Automorphy (Modularity) of Calabi–Yau Varieties over $\mathbb{Q}$

Noriko Yui

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Queen's University, Canada

Langlands Philosophy

(Motivic) L-functions of algebraic varieties over $\mathbb{Q}$ (or number fields) are automorphic L-functions

I will try to give some examples in support of this philosophy when varieties are Calabi–Yau varieties defined over $\mathbb{Q}$.

In this talk, we will consider Calabi–Yau varieties of dimension at most 3.

Calabi–Yau Varieties

Definition: A smooth projective variety $X/\mathbb{C}$ of dimension d is said to be *Calabi–Yau* if

(1) $H^i(X, \mathcal{O}_X) = 0$ for every i, $0 < i < d$, and

(2) The canonical bundle $\mathcal{K}_X$ is trivial.

Now introduce Hodge numbers:

$$h^{i,j}(X) := \dim_{\mathbb{C}} H^j(X, \Omega^i_X) \quad \text{for } 0 \leq i, j \leq d$$

Then

$$h^{i,j}(X) = h^{j,i}(X) \quad \text{by complex conjugation}$$

and

$$h^{i,j}(X) = h^{d-i,d-j}(X) \quad \text{by the Serre duality.}$$

Remark In terms of Hodge numbers, $X/\mathbb{C}$ is Calabi–Yau if

(1) $h^{i,0}(X) = 0 \quad$ for every $i, 0 < i < d$, and

(2) $h^{d,0}(X) = h^{0,d}(X) = \dim_{\mathbb{C}} H^0(X, \Omega_X^d) = \dim_{\mathbb{C}} H^0(X, \mathcal{K}_X) = \dim_{\mathbb{C}} H^0(X, \mathcal{O}_X) = 1$.

The number $h^{0,d}(X)$ is the *geometric genus* $p_g(X)$ of X.

Numerical characters

- The Betti numbers $B_k(X) := \dim_{\mathbb{C}} H^k(X, \mathbb{C})$.

$$B_k(X) = \sum_{i+j=k} h^{i,j}(X).$$

- The Euler characteristic $E(X) := \sum_{k=0}^{2d} (-1)^k B_k(X)$.

Hodge diamonds

The Hodge numbers of Calabi–Yau varieties are concocted to form the Hodge diamond.

$d = 1$: **Elliptic curves**

$$\begin{array}{ccccc} & 1 & & \qquad & B_0 = 1 \\ 1 & & 1 & \qquad & B_1 = 2 \\ & 1 & & \qquad & B_2 = 1 \\ & & & \qquad & E = 0 \end{array}$$

Dimension one Calabi–Yau varieties are elliptic curves.

$d = 2$: **K3 surfaces**

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ 1 & & 20 & & 1 \\ & 0 & & 0 & \\ & & 1 & & \end{array} \qquad \begin{array}{r} B_0 = 1 \\ B_1 = 0 \\ B_2 = 22 \\ B_3 = 0 \\ B_4 = 1 \\ E = 24 \end{array}$$

Examples: (1) Any quartic surface in $\mathbb{P}^3$.

(2) Double sextic surface.

(3) Elliptic K3 surfaces.

$d = 3$: Calabi–Yau threefolds

$$
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 0 & & 0 & & \\
 & 0 & & h^{1,1} & & 0 & \\
1 & & h^{2,1} & & h^{1,2} & & 1 \\
 & 0 & & h^{2,2} & & 0 & \\
 & & 0 & & 0 & & \\
 & & & 1 & & &
\end{array}
\qquad
\begin{aligned}
B_0 &= 1 \\
-\ B_1 &= 0 \\
B_2 &= h^{1,1} \\
-\ B_3 &= 2(1 + h^{2,1}) \\
B_4 &= h^{2,2} \\
-\ B_5 &= 0 \\
B_6 &= 1 \\
E &= 2(h^{1,1} - h^{2,1})
\end{aligned}
$$

Examples: (1) Quintic threefolds in $\mathbb{P}^4$.

