Hermitian dual frame $\{F^*(\lambda)\}$ define a QPR:

$$\mu_\rho(\lambda) = \mathrm{Tr}(F(\lambda)\rho)$$

$$\xi_k(\lambda) = \mathrm{Tr}(F^*(\lambda)\rho)$$

- Note: For any operator A, a dual frame satisfies

$$A = \int d\lambda F^*(\lambda)\mathrm{Tr}(F(\lambda)A)$$

Ref: C. Ferrie and J. Emerson (J. Phys. A, 2008)

Necessity of Negativity in any QPR

No-Go Theorem for a Fully Non-Negative Quasi-Probability Representation:

- All quantum states and measurements can not be represented by non-negative functions in any QPR.

- In other words: *quantum theory is not a probability theory*
- **Proof:** a frame of non-negative operators can not have a dual frame consisting of non-negative operators.

Refs:

C. Ferrie and J. Emerson (J. Phys. A, 2008);
C. Ferrie, R. Morris and J. Emerson, (Phys. Rev. A, 2010)

See also:

R. Spekkens (PRL, 2008).

Need to Motivate Choice of Quasi-Probability Representation

Different sets of states and/or measurements are non-negative in different QPRs

Key Idea

- Align choice of frame and dual frame to reflect *operational restrictions*!

- The Clifford/stabilizer subtheory: central to quantum error correction and fault-tolerance
- The stabilizer subtheory admits an efficient classical simulation scheme (Gottesman-Knill theorem): no quantum speed-up.
- In the Wootters/Gross DWF, the full Clifford subtheory is non-negative (for quopits)

Slice of the Quantum State Space and Stabilizer Polytope

- $\Lambda = \mathbb{Z}_3 \times \mathbb{Z}_3$

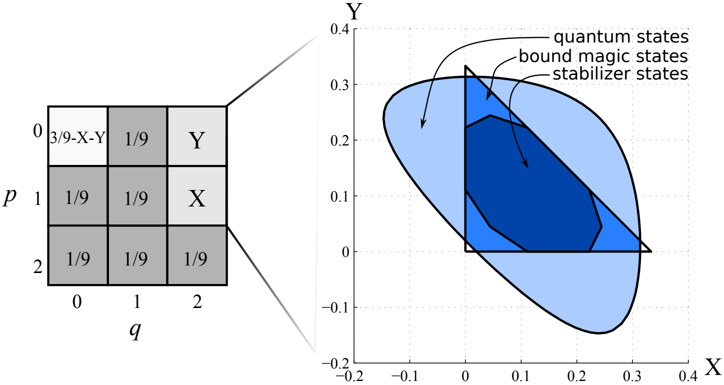


Figure: Slice defined by fixing six entries of the Wigner function and varying the remaining through their possible values to create the plot.

Clifford/Stabilizer Subtheory

- Let p be a prime number and define the boost and shift operators:

$$X\,|j\rangle = |\,j+1 \mod p\rangle$$
$$Z\,|j\rangle = \omega^j\,|j\rangle\,,\ \omega = \exp\left(\frac{2\pi i}{p}\right)$$

- The Heisenberg-Weyl operators for odd prime dimension

$$T_{(a,b)} = \omega^{-\frac{ab}{2}} Z^a X^b \quad (a,b) \in \mathbb{Z}_p \times \mathbb{Z}_p,\ \ p \neq 2$$

where $\mathbb{Z}_p$ are the integers modulo p.

- For composite Hilbert space of n *quopits*:

$$T_{(\vec{a},\vec{b})} \equiv T_{(a_1,b_1)} \otimes T_{(a_2,b_2)} \cdots \otimes T_{(a_n,b_n)}.$$

Clifford/Stabilizer Subtheory

- The Clifford operators are the unitaries that, up to a phase, take the Heisenberg-Weyl operators to themselves, ie.

$$U \in \mathcal{C}_d \iff \forall \boldsymbol{u}\ \exists \phi, \boldsymbol{u}' : \ U T_{\boldsymbol{u}} U^{\dagger} = \exp\left(i\phi\right) T_{\boldsymbol{u}'} .$$

- The set of such operators form the Clifford group $\mathcal{C}_d$ which is a subgroup of $U(d)$.
- The pure stabilizer states for dimension d are

$$\left\{\left|S_i\right\rangle\right\} = \left\{U\left|0\right\rangle : U \in \mathcal{C}_d\right\},$$

- The full set of stabilizer states is the convex hull of this set:

$$\mathrm{STAB}\left(\mathcal{H}_d\right) = \left\{\sigma \in L\left(\mathcal{H}_d\right) : \ \sigma = \sum_i p_i |S_i\rangle\langle S_i|\right\},$$

where p_i is some probability distribution.

