Gromov-Witten Invariants and Modular Forms

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Overview

- Background and motivation
- Solving topological string amplitudes in terms of quasi modular forms
- Example: $K_{\mathbb{P}^2}$
- Special geometry polynomial ring
- Conclusions and discussions

Background

Given a CY 3–fold Y, one of the most interesting problems to count the Gromov-Witten invariants of Y and consider the generating functions of genus g Gromov-Witten invariants of Y

$$F^g_{GW}(Y,t) = \sum_{\beta \in H_2(Y,\mathbb{Z})} \langle e^{\omega(t)} \rangle_{g,\beta} = \sum_{\beta \in H_2(Y,\mathbb{Z})} N^{GW}_{g,\beta}\, e^{\int_\beta \omega(t)} ,$$

$$\langle \omega_{i_1} \cdots \omega_{i_k} \rangle_{g,\beta} = \prod_{j=1}^{k} ev_j^* \omega_{i_j} \cap [\overline{\mathcal{M}}_{g,k}(Y,\beta)]^{vir} ,$$

here $\omega(t) = \sum_{i=1}^{h^{1,1}(Y)} t_i \omega_i$, where $\omega_i, i = 1, 2 \cdots h^{1,1}(Y)$ are the generators for the Kahler cone of Y.

For some special CY 3-folds, the generating functions $F^g_{GW}(Y,t)$ could be computed by localization technique, topological vertex, etc. For general CY 3-folds, they are very difficult to compute.

Background

- Physics (topological string theory)tells that $F^g_{GW}(Y,t)$ is the holomorphic limit of some non-holomorphic quantity called the A model genus g topological string partition function of Y

$$F^g_{GW}(Y,t) = \lim_{LVL} \mathcal{F}^g(Y,t,\bar{t})$$

  The above expression $\lim_{LVL}$ means the holomorphic limit based at the large volume limit $t = i\infty$: think of $t, \bar{t}$ as independent coordinates, fix t, send $\bar{t}$ to $\overline{i\infty}$.

- Mirror symmetry predicts the existence of the mirror manifold X of Y in the sense that $\mathcal{F}^g(Y)$ (and its holomorphic limit $F^g(Y)$) is identical to some quantity $\mathcal{F}^g(X)$ (and the holomorphic limit $F^g(X)$) called the B model genus g topological string partition function on X, under the mirror map.

Background

- The genus zero topological string partition function was studied intensively since the celerated work Candelas, de La Ossa, Green & Parkes (1991)
- The partition functions $\mathcal{F}^g(X), g \geq 1$ satisfy some differential equations called the holomorphic anomaly equations Bershadsky, Cecotti, Ooguri & Vafa (1993), and are easier to compute than $\mathcal{F}^g(Y)$.
- Thanks to mirror symmetry, one can try to extract Gromov-Witten invariants of Y by studying properties of (the moduli space of) X and solving $\mathcal{F}^g(X)$ from the equations.

Motivation

In some nicest cases, $\mathcal{F}^g(Y)$ (and its holomorphic limit $F^g_{GW}(Y)$) are expected to have some modular properties. Some examples include

- $Y =$ elliptic curve Rudd (1994), Dijkgraaf (1995), Kaneko & Zagier (1995)...

$$
\begin{aligned}
F^1_{GW}(t) &= -\log \eta(q), \quad q = \exp 2\pi i t \\
F^2_{GW}(t) &= \frac{1}{103680}(10E_2^3 - 6E_2E_4 - 4E_6) \\
F^g_{GW}(t) & \quad \text{is a quasi modular form of weight } 6g-6\ldots
\end{aligned}
$$

- STU model: $Y =$ a special $K3$ fibration
- FHSV model: $Y = K3 \times T^2/\mathbb{Z}^2$.
  $IIA - HE$ duality tells that $F^g(Y)$ have nice modular properties Kachru & Vafa (1995), Marino & Moore (1998), Klemm & Marino (2005), Maulik & Pandharipande (2006)...

Overview

In this talk, we shall work only on the B model of X. We shall

- solve $\mathcal{F}^g(X), g \geq 0$ from the holomorphic anomaly equations for certain noncompact CY 3–folds X and express them in terms of the generators of the ring of almost-holomorphic modular forms. The results we obtain predict the correct GW invariants of the mirror manifold Y under the mirror map.

- explore the duality of $\mathcal{F}^g(X)$ for these particular noncompact CY 3–folds

- construct the analogue of the ring of almost-holomorphic modular forms for general CY 3–folds by using quantities constructed out of the special Kahler geometry on the moduli space $\mathcal{M}_{complex}(X)$ of complex structures of X

Example

Here is an example we could compute :

$Y = K_{\mathbb{P}^2}$ Aganagic, Bouchard & Klemm (2006), [ASYZ]

$$
\begin{aligned}
F^1_{GW}(Y,t) &= -\frac{1}{2}\log \eta(q)\eta(q^3),\ q = \exp 2\pi i\tau,\ t \neq \tau \\
F^2_{GW}(Y,t) &= \frac{E(6A^4 - 9A^2E + 5E^2)}{1728B^6} + \frac{-\frac{8}{5}A^6 + \frac{2}{5}A^3B^3 + \frac{-8-3\chi}{10}B^6}{1728B^6} \\
&\dots
\end{aligned}
$$

where A, B, C, E are explicit quasi modular forms (with multiplier systems) of weights $1, 1, 1, 2$ respectively, with respect to the modular group $\Gamma_0(3)$. I will explain in detail how this modular group comes out.

Example

$$
\begin{aligned}
&F^3_{GW}(Y,t) \\
&= \frac{\left(-2532A^{10} + 3444A^7B^3 - 1140A^4B^6 + 48AB^9\right)E}{1244160B^{12}} \\
&+ \frac{\left(3516A^8 - 3708A^5B^3 + 732A^2B^6\right)E^2}{1244160B^{12}} \\
&+ \frac{\left(-2645A^6 + 1900A^3B^3 - 120B^6\right)E^3}{1244160B^{12}} \\
&+ \frac{\left(1200A^4 - 420AB^3\right)E^4}{1244160B^{12}} - \frac{25A^2E^5}{82944B^{12}} + \frac{5E^6}{82944B^{12}} \\
&+ \frac{5359A^{12} - 8864A^9B^3 + 4160A^6B^6 - 496A^3B^9 + 2(8-3\chi)B^{12}}{8709120B^{12}}
\end{aligned}
$$

Solving $\mathcal{F}^g$: Special Kahler metric

Consider the family of CY 3-folds $\pi : \mathcal{X} \to \mathcal{M}$, where $\mathcal{M}$ is some deformation space (of complex structures) of the CY 3–fold X. The base $\mathcal{M}$ is equipped with the Weil-Petersson metric whose Kahler potential is given by

$$e^{-K} = i \int \Omega \wedge \bar{\Omega} \tag{1}$$

where Ω is a holomorphic section of the Hodge line bundle $\mathcal{L} = R^0 \pi_* \Omega^3_{\mathcal{M}|\mathcal{X}}$. The curvature of the Weil-Petersson metric $G_{i\bar{j}}$ satisfies the so-called special geometry relation Strominger (1990)

$$-R_{i\bar{\jmath}}{}^{k}{}_{l} = \bar{\partial}_{\bar{\jmath}} \Gamma^k_{il} = \delta^k_l G_{i\bar{\jmath}} + \delta^k_i G_{l\bar{\jmath}} - C_{ilm} \overline{C}^{km}_{\bar{\jmath}} \tag{2}$$

where $C_{ijk} = -\int \Omega \wedge \partial_i \partial_j \partial_k \Omega$ (Yukawa coupling or three-point function) and $\overline{C}^{jk}_{\bar{\imath}} = e^{2K} G^{j\bar{\jmath}} G^{k\bar{k}} \overline{C}_{\bar{\imath}\bar{\jmath}\bar{k}}, i = 1, 2, \cdots n = dim\mathcal{M}$.