(2) Complete intersection threefolds.

(3) Toric Calabi–Yau threefolds ($\sim$ 600 million).

The largest possible Hodge numbers, or equivalently the Euler characteristic of a Calabi–Yau threefold is not known, but some known examples ($\sim$ 600 million) have $h^{1,1}$(or $h^{2,1}$)$\sim$ 500.

An implication in string theory is that string theory may have as many as 10^{500} vacua that can be described with various choices of branes and fluxes on homology cycles of a CY. Thus string theory has a vast number of different vacua, and only one should describe dynamics of our real world.

$|E(X)| \leq 960$

Today's lecture

We will consider Calabi–Yau varieties defined over $\mathbb{Q}$, say, by hypersurfaces or by complete intersections. We say that $X/\mathbb{Q}$ is Calabi–Yau if $X \otimes_{\mathbb{Q}} \mathbb{C}$ is Calabi–Yau. Let $X/\mathbb{Q}$ be a Calabi–Yau with a defining equation with coefficients in $\mathbb{Z}[1/m]$ for some $m \in \mathbb{N}$. Let p be a prime $(p, m) = 1$, let $X_p := X \bmod p$ be the reduction of X modulo p. We say that p is *good* if X_p is smooth over $\overline{\mathbb{F}}_p$, otherwise *bad*.

Let $\#X_p(\mathbb{F}_{p^k})$ be the number of rational points on X_p over $\mathbb{F}_{p^k}$. The *local (congruent) zeta function* of X_p is defined by taking the formal sum

$$Z_p(X,T) := \exp\left(\sum_{k=1}^{\infty} \frac{\#X(\mathbb{F}_{p^k})}{k} T^k\right) \in \mathbb{Q}[[T]]$$

where T is an indeterminate.

There were vast series of conjectures about $Z_p(X,T)$, known as the Weil conjectures, proved finally by Deligne.

Let ℓ be a prime $\neq p$. The Frobenius morphism Fr_p $(x \mapsto x^p)$ on X_p induces an endomorphism Fr_p^* on the étale cohomology groups $H^i_{et}(\overline{X}_p, \mathbb{Q}_\ell)$ for each i, $0 \leq i \leq 2d$. Grothendieck specialization theorem gives an isomorphism $H^i_{et}(\overline{X}_p, \mathbb{Q}_\ell) \simeq H^i_{et}(\overline{X}, \mathbb{Q}_\ell)$, where $\overline{X} = X \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$. By the comparison theorem, $H^i_{et}(\overline{X}, \mathbb{Q}_\ell) \simeq H^i(X \otimes_{\mathbb{Q}} \mathbb{C}, \mathbb{C})$ so that $\dim_{\mathbb{Q}_\ell} H^i_{et}(\overline{X}, \mathbb{Q}_\ell) = B_i(X)$ (the i-th Betti number). There is the Poincaré duality: $H^i(\overline{X}, \mathbb{Q}_\ell) \times H^{2d-i}_{et}(\overline{X}, \mathbb{Q}_\ell) \to \mathbb{Q}_\ell$ is a perfect pairing for every i, $0 \leq i \leq 2d$.

Let

$$P^i_p(T) := \det(1 - \mathrm{Fr}_p^* T \mid H^i_{et}(\overline{X}, \mathbb{Q}_\ell))$$

be the characteristic polynomial of Fr_p^*.