The Wootters/Gross DWF for Odd Dimension

Choose a frame of *phase space point operators*

$$A_{\mathbf{0}} \;=\; \frac{1}{d}\sum_{\boldsymbol{u}} T_{\boldsymbol{u}},\; A_{\boldsymbol{u}} = T_{\boldsymbol{u}} A_{\mathbf{0}} T_{\boldsymbol{u}}^{\dagger}.$$

- The frame operators in dimension p^n are n-fold tensor products of single system frame operators.
- There are d^2 such operators for d-dimensional Hilbert space, corresponding to the d^2 phase space points $\boldsymbol{u} \in \Lambda$.
- Let $d = p^n$ and p odd: the frame operators are Clifford covariant: for $U \in \mathcal{C}_d$,

$$U A_{\mathbf{u}} U^{\dagger} = A_{\mathbf{u}'}$$

  - There is a rich (symplectic) structure at play (suppressed here).
  - **Key point:** Cliffords are permutations on the phase space

Discrete Wigner Representation for Odd Dimension

- The DWF of a state is a QPR over $\Lambda = \mathbb{Z}_p^n \times \mathbb{Z}_p^n$, i.e., a set of $d \times d$ points, where

$$W_\rho(\boldsymbol{u}) = \frac{1}{d}\mathrm{Tr}(A_{\boldsymbol{u}}\rho),$$

- The DWF for a quantum measurement operator E_k is then the conditional (quasi-)probability function over Λ,

$$W_{E_k}(\boldsymbol{u}) = \mathrm{Tr}(A_{\boldsymbol{u}}E_k).$$

- Of course, the Born rule is reproduced by the law of total probability

$$\Pr(k) = \sum_{\boldsymbol{u}} W_\rho(\boldsymbol{u}) W_{E_k}(\boldsymbol{u}) = \mathrm{Tr}(\rho E_k)$$

Example of Discrete Wigner Representation for Qutrits

1/3	1/3	1/3
0	0	0
0	0	0

Figure: Wigner representation of qutrit $|0\rangle$ state

1/6	1/6	1/6
1/6	-1/3	1/6
1/6	1/6	1/6

Figure: Wigner representation of qutrit $|0\rangle - |1\rangle$ state

Resources for Quantum Computation?

Some Candidates

- Entanglement? ... Provably necessary in circuit model, but (largely) absent in DQC1.
- Purity/Coherence/Superposition? ... Unclear.
- Discord? ... Ok, probably not discord.
- Negative Wigner function and contextuality? ... Yes!

Quantum Resources

Resources arise naturally under operational restrictions, e.g., fundamental or practical restrictions on the quantum formalism.

Quantum Resources from operational restrictions

Limitations of fault-tolerant stabilizer computation give a set of resource-constraints for quantum computation!

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Resources for Fault Tolerance

Eastin-Knill, 2009

A transversal (and hence fault-tolerant) encoded gate set can not be universal.

Fault Tolerance with Stabilizer Operations

- Stabilizer operations are a typical choice of for fault tolerant gates - they form a subgroup of the unitary group.
- Stabilizer operations are not universal - this set is efficiently simulatable by the Gottesman-Knill theorem.
- This defines a natural restriction on the set of quantum operations.
- Thus an additional resource is needed for universal quantum computation - consumption of resource states.

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Magic State Computing (Bravyi, Kitaev 2005)

Magic State Model

- Operational restriction: only stabilizer operations (states, gates and projective measurement) can be realized
- Additional resource: preparation of non-stabilizer "magic" state ρ_R

Magic State Distillation

- Convert several noisy magic states ρ_R to produce a few very pure magic states $\tilde{\rho}_R$
- Consume pure magic states $\tilde{\rho}_R$ to perform non-stabilizer unitary gates (using only fault tolerant stabilizer operations)

An Open Question

Which non-stabilizer states promote stabilizer computation to universal quantum computation? Can answer this using DWF!