Solving $\mathcal{F}^g$: BCOV holomorphic anomaly equations

The genus g topological string partition function $\mathcal{F}^g$ is a section of $\mathcal{L}^{2-2g}$. For $g = 1$ case, it satisfies the holomorphic anomaly equation Bershadsky, Cecotti, Ooguri & Vafa (1993)

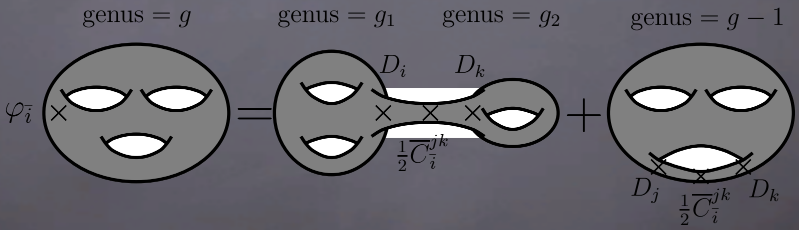
$$\overline{\partial}_{\bar{\imath}} \partial_j \mathcal{F}^1 = \frac{1}{2} C_{jkl} \overline{C}_{\bar{\imath}}^{jk} - \left(\frac{\chi}{24} - 1\right) G_{\bar{\imath} j} \tag{3}$$

where the quantity χ is the Euler characteristic of the mirror manifold Y(not of X).

Solving $\mathcal{F}^g$: BCOV holomorphic anomaly equations

For $g \geq 2$ Bershadsky, Cecotti, Ooguri & Vafa (1993)

$$\bar{\partial}_{\bar{i}}\mathcal{F}^g = \frac{1}{2}\bar{C}_{\bar{i}}^{jk}\left(\sum_{g_1+g_2=g,\, g_1,g_2\geq 1} D_j\mathcal{F}^{g_1}D_k\mathcal{F}^{g_2} + D_jD_k\mathcal{F}^{g-1}\right) \quad (4)$$



where D_i is the sum of the Chern connection associated to the Weil-Petersson metric $G_{i\bar{j}}$ and the Kahler connection K_i on the Hodge line bundle $\mathcal{L}$.

Solving $\mathcal{F}^g$: polynomial recursion

Using the special geometry relation and the holomorphic anomaly equation for genus 1, the equations could be solved recursively using integration by parts Bershadsky, Cecotti, Ooguri & Vafa (1993), Yamaguchi & Yau (2004), Alim & Länge (2007), for each genus g, the topological string partition function $\mathcal{F}^g$ is found to take the form

$$\bar{\partial}_{\bar{\imath}}\mathcal{F}^g = \bar{\partial}_{\bar{\imath}}\mathcal{P}^g,\ g \geq 2 \tag{5}$$

where $\mathcal{P}^g$ is a polynomial (coefficients are holomorphic functions) of the propagators S^{ij}, S^i, S defined via

$$\bar{\partial}_{\bar{k}} S^{ij} = \overline{C}^{ij}_{\bar{k}}, \quad \bar{\partial}_{\bar{k}} S^{i} = G_{k\bar{k}} S^{ik}, \quad \bar{\partial}_{\bar{k}} S = G_{k\bar{k}} S^{k} \tag{6}$$

and the Kahler connection K_i. The coefficients are kind of universal from recursion. These generators encode all of the anti-holomorphic dependence of $\mathcal{F}^g$.

Solving $\mathcal{F}^g$: differential ring of non-holomorphic generators

Moreover, the ring generated by these non-holomorphic generators S^{ij}, S^i, S, K_i is closed under the covariant derivative D_i Alim & Länge (2007)

$$
\begin{aligned}
D_i S^{jk} &= \delta_i^j S^k + \delta_i^k S^j - C_{imn} S^{mj} S^{nk} + h_i^{jk}, \\
D_i S^j &= 2\delta_i^j S - C_{imn} S^m S^{nj} + h_i^{jk} K_k + h_i^j, \\
D_i S &= -\frac{1}{2} C_{imn} S^m S^n + \frac{1}{2} h_i^{mn} K_m K_n + h_i^j K_j + h_i, \\
D_i K_j &= -K_i K_j - C_{ijk} S^k + C_{ijk} S^{kl} K_l + h_{ij},
\end{aligned}
\tag{7}
$$

where $h_i^{jk}, h_i^j, h_i, h_{ij}$ are holomorphic functions.

Solving $\mathcal{F}^g$: holomorphic ambiguities

From

$$\bar{\partial}_{\bar{i}}\mathcal{F}^g = \bar{\partial}_{\bar{i}}\mathcal{P}^g,\ g \geq 2 \tag{8}$$

we get

$$\mathcal{F}^g = \mathcal{P}^g(S^{ij}, S^i, S, K_i) + f^g,\ g \geq 2 \tag{9}$$

The function f^g is purely holomorphic and is called the holomorphic ambiguity. It can not be determined by only looking at the holomorphic anomaly equations. Boundary conditions are needed to fix it.

Solving $\mathcal{F}^g$: Boundary condition at LCSL

Mirror symmetry predicts that Bershadsky, Cecotti, Ooguri & Vafa (1993)

$$F^1_A(X,t) := \lim_{LCSL} \mathcal{F}^1(X,t,\bar{t})$$

$$\Leftrightarrow \quad F^1_{GW}(Y,t) = -\frac{1}{24}\sum_{i=1}^{h^{1,1}(Y)} t_i \int c_2(TY)\omega_i + \mathcal{O}(e^{2\pi i t_i})$$

$$F^g_A(X,t) := \lim_{LCSL} \mathcal{F}^g(X,t,\bar{t})$$

$$\Leftrightarrow \quad F^g_{GW}(Y,t) = (-1)^g \frac{\chi}{2}\frac{|B_{2g}B_{2g-2}|}{2g(2g-2)(2g-2)!} + \mathcal{O}(e^{2\pi i t_i}), \quad g \geq 2$$

where $t = (t^1, t^2, \cdots t^n)$ are the canonical coordinates on the moduli spaces and $\lim_{LCSL}$ means the holomorphic limit at the large complex structure limit. These conditions are obtained from computing the constant map contribution to $F^g_{GW}(Y,t)$.