Weil's Conjectures (Theorem)

- $P_p^i(T) \in 1 + T\mathbb{Z}[T]$.
- $P_p^i(T)$ does not depend on the choice of ℓ.
- $\deg P_p^i(T) = B_i(X)$ for every $i, 0 \le i \le 2d$.
- $P_p^{2d-i}(T) = \pm P_p^i(p^{d-i}T)$ for every i, $0 \le i \le d$.
- If we write $P_p^i(T) = \prod_{k=1}^{B_i}(1 - \alpha_k T) \in \overline{\mathbb{Q}}[T]$, then α_k is an algebraic integer with $|\alpha_k| = p^{i/2}$ (The Riemann Hypothesis).

$$Z_p(X,T) = \frac{\prod_{i=1}^{d} P_p^{2i-1}(X,T)}{\prod_{i=0}^{d} P_p^{2i}(X,T)}.$$

Let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group. There is a compatible system of ℓ-adic Galois representations

$$\rho^i_{X,\ell} : G_{\mathbb{Q}} \to GL(H^i_{et}(\overline{X}, \mathbb{Q}_\ell))$$

sending the (geometric) Frobenius $\mathrm{Fr}{*}_p^{-1}$ to $\rho^i(\mathrm{Fr}_p^{*-1})$ which has the same action as the Fr_p^* on $H^i_{et}(\overline{X}, \mathbb{Q}_\ell)$.

Definition: The i-th (cohomological) L-series (or L-function) of $X/\mathbb{Q}$ is defined by

$$L_i(X, s) := L(H^i_{et}(\overline{X}, \mathbb{Q}_\ell), s)$$

$$:= (*) \prod_{p \neq \ell:\text{good}} P^i_p(p^{-s})^{-1} \times (\text{factor corresponding to } \ell = p)$$

where the product is taken over all good primes different from ℓ and $(*)$ corresponds to factors of bad primes. For $\ell = p$ we use p-adic cohomology groups.

The most significant L-series is the L-series $L_d(X, s) =: L(X, s)$.

The vector space $H^i_{et}(\overline{X}, \mathbb{Q}_\ell)$ may decompose into a direct sum of subspaces, and we can define the *motivic* L-series corresponding to these subspaces.

Locally for each good prime, the characteristic polynomial $P^i_p(T)$ can be determined by geometric information and by counting the number of rational points on $\mathbb{F}_p$ by invoking the Lefschetz fixed point formula.

$$\#X(\mathbb{F}_p) = \sum_{k=0}^{2d} (-1)^k \mathrm{trace}(\mathrm{Fr}_p^* \mid H^i_{et}(\overline{X}, \mathbb{Q}_\ell))$$

Automorphy (Modularity) Question

Are there global functions that determine the L-series $L_i(X, s)$?

More concretely, are there automorphic (modular) forms that determine $L_i(X, s)$?

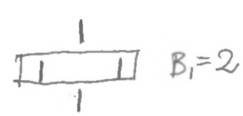
Result: $d = 1$

Theorem (Wiles et al.): *Every elliptic curve E over $\mathbb{Q}$ is modular. That is, there is a cusp form f of weight $2 = 1 + 1$ on $\Gamma_0(N)$ such that*

$$L_1(E, s) = \prod_p P_p^1(p^{-s})^{-1} = L(f, s)$$

Here N is the conductor of E and $\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \pmod{N} \right\} \subset PSL(2, \mathbb{Z})$.

There is a compatible system of 2-dimensional ℓ-adic Galois representations associated to E, and Wiles et al. established its modularity.



Some Results: $d = 2$

Let X be a K3 surface defined over $\mathbb{Q}$. Let $NS(X)$ denote the Néron–Severi group of X generated by algebraic cycles. It is a free finitely generated abelian group, and
$NS(X) = H^{1,1}(X, \mathbb{R}) \cap H^2(X, \mathbb{Z})$ so that the rank of $NS(X)$ (called the Picard number of X and denoted by $\rho(X)$) is bounded by 20. Let $T(X) = NS(X)^{\perp}$ be the orthogonal complement of $NS(X)$ in $H^2(X, \mathbb{Z})$. It has the $\mathbb{Z}$-rank $22 - \rho(X)$, and is called the group of transcendental cycles on X. We have the decomposition

$$H^2(X, \mathbb{Z}) \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = (NS(X) \otimes \mathbb{Q}_\ell) \oplus (T(X) \otimes \mathbb{Q}_\ell)$$

and we have the decomposition of the L-series:

$$L_2(X, s) = L(NS(X) \otimes \mathbb{Q}_\ell, s) L(T(X) \otimes \mathbb{Q}_\ell, s).$$

