On the Universal Rigidity of Tensegrity Frameworks

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A tensegrity framework has two aspects: a geometric one (p) and a combinatorial one (G).

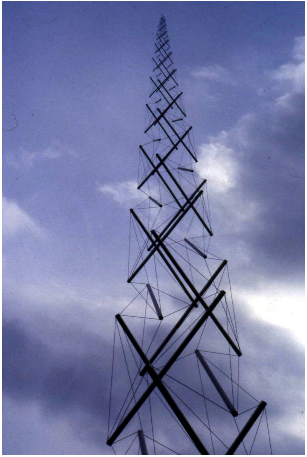
Applications

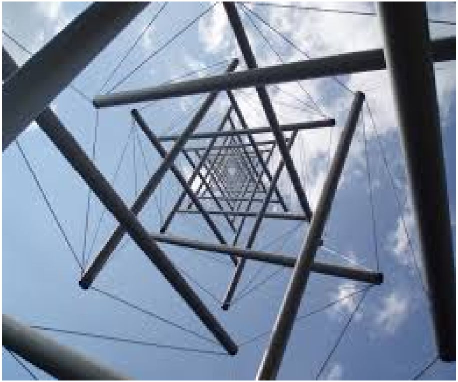
tensegrities have important applications in:

1. Molecular conformation theory.
2. Wireless sensor network localization problem.
3. Art.

Tensegrity as an Artwork

Kenneth Snelson needle tower sculpture in Washington D.C.





Tensegrity as an Artwork Cont'd

Kenneth Snelson Indexer II sculpture at the University of Michigan, Ann Arbor

Domination and Affine-Domination

Definition

Tensegrity (G, q) in $\mathbb{R}^s$ is said to be dominated by tensegrity (G, p) in $\mathbb{R}^r$ if

$$||q^i - q^j|| = ||p^i - p^j|| \text{ for all bar } \{i,j\}.$$
$$||q^i - q^j|| \leq ||p^i - p^j|| \text{ for all cable } \{i,j\}.$$
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Definition

Tensegrity (G, q) in $\mathbb{R}^r$ is said to be affinely-dominated by tensegrity (G, p) in $\mathbb{R}^r$ if (G, q) is dominated by (G, p) and

$$q^i = Ap^i + b \text{ for all } i = 1, \ldots, n$$

for some $r \times r$ matrix A and an r-vector b.

Dimensional and Universal Rigidities

Definition

Tensegrity (G, q) in $\mathbb{R}^r$ is said to be congruent to tensegrity (G, p) in $\mathbb{R}^r$ if $||q^i - q^j|| = ||p^i - p^j||$ for every $i = 1, \ldots, n$.

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An r-dimensional tensegrity (G, p) in $\mathbb{R}^r$ is said to be dimensionally rigid if no s-dimensionl tensegrity (G, q), for any $s \geq r + 1$, is dominated by (G, p).

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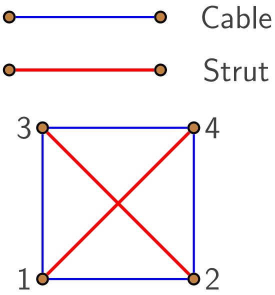
Definition

An r-dimensional tensegrity (G, p) in $\mathbb{R}^r$ is said to be universally rigid if every s-dimensionl tensegrity (G, q), for any s, that is dominated by (G, p) is in fact congruent to (G, p).

Example



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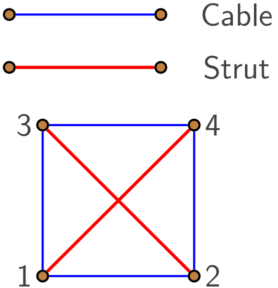

universally rigid.

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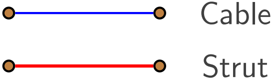

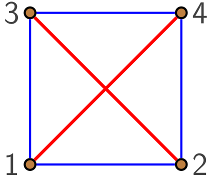

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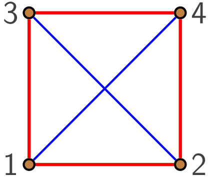

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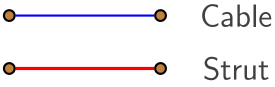

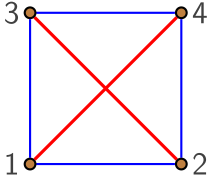

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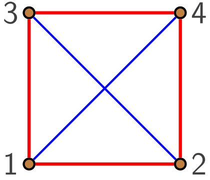

Not universally rigid. It folds on the diagonal.

Characterization of Universal Rigidity

Theorem

An r-dimensional Tensegrity (G, p) in $\mathbb{R}^r$ is universally rigid if and only if

1. *(G, p) is dimensionally rigid.*
2. *There does not exist an r-dimensional tensegrity (G, q) in $\mathbb{R}^r$ affinely-dominated by, but not congruent to, (G, p).*

Condition 2 is known as the "no conic at infinity" condition.

In This Talk, I'll:

1. Present the well-known sufficient condition for dimensional rigidity.

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2. Present conditions under which the "no conic at infinity" holds.