Solving $\mathcal{F}^g$: boundary condition at conifold loci

Moreover, at the conifold loci Bershadsky, Cecotti, Ooguri & Vafa (1993), Ghoshal & Vafa (1995), Huang & Klemm (2006)

$$F^1_{con}(t^i_c) := \lim_{con_i} \mathcal{F}^1 = -\frac{1}{12}\log\Delta_i + regular\ \ terms$$

$$F^g_{con}(t^i_c) := \lim_{con_i} \mathcal{F}^g = \frac{c^{g-1}B_{2g}}{2g(2g-2)(t^i_c)^{2g-2}} + \mathcal{O}((t^i_c)^0), \quad g \geq 2$$

where $\lim_{con_i}$ means the holomorphic limit at the conifold locus which is defined as the discriminant locus $\Delta_i = 0$, $i = 1, 2, \cdots, m$. The quantity t^i_c is a suitably chosen coordinate normal to the conifold locus $\Delta_i = 0$. This is called the gap condition.

Solving $\mathcal{F}^g$: traditional approach in fixing f^g

According to the above boundary conditions at the LCSL and conifold loci, one can try the following *ansatz* for f^g Bershadsky, Cecotti, Ooguri & Vafa (1993), Katz, Klemm & Vafa (1999), Yamaguchi & Yau (2004)

$$f^g(z) = \sum_{i=1}^{m} \frac{A_i^g}{\Delta_i^{2g-2}} \tag{10}$$

where A_i^g is a polynomial of $z_1, \cdots z_n$ of degree $(2g-2)deg\Delta_i$ Then one aims to solve for the coefficients in A_i^g from the boundary conditions. For noncompact CY 3–folds, a dimension counting Haghighat, Klemm & Rauch (2008) suggests that the numbers of unknowns is the same as the number of boundary conditions. So in principle, f^g could be completely determined. For compact CY 3–folds, further inputs, e.g., boundary conditions at the orbifold loci are needed Katz, Klemm & Vafa (1999), Huang & Klemm (2006).

Solving $\mathcal{F}^g$: difficulties

- The boundary conditions are applied to the different holomorphic limits of the same function $\mathcal{F}^g$, namely, to F^g_A and F^g_{con}. But the relation between these two different functions is not quite clear. They are not related by a simple analytic continuation (recall that the definition of a holomorphic limit requires a choice of the base point).

- The *ansatz* for the ambiguity f^g is made based on regularity at the orbifold points (loci), which is not ensured for general CYs.

- In practice, the generators S^{ij}, S^i, S, K_i are computed as infinite series in the complex coordinates on $\mathcal{M}$, so the expression for $\mathcal{F}^g$ is not compact.

- Modularity is not manifest.

Solving $\mathcal{F}^g$: implement of modularity

In our work, we could overcome these difficulties by making use of the arithmetic properties of the moduli spaces, for certain noncompact CY 3–folds X. We could solve $\mathcal{F}^g$ in terms of modular forms according to the following procedure:

- identify the moduli space $\mathcal{M}$ with a certain modular curve $X_\Gamma = \mathcal{H}^*/\Gamma$, $\Gamma \subseteq PSL(2,\mathbb{Z})$
- construct the ring of quasi modular forms and almost-holomorphic modular forms attached to X_Γ, the latter turns out to be equivalent to the special geometry polynomial ring constructed out of purely geometric quantities (periods, connections, Yukawa couplings...)

Solving $\mathcal{F}^g$: implement of modularity

Then we

1. express the quantities S^{ij}, S^i, S, K_i and $\mathcal{F}^g, g = 0, 1$ in terms of the generators of the ring of almost-holomorphic modular forms

2. solve $\mathcal{P}^g$ via polynomial recursion in terms of these generators, modularity then gives a very strong constraint and also a natural *ansatz* for f^g in terms of modular forms, with under-determined coefficients

3. explore the relation between F_A^g and F_{con}^g, realize the boundary conditions as certain regularity conditions imposed on the modular objects $\mathcal{F}^g$ and F_A^g

4. solve the under-determined coefficients in f^g and thus express $\mathcal{F}^g$s (F_A^gs) as almost-holomorphic (quasi) modular forms

Local $\mathbb{P}^2$ example: mirror CY 3–fold family

The mirror CY family of $Y = K_{\mathbb{P}^2}$ (called local $\mathbb{P}^2$ below) is a family of noncompact CY 3-folds $\pi : \mathcal{X} \to \mathcal{M} \cong \mathbb{P}^1$ given by Chiang, Klemm, Yau & Zaslow (1999), Hori & Vafa (2000). More precisely, choose z as the parameter for the base $\mathcal{M}$. For each z, the CY 3–fold $\mathcal{X}_z$ is itself a conic fibration

$$uv - H(y_i; z) := uv - (y_0 + y_1 + y_2 + y_3) = 0, (u, v) \in \mathbb{C}^2$$

over the base $\mathbb{C}^2$ parametrized by $y_i, i = 0, 1, 2, 3$ with the following conditions:

a. there is a $\mathbb{C}^*$ action: $y_i \mapsto \lambda y_i, \lambda \in \mathbb{C}^*, i = 0, 1, 2, 3$;
b. $z = -\frac{y_1 y_2 y_3}{y_0^3}$.

Straightforward computation shows that $z = 0, 1/27, \infty$ corresponds to the large complex structure limit, conifold point, orbifold of the CY 3–fold family, respectively.

Local $\mathbb{P}^2$ example: mirror curve family

The degeneration locus of this conic fibration is a curve $\mathcal{E}_z$ (called the mirror curve) sitting inside $\mathcal{X}_z$:

$$H(y_i; z) = y_0 + y_1 + y_2 + y_3 = 0,\ z = -\frac{y_1 y_2 y_3}{y_0^3}\,.$$

This way, we get the mirror curve family $\pi : \mathcal{E} \to \mathcal{M}$. It is equivalent to

$$y_3^2 - (y_0 + 1)y_3 = z y_0^3 \tag{11}$$

with

$$\Delta = -27 + \frac{1}{z},\ j(z) = \frac{(1 - 24z)^3}{z^3(1 - 27z)}$$

Local $\mathbb{P}^2$ example: arithmetic of moduli space

Comparing it with the elliptic modular surface associated to $\Gamma_0(3)$, i.e., the Hesse family $\pi_{\Gamma_0(3)} : \mathcal{E}_{\Gamma_0(3)} \to X_{\Gamma_0(3)} = \mathcal{H}^*/\Gamma_0(3)$

$$x_1^3 + x_2^3 + x_3^3 - z^{-\frac{1}{3}} x_1 x_2 x_3 = 0, \quad j(z) = \frac{(1+216z)}{z(1-27z)^3}$$

One can see that Aganagic, Bouchard & Klemm (2006), Haghihat, Klemm & Rauch (2008), [ASYZ]

$$\mathcal{M} \cong X_0(3) := \mathcal{H}^*/\Gamma_0(3)$$

In fact these two families are related by a 3–isogeny, see e.g., Elliptic Curves (GTM) Husemöller. (2004).
Taking the generator of the rational functional field (Hauptmodul) of $X_0(3)$ to be $\alpha = 27z$, then the points $\alpha = 0, 1, \infty$ correspond to the large complex structure limit, conifold point, orbifold of the CY 3–fold family respectively.