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Discrete Wigner Representation for Odd Dimension

1. Discrete Hudson's theorem (Gross, 2006): a pure state $|S\rangle$ has positive representation if and only if it is a stabilizer state. Hence for any state in STAB we know $\mathrm{Tr}(A_{\boldsymbol{u}}S) \geq 0 \ \forall \boldsymbol{u}$.
2. Clifford unitaries act as permutations of phase space. This means that if U is a Clifford then,

   $$W_{U\rho U^\dagger}(\boldsymbol{v}) = W_\rho(\boldsymbol{v}'),$$

   for each point $\boldsymbol{v}$.
3. Hence Clifford operations preserve non-negativity.
4. Note: only a small subset of the possible permutations of phase space correspond to Clifford operations.

Stabilizer Operations Preserve Positive Representation

Observation

Negative Wigner representation is a resource that can not be created by stabilizer operations.

Proof

Let $\rho \in L(\mathbb{C}_{d^n})$ be an n qudit quantum state with positive Wigner representation. Observe the following:

1. $U\rho U^\dagger$ is positively represented for any Clifford (stabilizer) unitary U.
2. $\rho \otimes S$ is positively represented for any stabilizer state S.
3. state-update, $M\rho M^\dagger / \mathrm{Tr}\left(M\rho M^\dagger\right)$, is positively represented for any stabilizer projector M.

A question

Positive Representation $\equiv$ Stabilizer State?

Do all non-stabilizer states have negative Wigner representation?

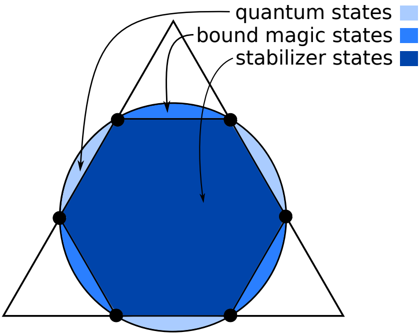
Stabilizer Polytope

Stabilizer Polytope

- Convex polytope with stabilizer states as vertices
- Can be defined from set of "facets"

Wigner Facets

The Wigner simplex has d^2 facets = small subset of stabilizer polytope facets



This is a cartoon.

Slice of the Quantum State Space and Stabilizer Polytope

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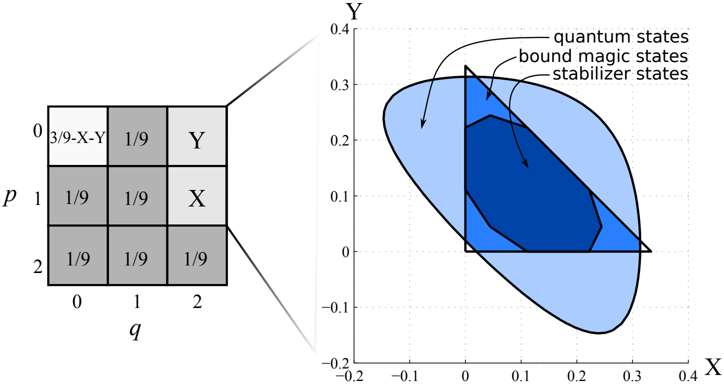


Figure: Slice defined by fixing six entries of the Wigner function and varying the remaining through their possible values to create the plot.

Magic States and Negative Quasi-Probability

Distillable Magic States for Odd Dimensional Qudits

- There is a large class of non-stabilizer quantum states (*bound magic states*) that are not useful for magic state distillation.
- Hence negative quasi-probability is necessary condition for a state to be distillable
- Is the boundary for negativity also a boundary for contextuality?

State-dependent contextuality

Use the graph-based contextuality formalism in Cabello, Severini and Winter (2010):

- Consider a set of binary yes-no tests, which we quantum mechanically represent by a set of rank-one projectors, Π, with eigenvalues $\lambda(\Pi) \in \{1, 0\}$.
- Compatible tests are those whose representative projectors commute, and a context is a set of mutually compatible tests.
- Commuting rank-1 projectors cannot both take on the value $+1$ i.e., the respective propositions are mutually exclusive and cannot both be answered in the affirmative.
- These (mutual orthogonality) relations can be represented by a graph Γ where connected vertices correspond to compatible and exclusive tests.