The Tate conjecture is valid for any K3 surface over $\mathbb{Q}$, which asserts that

$$H^2_{et}(\overline{X}, \mathbb{Q}_\ell)^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})} = NS(X)_{\mathbb{Q}}.$$

Thus the L-series $L(NS(X) \otimes \mathbb{Q}_\ell, s)$ is well-understood, e.g., expressed in terms of $\zeta(s-1)^{\rho(X)}$ if all algebraic cycles are defined over $\mathbb{Q}$, otherwise $\zeta_{\mathbb{L}}(s-1)^{\rho(X)}$ if all algebraic cycles are defined over a number field $\mathbb{L}$. Here $\zeta(s)$ is the Riemann zeta-function, and $\zeta_{\mathbb{L}}(s)$ is the Dedekind zeta-function of $\mathbb{L}$.

Therefore, for K3 surfaces, we will address the automorphy (modularity) of the motivic L-series, namely, that of

$$L(T(X) \otimes \mathbb{Q}_\ell, s).$$

Definition: A K3 surface X over $\mathbb{Q}$ is *singular* (or *extremal)* if $\rho(X) = 20$.

Theorem (Livné):*Every singular K3 surface X over $\mathbb{Q}$ is motivically modular. That is, there is a cusp form f of weight $3 = 2 + 1$ on $\Gamma_1(N)$ or $\Gamma_0(N)$ with a character χ such that*

$$L(T(X) \otimes \mathbb{Q}_\ell, s) = L(f, s)$$

There is a compatible system of 2-dimensional ℓ-adic Galois representations associated to $T(X)$, and Livné established the modularity of such representations.

Problem: Can we establish automorphy (modularity) of K3 surfaces X over $\mathbb{Q}$ when $\rho(X) \leq 19$ or $\mathbb{Z}$-rank of $T(X)$ is ≥ 3?

Answer: Yes, we can in some cases. These will be discussed later in this talk.

Some Results: $d = 3$

Definition: A Calabi–Yau threefold X over $\mathbb{Q}$ is said to be *rigid* if $h^{2,1}(X) = 0$. Thus, the Hodge diamond of any rigid Calabi–Yau threefold is given by

$$
\begin{array}{ccccccccc}
 & & & 1 & & & & & B_0 = 1 \\
 & & 0 & & 0 & & & & B_1 = 0 \\
 & 0 & & h^{1,1} & & 0 & & & B_2 = h^{1,1} \\
1 & & 0 & & 0 & & 1 & & B_3 = 2 \\
 & 0 & & h^{2,2} & & 0 & & & B_4 = h^{2,2} \\
 & & 0 & & 0 & & & & B_5 = 0 \\
 & & & 1 & & & & & B_6 = 1 \\
 & & & & & & & & E = 2h^{1,1}
\end{array}
$$

Problem: Can we establish automorphy (modularity) when $B_3(X) \geq 4$?

Accessible cases: When $H^3_{et}(\overline{X}, \mathbb{Q}_\ell)$ decomposes into a direct sum of 2-dimensional pieces, the Galois representations are highly reducible over $\mathbb{Q}$, and we can establish motivic modularity. There are several examples of this kinds by several authors (e.g, E. Lee, Hulek–Verrill, and others).

Challenging cases: When $H^3_{et}(\overline{X}, \mathbb{Q}_\ell)$ does not decompose, so that the Galois representations associated to X are irreducible over $\mathbb{Q}$, the automorphy (modularity) is a really challenging problem. We will discuss some examples below.

Some Results: $d = 2$

We now return to the question of automorphy (modularity) of K3 surfaces X over $\mathbb{Q}$ when $T(X)$ has $\mathbb{Z}$-rank≥ 3.