Local $\mathbb{P}^2$ example: periods of elliptic curve family

The Picard-Fuchs operator attached to the Hesse elliptic curve family is the hypergeometric operator Klemm, Lian, Roan & Yau (1994), Lian & Yau (1994)

$$\mathcal{L}_{elliptic} = \theta^2 - \alpha(\theta + \frac{1}{3})(\theta + \frac{2}{3}) \tag{12}$$

where $\theta = \alpha \frac{\partial}{\partial \alpha}$. A basis of the space of periods could be chosen to be

$$\omega_0 = {}_2F_1(\frac{1}{3}, \frac{2}{3}, 1; \alpha), \quad \omega_1 = \frac{i}{\sqrt{3}} {}_2F_1(\frac{1}{3}, \frac{2}{3}, 1; 1-\alpha) \tag{13}$$

Then we get

$$\tau = \frac{\omega_1}{\omega_0} = \frac{i}{\sqrt{3}} \frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}, 1; 1-\alpha)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}, 1; \alpha)} \tag{14}$$

It follows then the points $\alpha = 0, 1, \infty$ correspond to $[\tau] = [i\infty], [0], [\exp 2\pi i/3]$, respectively.

Local $\mathbb{P}^2$ example: periods of elliptic curve family

Define Maier (2006)

$$A = \omega_0, \quad B = (1-\alpha)^{\frac{1}{3}} A, \quad C = \alpha^{\frac{1}{3}} A \tag{15}$$

then

$$A^3 = B^3 + C^3$$

Moreover, one can show

$$Div\ A = \frac{1}{3}(\alpha = \infty), \quad Div\ B = \frac{1}{3}(\alpha = 1), \quad Div\ A = \frac{1}{3}(\alpha = 0). \tag{16}$$

Local $\mathbb{P}^2$ example: modular forms

It turns out that the A, B, C defined out of periods are modular forms (with multiplier systems) with respect to $\Gamma_0(3)$ (see e.g., Maier (2006), The 1-2-3 of Modular Forms (2006) and references therein). Moreover, they have very nice θ or η expansions ($q = \exp 2\pi i\tau$)

$$
\begin{aligned}
A(\tau) &= \theta_2(2\tau)\theta_2(6\tau) + \theta_3(2\tau)\theta_3(6\tau) = \sum_{(m,n)\in\mathbb{Z}^2} q^{m^2-mn+n^2} \\
B(\tau) &= \frac{\eta(\tau)^3}{\eta(3\tau)} = \sum_{(m,n)\in\mathbb{Z}^2} (\exp(\frac{2\pi i}{3}))^{m+n} q^{m^2-mn+n^2} \\
C(\tau) &= 3\frac{\eta(3\tau)^3}{\eta(\tau)} = q^{\frac{1}{3}} \sum_{(m,n)\in\mathbb{Z}^2} q^{m^2-mn+n^2+m+n} = \frac{1}{2}(A(\frac{\tau}{3}) - A(\tau))
\end{aligned}
$$

Local $\mathbb{P}^2$ example: quasi modular forms

Now we consider the differential ring structure. One can show that

$$D\alpha = \alpha\beta A^2 \tag{17}$$

where $D = \frac{1}{2\pi i}\frac{\partial}{\partial\tau}, \beta := 1-\alpha$. This is equivalent to

$$A^2 = D\log\frac{C^3}{B^3}(=\frac{3E_2(3\tau)-E_2(\tau)}{2}) \tag{18}$$

which transforms as an honest modular form under $\Gamma_0(3)$. Now we define the analogue of the quasi modular form E_2 by

$$E = D\log C^3B^3(=\frac{3E_2(3\tau)+E_2(\tau)}{4}) \tag{19}$$

It is easy to see that it transforms as a quasi modular form under $\Gamma_0(3)$. We denote its modular completion $E + \frac{-3}{\pi Im\tau}(\frac{3}{4}\cdot\frac{1}{3}+\frac{1}{4})$ by $\hat{E}$.

Local $\mathbb{P}^2$ example: ring of quasi modular forms

It follows from the definitions and the Picard-Fuchs equation that the ring generated by A, B, C, E is closed upon taking derivative $D = \frac{1}{2\pi i}\frac{\partial}{\partial \tau}$ [ASYZ]

$$\begin{aligned} DA &= \frac{1}{6}A(E + \frac{C^3 - B^3}{A}) \\ DB &= \frac{1}{6}B(E - A^2) \\ DC &= \frac{1}{6}A(E + A^2) \\ DE &= \frac{1}{6}(E^2 - A^4) \end{aligned} \tag{20}$$

This is the ring of quasi modular forms (with multiplier systems) for $\Gamma_0(3)$. The ring generated by $\hat{E}, A, B, C$ is then the ring of almost-holomorphic modular forms (with multiplier systems) .

Local $\mathbb{P}^2$ example: Fricke involution on modular curve

There is a natural automorphism, called Fricke involution W_N, of the family $\pi_{\Gamma_0(3)} : \mathcal{E}_{\Gamma_0(3)} \to X_{\Gamma_0(3)}$. Here $N = 3$.
In terms of coordinates, it is described by

$$W_N : \tau \mapsto -\frac{1}{N\tau}, \alpha \mapsto \beta := 1 - \alpha \tag{21}$$

Using the interpretation of the modular curve as a moduli space

$$X_0(3) = \{(E, C) |\ C < E_N \cong \mathbb{Z}_N^2, |C| = N\}$$

then the Fircke involution has the following description

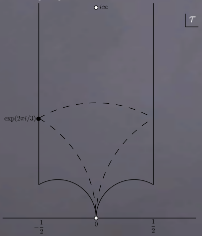
$$W_N : (E, C) \mapsto (E/C, E_N/C)$$

The mirror curve family $\mathcal{X}$ is related to $\mathcal{E}_{\Gamma_0(3)}$ by a 3–isogeny and the Fricke involution is also an automorphism of $\mathcal{X}$.

Local $\mathbb{P}^2$ example: Fricke involution on modular curve

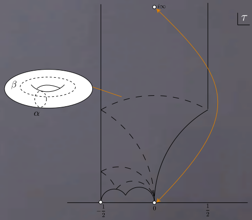
$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \; c \equiv 0 \mod N \right\} \subseteq PSL(2, \mathbb{Z})$$

$\Gamma_0(3)$



Fricke involution

$$W_N : \tau \rightarrow -\frac{1}{N\tau}$$



For $X_0(N) = \mathcal{H}^*/\Gamma_0(N)$, $N = 4, 3, 2, 1^*$ which has three singular points, the Fricke involution exchanges the two distinguished cusps $[i\infty] = [1/N]$ and $[0]$ on the modular curve, and fixes the last of the singularities on $X_0(N)$.

Local $\mathbb{P}^2$: Fricke involution on modular forms

Under this transformation, one has Maier (2006), [ASYZ]

$$
\begin{aligned}
A(\tau) &\mapsto \frac{\sqrt{N}}{i}\tau A(\tau) \\
B(\tau) &\mapsto \frac{\sqrt{N}}{i}\tau C(\tau) \\
C(\tau) &\mapsto \frac{\sqrt{N}}{i}\tau B(\tau) \\
\hat{E}(\tau,\bar{\tau}) &\mapsto -(\frac{\sqrt{N}}{i}\tau)^2 \hat{E}(\tau,\bar{\tau})
\end{aligned}
\tag{22}
$$

This involution turns out to be a "duality" for the topological string partition functions $\mathcal{F}^g$, as I shall explain below.