State-dependent contextuality

- Define an operator $\Sigma_\Gamma = \sum_{\Pi \in \Gamma} \Pi$
- Cabello, Severini and Winter (2010) show that
  - The maximum classical (non-contextual) assignment is
    $$\langle \Sigma_\Gamma \rangle^{\mathsf{NCHV}}_{\max} = \alpha(\Gamma)$$
    where $\alpha(\Gamma)$ is the independence number of the graph.
  - An independent set of a graph is a set of vertices, no two of which are adjacent. The independence number $\alpha(\Gamma) \in \mathbb{N}$ is the size of the largest such set.
  - The maximum quantum value
    $$\langle \Sigma_\Gamma \rangle^{\mathsf{QM}}_{\max} = \vartheta(\Gamma)$$
    where $\vartheta(\Gamma) \in \mathbb{R}$ is the Lovasz theta number which is the solution of a certain semidefinite program.

Graph of Stabilizer Projectors

We construct a set of stabilizer projectors for a system of two p-dimensional qudits such that:

$$\langle \Sigma_{\text{tot}} \rangle_{\text{max}}^{\text{QM}} = p^3 + 1.$$

Let

$$\Sigma_{\text{tot}} = \Sigma_{\text{sep}} + \Sigma_{\text{ent}} = p^3 \mathbb{I}_{p^2} - \left(A_{(0,0)} \otimes \mathbb{I}_p \right)$$

- Then for any state $\sigma \in \mathcal{H}_p$ we have

$$\text{Tr}\left[\Sigma_{\text{tot}} \left(\rho \otimes \sigma\right)\right] > p^3 \iff \text{Tr}\left[A_{(0,0)}\rho\right] < 0.$$

- Let $|\nu\rangle = \frac{|1\rangle - |p-1\rangle}{\sqrt{2}}$ we get

$$\text{Tr}\left[A_{(0,0)} |\nu\rangle\langle\nu|\right] = -1,$$

Graph of Stabilizer Projectors

What about the maximal NCHV assignment of 0 and 1 to vertices of the graph?

- Via exhaustive numerical search for $p = 3$ and $p = 5$ we show that

$$\alpha(\Gamma_{\mathrm{tot}}) = p^3 \Rightarrow \langle \Sigma_{\mathrm{tot}} \rangle^{\mathsf{NCHV}}_{\mathsf{max}} = p^3$$

- We conjecture this holds in general for all odd prime p.

Graph of Stabilizer Projectors

Hence for $p = 3$ and $p = 5$ and we conjecture for all odd p:

$$\langle \Sigma_{\mathrm{tot}} \rangle_{\mathsf{max}}^{\mathsf{NCHV}} = p^3 < \langle \Sigma_{\mathsf{tot}} \rangle_{\mathsf{max}}^{\mathsf{QM}} = p^3 + 1.$$

From the above it follows that:

- (i) a state is non-contextual if and only if it is positively represented in the discrete Wigner function,
- (ii) maximally negative states exhibit the maximum possible amount of contextuality

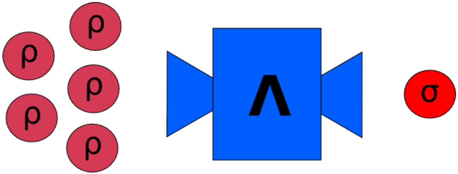
Magic State Computing (Bravyi, Kitaev 2005)

Magic State Model

- Operational restriction: perfect stabilizer operations (states, gates and projective measurement)
- Additional resource: preparation of non-stabilizer state ρ_R

Magic State Distillation

- Consume many resource states ρ_R to produce a few very pure resource states $\sigma \approx |\psi\rangle\langle\psi|$
- Inject $\sigma \approx |\psi\rangle\langle\psi|$ to perform non-stabilizer unitary gates (using only fault tolerant stabilizer operations)



Importance of Efficiency

Example

Fowler et al.[a] analyze the requirements to use Shor's algorithm to factor a 2000 bit number using physical qubits with realistic error rates[b]. Using a 2D surface code they find:

- Approximately one billion physical qubits are required.
- About 94% of these are used for magic state distillation.

[a] Fowler, Mariantoni, Martinis and Cleland (2012)
[b] Physical qubit error rate 0.1%, ancilla preparation error rate 0.5%

Main Result

Main Result: Magic Monotones

We identify and study two magic monotones:

- The *(regularized) relative entropy of magic*. This is most interesting in the asymptotic regime.
- The *mana*, a computable monotone based on the discrete Wigner function defined for odd dimensional systems.

As a corollary we find explicit, absolute bounds on the efficiency of magic state distillation.

Mana - Overview

Bound States

- Previous work: states with positive discrete Wigner function are not distillable.
- Positively represented states also not useful for quantum computation.
- Is the "amount" of negativity of the Wigner function meaningful?