Proposition:*Let X be a K3 surface over $\mathbb{Q}$ with $\rho(X) = 19$. Then X has a Shioda–Inose structure, i.e., there is an involution ι on X such that X/ι is a Kummer surface Y over $\mathbb{C}$. Suppose that $Y = E \times E$ of a non-CM elliptic curve E over $\mathbb{Q}$. Then $T(X)$ is potentially modular in the sense that $L(T(X) \otimes \mathbb{Q}_\ell, s)$ is determined over some number field $\mathbb{L}$ by the symmetric square of a modular form f associated to E, and over $\mathbb{L}$*

$$L(T(X) \otimes \mathbb{Q}_\ell, s) = L(Sym^2(f), s)$$

We are not able to establish the modularity over $\mathbb{Q}$. Since X is defined over $\mathbb{Q}$, the Galois representation on $T(X)$ is also defined over $\mathbb{Q}$, but the isomorphism to $\mathrm{Sym}^2(f)$ may not be.

K3 surfaces with non-symplectic automorphisms

Now we consider K3 surfaces with non-symplectic automorphisms. Let ω_X be a holomorphic 2-form on X, fixed once and for all. Then $H^{2,0}(X) = \mathbb{C}\omega_X$. Let $g \in \mathrm{Aut}(X)$. Then g induces a map

$$g^* : H^{2,0}(X) \to H^{2,0}(X) : g^*\omega_X = \alpha(g)\omega_X$$

for some $\alpha(g) \in \mathbb{C}^*$. We say that g is *non-symplectic* if $\alpha(g) \neq 1$.

Suppose that $T(X)$ is unimodular, i.e, $\det T(X) = \pm 1$. Then the following assertions hold:

- $\alpha(\mathrm{Aut}(X))$ is a finite cyclic group of order k where $k \leq 66$ (Nikulin).
- k is a divisor of $66, 44, 42, 36, 28, 12$ (Kondo).
- If $\mathbb{Z}$-rank of $T(X) = \varphi(k)$ (where φ is the Euler function), then $k = 66, 44, 42, 36, 28, 12$. For a given k there is a unique K3 surface.

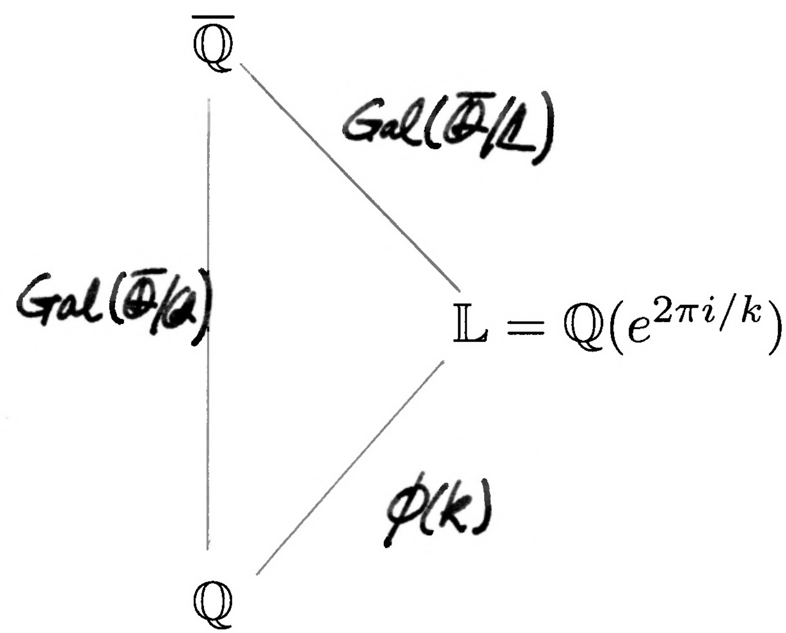
Theorem (Livné–Schütt–Yui):*Let X be a K3 surface corresponding to one of the above values of k. Then the following assertions hold:*

(1) The ℓ-adic Galois representation associated to $T(X)$ is irreducible over $\mathbb{Q}$ of dimension $\varphi(k) \in \{20, 12, 4\}$.