Local $\mathbb{P}^2$ example: Picard-Fuchs and periods of the CY 3-fold

The Picard-Fuchs operator of the corresponding CY 3-fold $X : uv - H(y_i, z) = 0$ is Lerche, Mayr, & Warner (1996), Chiang, Klemm, Yau & Zaslow (1999), Haghighat, Klemm & Rauch (2008)

$$\mathcal{L}_{CY} = \mathcal{L}_{elliptic} \circ \theta \tag{23}$$

The periods are given by $X^0 = 1, t, t_c = \kappa^{-1} F_t$, where $t \sim \log \alpha + \cdots$ near the LCSL $\alpha = 0$, t_c is the vanishing period at the conifold point $\alpha = 1$, and κ is the classical triple intersection number of the mirror $Y = K_{\mathbb{P}^2}$ of X. Here we have chosen the normalization of t_c so that

$$\theta t = \omega_0, \quad \theta t_c = \omega_1 \tag{24}$$

Then

$$\tau = \frac{\omega_1}{\omega_0} = \frac{\theta t_c}{\theta t} = \kappa^{-1} F_{tt} \tag{25}$$

Local $\mathbb{P}^2$ example: holomorphic limit at the LCSL

Recall the large complex structure limit is given by $\alpha = 0$ or equivalently $[\tau] = [i\infty]$ on $X_0(3)$.

In the following we shall compute the special geometry quantities (connections, Yukawa coupling$\cdots$) in the holomorphic limit at this particular point. It is in this limit that $\mathcal{F}^g$ becomes $F^g_A(X)$ and is mirror to the generating function $F^g_{GW}(Y)$ of the GW invariants on Y.

Using the modularity. we shall see that the full non-holomorphic partition function $\mathcal{F}^g$ could be recovered by its holomorphic limit F^g_A Huang & Klemm (2006), Aganagic, Bouchard & Klemm (2006)

Local $\mathbb{P}^2$ example: special geometry quantities in the holomorphic limit

Take the coordinate $x = \ln \alpha$ near the LCSL. In this coordinate the holomorphic limit of the connections are

$$\lim K_x = 0, \lim \Gamma^x_{xx} = \partial_x \log \frac{\partial t}{\partial x}$$

The Yukawa coupling is

$$C_{xxx} = \frac{\kappa}{\beta}$$

where κ is the classical triple intersection number, it is $-\frac{1}{3}$ in the local $\mathbb{P}^2$ case. Using the boundary conditions for $\mathcal{F}^1$ at LCSL and at conifold, the holomorphic limit of the genus one partition function is solved to be

$$F^1_A = -\frac{1}{2} \log \theta t + \log \beta^a \alpha^{b+\frac{1}{2}}$$

where $a = -\frac{1}{12}, b + \frac{1}{2} = -\frac{1}{24} \int c_2(TY) J = -\frac{2}{24}$.

Local $\mathbb{P}^2$ example: special geometry quantities in terms of modular forms

Recall $\alpha = \frac{C^3}{A^3}, \beta = \frac{B^3}{A^3}$, using the η expansions

$$C = 3\frac{\eta(3\tau)^3}{\eta(\tau)}, B = \frac{\eta(\tau)^3}{\eta(3\tau)}$$

and the definition of A by $A = \omega_0 = \theta t$, we then get

$$C_{ttt} = \frac{1}{(X^0)^2} C_{xxx} (\frac{\partial x}{\partial t})^3 = \frac{\kappa}{B^3} = -\frac{1}{3}\frac{\eta(3\tau)^3}{\eta(\tau)^9}$$

$$F_A^1 = -\frac{1}{12}\log B^3 C^3 = -\frac{1}{2}\log \eta(\tau)\eta(3\tau),\ DF_A^1 = -\frac{1}{12}E$$

here as before $D = \frac{1}{2\pi i}\frac{\partial}{\partial \tau}$. In particular,

$$\mathcal{F}^1 = -\frac{1}{2}\log \sqrt{Im\tau}\sqrt{Im3\tau}\,\eta(\tau)\overline{\eta(\tau)}\eta(3\tau)\overline{\eta(3\tau)},\ D\mathcal{F}^1 = -\frac{1}{12}\hat{E}$$

Local $\mathbb{P}^2$ example: generators in terms of modular forms

The generators S^x, S are chosen so that:

$$\lim S^x = \lim S = 0, \lim K_x = 0$$

while $\lim S^{xx}$ is solved according to the integrated special geometry relation

$$\Gamma^k_{ij} = \delta^k_i K_j + \delta^k_j K_i - C_{ijm} S^{mk} + s^m_{ij}$$

In this case, it simplifies to

$$\lim \Gamma^x_{xx} = \lim 2K_x - C_{xxx} \lim S^{xx} + s^x_{xx}$$

That is,

$$\theta \log A = -\frac{\kappa}{\beta} \lim S^{xx} + s^x_{xx}$$

Local $\mathbb{P}^2$ example: generators in terms of modular forms

Recall that $D\alpha = \alpha\beta A^2$ with $D = \frac{1}{2\pi i}\frac{\partial}{\partial\tau}$, one has

$$\theta = \alpha\frac{\partial}{\partial\alpha} = 2\pi i\alpha\frac{\partial\tau}{\partial\alpha}D = \frac{1}{\beta A^2}D$$

Then the above equation becomes

$$\frac{1}{\beta A^2}D\log A = \frac{1}{\beta A^2}\frac{1}{6}(E + \frac{C^3 - B^3}{A}) = -\frac{\kappa}{\beta}\lim S^{xx} + s^x_{xx}$$

A natural choice for s^x_{xx} is

$$s^x_{xx} = \frac{C^3 - B^3}{6\beta A^3} = \frac{C^3 - B^3}{6B^3}$$

so that

$$\lim S^{xx} = \frac{E}{-6\kappa A^2} = \frac{1}{2}\frac{E}{A^2}$$

Local $\mathbb{P}^2$ example: differential ring of generators

Recall the differential ring of generators

$$
\begin{aligned}
D_i S^{jk} &= \delta_i^j S^k + \delta_i^k S^j - C_{imn} S^{mj} S^{nk} + h_i^{jk}, \\
D_i S^j &= 2\delta_i^j S - C_{imn} S^m S^{nj} + h_i^{jk} K_k + h_i^j, \\
D_i S &= -\frac{1}{2} C_{imn} S^m S^n + \frac{1}{2} h_i^{mn} K_m K_n + h_i^j K_j + h_i, \\
D_i K_j &= -K_i K_j - C_{ijk} S^k + C_{ijk} S^{kl} K_l + h_{ij},
\end{aligned}
\tag{26}
$$

where $h_i^{jk}, h_i^j, h_i, h_{ij}$ are holomorphic functions.

According to our choices, the only nontrivial equation in the above is

$$DS^{xx} = -C_{xxx} S^{xx} S^{xx} + h_{xx}^x$$

Local $\mathbb{P}^2$ example: differential ring of generators

Taking the holomorphic limit, one then gets

$$\partial_x \lim S^{xx} + 2 \lim \Gamma_{xx} \lim S^{xx} = -C_{xxx} \lim S^{xx} S^{xx} + h_x^{xx}$$

It follows that

$$h_x^{xx} = -\frac{1}{12} \frac{A^3}{B^3}$$

Note that the quantities $s = s_{xx}^x = \frac{C^3 - B^3}{6B^3}, h = h_x^{xx} = -\frac{1}{12}\frac{A^3}{B^3}$ are modular and thus honest holomorphic objects in the non-holomorphic completion (due to the fact that the non-holomorphic completion is the same as the completion to almost-holomorphic modular objects.) That is, these holomorphic ambiguities are really holomorphic as they should be.