Mana

- The *sum negativity* $\mathrm{sn}\rho$ is the sum of the negative entries of the Wigner function of ρ
- The *mana* is the additive variant of the sum negativity, $\mathcal{M}(\rho) = \log\left(2\mathrm{sn}(\rho) + 1\right)$

Mana - Definition

Magic Monotones

- *Mana*

  $\mathcal{M}(\rho) = \log\left(2\,\mathrm{sn}\,\rho + 1\right)$

Wigner negativity

The negativity of the DWF gives a *computable*, *quantitative* measure of resource for *universal* quantum computation.

-1/3	1/6	1/6
1/6	1/6	1/6
1/6	1/6	1/6

Figure: Sum negativity $= \frac{1}{3}$

2/9	2/9	1/9
2/9	-1/9	1/9
2/9	-1/9	1/9

Figure: Sum negativity $= \frac{2}{9}$

Quantum Foundations

Quantum mechanics has unfamiliar features

- Superposition, entanglement, collapse under measurement, tensor product structure of Hilbert space, non-locality, contextuality, negative (quasi-)probability ...

Which of these concepts are truly quantum and which are classical?

- **Classical concepts:** superposition, entanglement, collapse under measurement, tensor product structure of Hilbert space, ...
- **Quantum concepts:** Non-locality, contextuality, negative (quasi-)probability.

Summary and Open Questions

Summary

- Bound states for magic state distillation
- Negative Wigner function is a resource for FT stabilizer computation
- Negative quasi-probability and contextuality are equivalent resources

Related Results:

- Extension of Gottesman-Knill
- Entanglement in a LHV

Future Work

- Should we compute with qudits (quopits)?
- Is contextuality sufficient for distillability?
- How to extend the QPR approach to other operational restrictions?

Main Refs:

Veitch et al, NJP (2012)
Veitch et al, arxiv:1307.7171
Howard et al, forthcoming.

Entanglement from Epistemic Restriction

Entanglement without non-locality:

- The two qutrit Bell state

$$|B\rangle = \frac{|00\rangle + |11\rangle + |22\rangle}{\sqrt{3}}$$

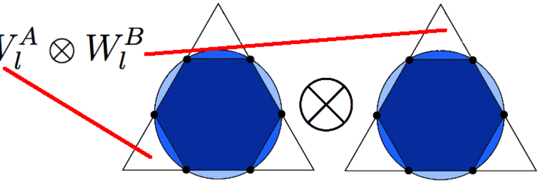
  is an entangled stabilizer state
- Its density operator does *not* admit a convex decomposition into factored qutrit states
- But under stabilizer measurements it can not exhibit any form of contextuality
- Morever, its discrete Wigner function *must* admit the decomposition

$$W_{|B\rangle\langle B|} = \Sigma_l p_l W_l^A \otimes W_l^B$$

Entanglement from Epistemic Restriction

- Note that W_l^A and W_l^B come from *forbidden regions* of the single-qutrit Wigner probability simplex – that is, W_l^A and W_l^B are not valid single qutrit quantum states

$$\sum_l p_l W_l^A \otimes W_l^B$$



- Entanglement arises naturally from the epistemic restriction, i.e. from incompleteness of quantum states!

Extended Gottesman-Knill Theorem

Weak simulation protocol for all states inside and some mixed states *outside* the stabilizer polytope!

Scope

- Prepare ρ with positive representation
- Act on input with Clifford U_F (corresponding to linear size F)
- Perform measurement $\{E_k\}$ with positive representation

Simulation Protocol

- Sample phase space point (u, v) according to distribution $W_\rho(u, v)$
- Evolve phase space point according to $(u, v) \rightarrow \boldsymbol{F}^{-1}(u, v)$
- Sample from measurement outcome according to $\tilde{W}_{\{E_k\}}(u, v)$

Continuous Variable Simulation for Linear Optics

Odd Dimension	*Infinite Dimension*
Stabilizer Operations	Linear Optics
Stabilizer States	Gaussian States
Discrete Wigner Function	Wigner Function

Table: Comparison of Odd and Infinite Dimensional Formalisms

Results

- There exist mixed states with positive Wigner representation that are not convex combinations of gaussian states (Brocker and Werner, 1995)
- Computations using linear optical transformations and measurements as well as preparations with positive Wigner function can be efficiently classically simulated.

Ref: Veitch, Wiebe, Ferrie and Emerson, NJP 15, 013037 (2013)