(2) Furthermore, this $G_{\mathbb{Q}}$-Galois representation is induced from a 1-dimensional Galois representation of the cyclotomic field $\mathbb{Q}(e^{2\pi i/k})$.

(3) The motivic L-series $L(T(X) \otimes \mathbb{Q}_\ell, s)$ is automorphic.

Idea of Proof: Automorphic induction.



The Gal($\overline{\mathbb{Q}}/\mathbb{L}$) representation defined by $T(X)$ is a direct sum of 1-dimensional representations, which are simply permuted transitively by Gal($\mathbb{L}/\mathbb{Q}$). The 1-dimensional representations are determined by Jacobi sum Grossencharacters of $\mathbb{L}$.

Calabi–Yau varieties of CM type

Definition: A Calabi–Yau variety $X/\mathbb{Q}$ of dimension d is said to be of *CM type* if the Hodge group $\mathrm{Hdg}(X)$ associated to a rational Hodge structure on $H^d(X,\mathbb{Q})$ is commutative, so $\mathrm{Hdg}(X)\otimes\mathbb{C}\simeq$ copies of $\mathbb{G}_m$.

Remark: $\mathrm{Hdg}(X)$ is, in general, not computable.

Examples: (1) Fermat surface of degree m:
$x_0^m+x_1^m+x_2^m+x_3^m=0$ is of CM type.

(2) Delsarte surfaces are defined by four-term equations of the form $\sum_{i=0}^3\prod_{j=0}^3 x_j^{a_{ij}}=0$. They are of CM type.

(3) Invertible polynomials in $\mathbb{P}^3$ are of CM type. (Here invertible polynomials means that # of monomials = # of variables (4).

(4) All K3 surfaces in the above theorem of Livné–Schütt-Yui are of CM type.

Idea of Proof: All K3 surfaces have defining equations with 4 monomials (Kondo). They can be realized as Fermat quotients by finite groups. Since Fermat surfaces are of CM type, our K3 surfaces are also of CM type.

Some Results: $d = 3$

Here I will just announce automorphy (modularity) result for Calabi–Yau threefolds of Borcea–Voisin type defined over $\mathbb{Q}$. Details will be presented in Workshop 1: Modular Forms around String Theory, September 16–20, 2013.

A Calabi–Yau threefold X of Borcea–Voisin type is the crepant resolution of the quotient form

$$E \times S/\iota \times \sigma$$

where

- (E, ι) is an elliptic curve with a non-symplectic involution ι
- (S, σ) is a K3 surface with a non-symplectic involution acting by -1 on $H^{2,0}(S)$
- X is the crepant resolution of this quotient.

Theorem(Goto–Livné–Yui) : *Suppose that* S *is of CM type, then the (motivic)* L*-series of a Calabi–Yau threefold* X *of Borcea–Voisin type is automorphic.*

Problems

- Determine various zeta-functions for families of Calabi–Yau varieties over $\mathbb{Q}$.
- Arithmetic mirror symmetry: Capture mirror symmetry phenomenon in terms of zeta-functions and L-series of mirror pairs of Calabi–Yau varieties over $\mathbb{Q}$.
- For families of Calabi–Yau varieties over $\mathbb{Q}$, how do we interpret "automorphy (modularity)"?

Modular forms around string theory

- Modularity of Galois representatations associated to Calabi-Yau varieties over $\mathbb{Q}$
- Modularity of mirror maps $\mapsto$ Mirror moonshine

  Elliptic genera of K3 surfaces and M24 moonshne
- Modularity of generating functions of Gromov–Witten (and related) invariants
- Various modular forms (e.g., quasi-modular forms, mock modular forms, Jacobi forms, Siegel modular forms), appearing in string theory (black holes, string amplitudes, holomorphic anomaly, $\cdots$)