Local $\mathbb{P}^2$ example: polynomial part $\mathcal{P}^g$

Polynomial recursion gives

$$\mathcal{P}^2 = \frac{5}{24}C^2S^3 + \frac{1}{4}ChS - \frac{3}{8}CS^2s + \frac{1}{8}S^2\partial C \qquad (27)$$

Straightforward computations show that its holomorphic limit at LCSL is

$$P_A^2 = \frac{E(6A^4 - 9A^2E + 5E^2)}{1728B^6} \qquad (28)$$

This implies in return that the non-holomorphic quantity $\mathcal{P}^2$ is

$$\mathcal{P}^2 = \frac{\hat{E}(6A^4 - 9A^2\hat{E} + 5\hat{E}^2)}{1728B^6} \qquad (29)$$

That is, the non-holomorphic quantity $\mathcal{P}^2$ could be obtained from its holomorphic limit P_A^2, thanks to (quasi) modularity.

Local $\mathbb{P}^2$ example: holomorphic ambiguity in terms of modular forms

By induction, one can prove that $\mathcal{P}^g$ and $\mathcal{F}^g$ are non-holomorphic completions of quasi modular functions (that is, modular weights are 0) for $g \geq 2$. That is, they are almost-holomorphic modular functions.

The above result for $\mathcal{P}^2$ suggests the following ansatz for the holomorphic ambiguity f^2

$$f^2 = \frac{c_1 A^6 + c_2 A^3 B^3 + c_3 B^6}{1728 B^6} \tag{30}$$

(In fact, this form could be obtained by considering the singularities of $\mathcal{F}^g$ on the deformation space $\mathcal{M}$.)

Local $\mathbb{P}^2$ example: Boundary condition at LCSL

Boundary condition at the LCSL given by $\alpha = 0$ or equivalently $t = i\infty$ is

$$F_A^g = (-1)^g \frac{\chi}{2} \frac{|B_{2g} B_{2g-2}|}{2g(2g-2)(2g-2)!} + \mathcal{O}(e^{2\pi i t}), \quad g \geq 2$$

It is easy to apply this boundary condition since the holomorphic limits of the quantities $A, B, C, \hat{E}$ based at the LCSL are very easy to compute from their expressions in terms of hypergeometric functions. For example,

$$A(\alpha) = {}_2F_1(\frac{1}{3}, \frac{2}{3}, 1; \alpha) = 1 + \frac{2\alpha}{9} + \frac{10\alpha^2}{81} + \frac{560\alpha^3}{6561} + \frac{3850\alpha^4}{59049} + \frac{28028\alpha^5}{531441} + \cdots$$

$$B(\alpha) = (1-\alpha)^{\frac{1}{3}} {}_2F_1(\frac{1}{3}, \frac{2}{3}, 1; \alpha) = 1 - \frac{\alpha}{9} - \frac{5\alpha^2}{81} - \frac{277\alpha^3}{6561} - \frac{1880\alpha^4}{59049} + \cdots$$

Local $\mathbb{P}^2$ example: Gap condition at the conifold point

The expression $t_c(\beta)$ is clear near the conifold point $\alpha = 1$ or equivalently $\beta = 0$ since t_c is chosen to be the vanishing period of the Picard-Fuchs equation of the CY 3–fold at the this point.

From the expansion of the period

$$t_c(\beta) = \beta + \frac{11\beta^2}{18} + \frac{109\beta^3}{243} + \frac{9389\beta^4}{26244} + \frac{88351\beta^5}{295245} + \cdots$$

solved as the vanishing period of $\mathcal{L}_{CY} = \mathcal{L}_{elliptic} \circ \theta$, we can invert the series to get $\beta = \beta(t_c)$:

$$\beta(t_c) = t_c - \frac{11t_c^2}{18} + \frac{145t_c^3}{486} - \frac{6733t_c^4}{52488} + \frac{120127t_c^5}{2361960} + \cdots$$

Local $\mathbb{P}^2$ example: Gap condition at the conifold point

To apply the gap condition

$$F^g_{con}(t_c) = \frac{c^{g-1} B_{2g}}{2g(2g-2)(t_c)^{2g-2}} + \mathcal{O}(t_c^0), \quad g \geq 2,$$

one needs to evaluate the holomorphic limit F^g_{con} from $\mathcal{F}^g = \mathcal{P}^g + f^g$. In the holomorphic limit based at the conifold, the holomorphic limit of f^g is itself, while the holomorphic limit P^g_{con} of $\mathcal{P}^g$ is different from (the analytic continuation of) P^g_A since a different base point is taken. Then one needs to compute the β or t_c expansion of F^g_{con}.

Local $\mathbb{P}^2$ example: series expansion in dual coordinates

We have obtained the $\alpha, \bar{\alpha}$ expansions of $\mathcal{F}^g(\alpha, \bar{\alpha})$, we could try to do analytic continuation to get $(\mathcal{F}^g \circ (\alpha, \bar{\alpha}))(\beta, \bar{\beta})$. But it is extremally complicated to do this directly.

Local $\mathbb{P}^2$ example: series expansion in dual coordinates

Instead of doing analytic continuation to get $(\mathcal{F}^g \circ (\alpha, \bar{\alpha}))(\beta, \bar{\beta})$, we make use of the Fricke involution as follows

$$
\begin{aligned}
A(\alpha) &= \frac{1}{\frac{\sqrt{N}}{i}\tau} A(\beta) \\
B(\alpha) &= (1-\alpha)^{\frac{1}{3}} A(\alpha) = \beta^{\frac{1}{3}} \frac{1}{\frac{\sqrt{N}}{i}\tau} A(\beta) = \frac{1}{\frac{\sqrt{N}}{i}\tau} C(\beta) \\
C(\alpha) &= \alpha^{\frac{1}{3}} A(\alpha) = \frac{1}{\frac{\sqrt{N}}{i}\tau} B(\beta) \\
\hat{E}(\alpha, \bar{\alpha}) &= -\left(\frac{1}{\frac{\sqrt{N}}{i}\tau}\right)^2 \hat{E}(\beta, \bar{\beta})
\end{aligned}
$$

here we treat A, B, C, E as functions $A(\bullet), B(\bullet), \cdots$. More precisely, $A(\bullet) = {}_2F_1(\frac{1}{3}, \frac{2}{3}, 1; \bullet), \cdots$

Local $\mathbb{P}^2$ example: Fricke involution

At first glance, it seems that we only used the definition of τ in terms of $A(\alpha), A(\beta)$ to get the series expansion in the dual β coordinate for all of the generators. However, what is really working is the Fricke involution:

$$W_N : \tau \leftrightarrow -\frac{1}{N\tau}$$

$$\alpha \leftrightarrow \beta$$

$${}_2F_1(\frac{1}{3}, \frac{2}{3}, 1; \alpha) \leftrightarrow {}_2F_1(\frac{1}{3}, \frac{2}{3}, 1; \beta)$$

$$A(\tau) \leftrightarrow A(W_N\tau)$$

$$\mathcal{F}^g(\tau, \bar{\tau}) \leftrightarrow \mathcal{F}^g|_{W_N}(W_N\tau, \overline{W_N\tau})$$

with

$$\frac{\sqrt{N}}{i}\tau = \frac{A(W_N\tau)}{A(\tau)} = \frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}, 1; \beta)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}, 1; \alpha)}$$

Local $\mathbb{P}^2$ example: Fricke involution

Since $\mathcal{F}^g$ has weight 0, the $\frac{\sqrt{N}}{i}\tau$ factors are cancelled out. From

$$\mathcal{P}^2(\alpha,\bar{\alpha}) = \frac{\hat{E}(\alpha,\bar{\alpha})(6A(\alpha)^4 - 9A(\alpha)^2\hat{E}(\alpha,\bar{\alpha}) + 4\hat{E}(\alpha,\bar{\alpha})^2)}{1728B^6(\alpha)}$$

$$f^2(\alpha) = \frac{c_1A^6(\alpha) + c_2A^3(\alpha)B^3(\alpha) + c_3B^6(\alpha)}{1728B^6(\alpha)}$$

we then get

$$\mathcal{P}^2(\alpha(\beta,\bar{\beta}),\bar{\alpha}(\beta,\bar{\beta}))$$
$$= \frac{(-\hat{E}(\beta,\bar{\beta}))(6A(\beta)^4 - 9A(\beta)^2(-\hat{E}(\beta,\bar{\beta})) + 4(-\hat{E}(\beta,\bar{\beta})^2))}{1728C^6(\beta)}$$

$$f^2(\alpha(\beta)) = \frac{c_1A^6(\beta) + c_2A^3(\beta)C^3(\beta) + c_3C^6(\beta)}{1728C^6(\beta)}$$

Local $\mathbb{P}^2$ example: solving the unknowns

Therefore, we can easily get the β expansion of

$$
\begin{aligned}
F^2_{con}(\beta) &= \lim(\mathcal{P}^2(\alpha(\beta,\bar{\beta}),\bar{\alpha}(\beta,\bar{\beta})) + f^2(\alpha(\beta)) \\
&= \frac{(-E(\beta)(6A(\beta)^4 - 9A(\beta)^2(-E(\beta)+4(-E(\beta)^2))}{1728C^6(\beta)} \\
&\quad + \frac{c_1 A^6(\beta) + c_2 A^3(\beta)C^3(\beta) + c_3 C^6(\beta)}{1728C^6(\beta)}
\end{aligned}
$$

From the expansion

$$\beta(t_c) = t_c - \frac{11t_c^2}{18} + \frac{145t_c^3}{486} - \frac{6733t_c^4}{52488} + \frac{120127t_c^5}{2361960} + \cdots$$

we can express $F^2_{con}(\beta)$ in terms of t_c series. We can then make use of the gap conation to get linear equations satisfied by c_1, c_2, c_3.

Local $\mathbb{P}^2$ example: predicting GW (GV) invariants

It turns out that

$$c_1 = -\frac{8}{5},\ c_2 = \frac{2}{5},\ c_3 = \frac{-8-3\chi}{10} \tag{31}$$

By using the mirror map

$$\alpha = -27q - 162q^2 - 243q^3 - 1512q^4 + 8100q^5 + \cdots,\ q = \exp 2\pi it$$

we then get the $q = \exp 2\pi it$ expansion of

$$F^2_{GW}(Y,t) = \frac{E(6A^4 - 9A^2E + 5E^2)}{1728B^6} + \frac{-\frac{8}{5}A^6 + \frac{2}{5}A^3B^3 + \frac{-8-3\chi}{10}B^6}{1728B^6}$$

This gives exactly the generating function of genus 2 GW (GV) invariants listed in Katz, Klemm, & Vafa (1999).
For example, the first few GV invariants $n_d^{g=2}, d = 1, 2, \cdots$ are

$$0, 0, 0, -102, 5430, -194022, 5784837, -155322234, 3894455457, \cdots$$

Other examples: local $dP_n, n = 5, 6, 7, 8$

One can easily work out the higher genus cases for local $\mathbb{P}^2$ using the same approach.

For the local del Pezzo geometries $K_{dP_n}, n = 5, 6, 7, 8$ with corresponding modular groups being $\Gamma_0(N)$ with $N = 4, 3, 2, 1^*$ respectively, the same procedure we outlined above constructs the ring of quasi modular forms from the periods, and solves $\mathcal{F}^g$. The $\mathcal{F}^g$s also predict the correct GV invariant.

This approach making use of modularity works for noncompact CY 3-fold families whose mirror curves are of genus one, and whose base $\mathcal{M}$ could be identified with some modular curves.

Local CY examples: differential ring of special geometry generators vs differential ring of almost-holomorphic modular forms

In these examples, first we constructed the differential ring of almost-holomorphic modular forms, then we expressed the differential ring of generators S^{ij}, S^{i}, S, K_i in terms of these almost-holomorphic modular forms. After that we did the polynomial recursion.

We could have started from the differential ring of generators S^{ij}, S^{i}, S, K_i constructed from special geometry without knowing their relations to the generators A, B, C, E, provided that we know the correct notion of τ and their gradings as "modular weights" which tell how they transform.

Local CY examples: special geometry polynomial ring

This ring of generators S^{ij}, S^i, S, K_i constructed from special geometry (called the special geometry polynomial ring below) is as follows (for $K_{\mathbb{P}^2}$, $\kappa = -\frac{1}{3}$):

$$\lim S^{tt} (= \frac{1}{2} E), \quad \theta t (= A), \quad C_{ttt}^{-1} = \kappa^{-1} \frac{\partial t}{\partial \tau} (= \kappa^{-1} B^3)$$

$$DS^{tt} = -S^{tt} S^{tt} - \frac{1}{12\kappa} (\theta t)^4$$

$$D\theta t = C_{ttt}^{-1} \theta^2 t = -S^{tt} \theta t + C_{ttt}^{-1} s_{xx}^x$$

$$DC_{ttt}^{-1} = -3 C_{ttt}^{-1} S^{tt} + C_{ttt}^{-1} (\theta t)^2 (\partial_x \log C_{xxx}^{-1} + 3 s_{xx}^x)$$

where $s_{xx}^x = \frac{1}{6}\frac{\alpha}{\beta} - \frac{1}{3}, \partial_x \log C_{xxx}^{-1} = -\frac{\alpha}{\beta}$. To make the ring closed, we add the holomorphic quantity $\partial_x \log C_{xxx} = \frac{\alpha}{\beta} = \frac{C^3}{B^3}$. Now its derivative lies in the ring of the above generators:

$$D\partial_x \log C_{xxx} = (\partial_x \log C_{xxx}) D \log \partial_x \log C_{xxx} = (\partial_x \log C_{xxx})(\theta t)^2$$

Local CY examples: special geometry polynomial ring vs ring of quasi modular forms

The holomorphic limit of the special geometry polynomial ring is essentially equivalent to the differential ring of quasi modular forms [ASYZ]

$$\begin{aligned} DA &= \frac{1}{2r} A\left(E + \frac{C^r - B^r}{A^r} A^2\right) \\ DB &= \frac{1}{2r} B(E - A^2) \\ DC &= \frac{1}{2r} C(E + A^2) \\ DE &= \frac{1}{2r} (E^2 - A^4) \end{aligned} \tag{32}$$

where $D = \frac{1}{2\pi i} \frac{\partial}{\partial \tau}$ and $r = 2, 3, 4, 6$ for $N = 4, 3, 2, 1^*$ respectively.

Special geometry polynomial ring

This seems to suggest that the special geometry polynomial ring constructed using connections, Yukawa couplings, etc. is a natural candidate of the ring of almost-holomorphic modular forms, even for the cases in which the arithmetic properties of the moduli space is unknown.

This leads us to define the following "special" special geometry polynomial ring on the moduli space ($\dim\mathcal{M}=1$)[ASYZ]

$$\begin{aligned} K_0 &= \kappa\, C_{ttt}^{-1}\,(\theta t)^{-3}\,, & G_1 &= \theta t\,, & K_2 &= \kappa\, C_{ttt}^{-1} K_t \\ T_2 &= S^{tt}\,, & T_4 &= C_{ttt}^{-1}\tilde{S}^t\,, & T_6 &= C_{ttt}^{-2}\tilde{S}_0\,, \end{aligned} \qquad (33)$$

with $\theta = z\frac{d}{dz}$, z is the algebraic coordinate. We furthermore need a generator $C_0 = \theta \log z^3 C_{zzz}$ for the coefficients from their derivatives.

Special geometry polynomial ring

This differential ring, which has a nice grading called the weight, is given by

$$
\begin{aligned}
\partial_\tau K_0 &= -2K_0\, K_2 - K_0^2\, G_1^2\, (\tilde{h}^z_{zzz} + 3(s^z_{zz} + 1))\,, \\
\partial_\tau G_1 &= 2G_1\, K_2 - \kappa G_1\, T_2\, + K_0 G_1^3 (s^z_{zz} + 1)\,, \\
\partial_\tau K_2 &= 3K_2^2 - 3\kappa K_2\, T_2 - \kappa^2\, T_4 + K_0^2\, G_1^4\, k_{zz} - K_0\, G_1^2\, K_2\, \tilde{h}^z_{zzz}\,, \\
\partial_\tau T_2 &= 2K_2\, T_2 - \kappa T_2^2 + 2\kappa T_4 + \frac{1}{\kappa} K_0^2 G_1^4 \tilde{h}^z_{zz}\,, \\
\partial_\tau T_4 &= 4K_2 T_4 - 3\kappa T_2\, T_4 + 2\kappa T_6 - K_0\, G_1^2\, T_4 \tilde{h}^z_{zzz} - \frac{1}{\kappa} K_0^2\, G_1^4\, T_2 k_{zz} \\
&\quad + \frac{1}{\kappa^2} K_0^3\, G_1^6 \tilde{h}_{zz}\,, \\
\partial_\tau T_6 &= 6K_2\, T_6 - 6\kappa T_2\, T_6 + \frac{\kappa}{2} T_4^2 - \frac{1}{\kappa} K_0^2\, G_1^4\, T_4\, k_{zz} + \frac{1}{\kappa^3} K_0^4\, G_1^8 \tilde{h}_z \\
&\quad - 2\, K_0\, G_1^2\, T_6 \tilde{h}^z_{zzz}\,,
\end{aligned}
\tag{34}
$$

where $\tau = \kappa^{-1} F_{tt}$ and the sub-indices are the weights [ASYZ].
For each $g \geq 2$, the normalized topological string partition function $(X^0)^{2-2g} \mathcal{F}^g$ is a rational function of weight 0 in these generators.

Candidate of the ring of almost-holomorphic modular forms

As we have discussed, for local $\mathbb{P}^2$ and local del Pezzo cases, this ring, in the holomorphic limit, is essentially equivalent to the ring of quasi modular forms.

For a general CY 3–fold X (compact or noncompact), we could use this ring, as a guidance for the study of modular forms.

Candidate of the ring of almost-holomorphic modular forms

For example, for the mirror quintic family, one gets

$$
\begin{aligned}
\partial_\tau C_0 &= C_0\,(1+C_0)\,K_0\,G_1^2\,,\\
\partial_\tau K_0 &= -2K_0\,K_2 - C_0 K_0^2\,G_1^2\,,\\
\partial_\tau G_1 &= 2G_1\,K_2 - 5G_1\,T_2 - \frac{3}{5}K_0\,G_1^3\,,\\
\partial_\tau K_2 &= 3K_2^2 - 15K_2\,T_2 - 25\,T_4 + \frac{2}{25}\,K_0^2\,G_1^4 - \left(\frac{9}{5}+C_0\right)K_0\,G_1^2\,K_2\,,\\
\partial_\tau T_2 &= 2K_2\,T_2 - 5\,T_2^2 + 10\,T_4 + \frac{1}{25}(1+C_0)\,K_0^2\,G_1^4\,,\\
\partial_\tau T_4 &= 4K_2\,T_4 - 15\,T_2\,T_4 + 10\,T_6 - \left(\frac{9}{5}+C_0\right)K_0\,G_1^2\,T_4 - \frac{2}{125}K_0^2\,G_1^4\,T_2\\
&\quad - \frac{1}{625}(1+C_0)K_0^3\,G_1^6\,,\\
\partial_\tau T_6 &= 6K_2\,T_6 - 30\,T_2\,T_6 + \frac{5}{2}\,T_4^2 - \frac{2}{125}K_0^2\,G_1^4\,T_4 + \frac{2}{78125}(1+C_0)K_0^4\,G_1^8\\
&\quad - 2\left(\frac{9}{5}+C_0\right)K_0\,G_1^2\,T_6\,.
\end{aligned}
\tag{35}
$$

Candidate of the ring of almost-holomorphic modular forms

Since we know how to take the holomorphic limit in this special geometry polynomial ring, we can get the candidate for the ring of quasi modular forms from it.

But if the arithmetic properties of the moduli space is not clear, we cann't really say that the special polynomial ring is *the* ring of almost-holomorphic modular forms.

Also if there is no suitable description of the modular group, it is not clear whether there is an analogue of Fricke involution as an duality of the topological string partition functions.

Conclusions

In summary, we have

- computed $\mathcal{F}^g$s in terms of modular forms for certain CY manifolds whose moduli spaces have nice arithmetic descriptions: $\mathcal{M} \cong \mathcal{H}^*/\Gamma_0(N)$

- found a duality acting on topological string partition functions for these examples: $\mathcal{F}^g_{con} = \mathcal{F}^g_A|_{W_N}$

- constructed the special geometry polynomial ring which has a nice grading. The normalized topological string partition functions are rational functions of degree zero in these generators. This ring is a natural candidate for the ring of almost-holomorphic modular forms.

Discussions and future directions

There are some interesting questions

- interpretation of τ coordinate in generality, e.g. for compact CY, multi-moduli cases?
- enumerative meaning of the $q_\tau = \exp 2\pi i\tau$ expansions? (example: quintic, where $q_t = \exp(2\pi i t)$ is used in GW generating functions)

$$q_t = q_\tau - 575q_\tau^2 + 8250q_\tau^3 + 43751250q_\tau^4 + \dots$$

- enumerative meaning of holomorphic limits of $\mathcal{F}^g$ at some other points on the moduli space.
- interpretation of gap condition in mathematics
- Fricke involution on the level of moduli space of CYs
- How exactly is the special geometry polynomial ring related to the ring of quasi modular forms for general CYs, e.g, mirror quintic?

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