Stress Matrices

- A stress of a tensegrity (G, p) is a real-valued function ω on $E(G) = B \cup C \cup S$ such that:

$$\sum_{j:\{i,j\}\in E(G)} \omega_{ij}(p^i - p^j) = 0 \text{ for all } i = 1, \ldots, n.$$

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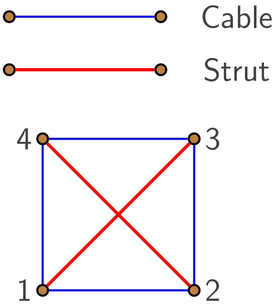
- A stress ω is proper if $\omega_{ij} \geq 0$ for every $\{i,j\} \in C$ and $\omega_{ij} \leq 0$ for very $\{i,j\} \in S$.
- The stress matrix associated with stress ω is the $n \times n$ symmetric matrix Ω where

$$\Omega_{ij} = \begin{cases} -\omega_{ij} & \text{if } (i,j) \in E(G), \\ 0 & \text{if } (i,j) \notin E(G), \\ \sum_{k:\{i,k\}\in E(G)} \omega_{ik} & \text{if } i = j. \end{cases}$$

Example



$\omega_{12} = 1$, $\omega_{14} = 1$,
$\omega_{13} = -1$.

$$\Omega = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}.$$

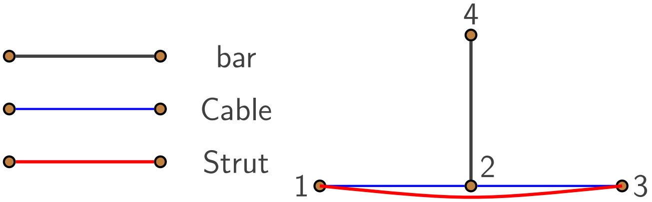
Ω is proper positive semidefinite of rank 1.

Sufficient Condition for Dimensional Rigidity

Theorem (Connelly '82)

An r-dimensional Tensegrity (G, p) on n nodes in $\mathbb{R}^r$ ($r \leq n - 2$) is dimensionally rigid if there exists a proper positive semidefinite stress matrix Ω of (G, p) of rank $n - r - 1$.

Example



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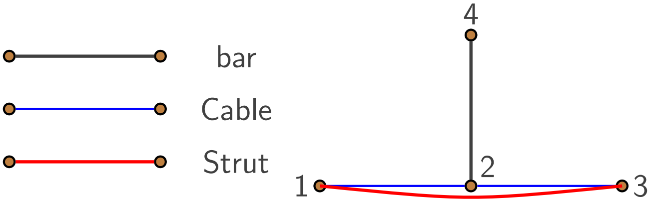

A dimensionally but not universally rigid tensegrity.

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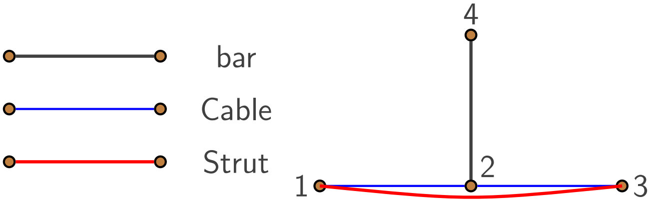

A dimensionally but not universally rigid tensegrity.
The "No Conic at Infinity" Condition does not hold.
In the sequel we concentrate on this condition.

Generic Configurations

Definition

A configuration $p = (p^1, \ldots, p^n)$ in $\mathbb{R}^r$ is generic if the coordinates of $p^1, \ldots, p^n$ are algebraically independent over the rationals, i.e., the coordinates of $p^1, \ldots, p^n$ do not satisfy any nonzero polynomial with rational coefficients.

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Lemma (Connelly '05)

Let (G, p) be an r-dimensional tensegerity. If configuration p is generic and every node of G has degree at least r, then the "no conic at infinity "condition holds. Consequently, dimensional rigidity implies universal rigidity.

Configurations in General Position

Definition

A configuration $p = (p^1, \ldots, p^n)$ in $\mathbb{R}^r$ is in general position if every subset of $\{p^1, \ldots, p^n\}$ of cardinality $r + 1$ is affinely independent.

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Lemma (A. and Ye '13)

Let (G, p) be an r-dimensional bar framework. If (G, p) admits a stress matrix Ω of rank $n - r - 1$ and configuration p is in general position, then the "no conic at infinity "condition holds. Consequently, dimensional rigidity implies universal rigidity.

Let C^* and S^* be the sets of stressed cables and stressed struts respectively, i.e,
$C^* = \{\{i,j\} \in C : \omega_{ij} \neq 0\}$ and $S^* = \{\{i,j\} \in S : \omega_{ij} \neq 0\}$.

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Theorem (A. and V-T Nguyen '13)

Let (G,p) be an r-dimensional tensegrity in $\mathbb{R}^r$. If the following conditions hold:

1. *there exists a proper stress matrix Ω of (G,p) of rank $n-r-1$.*
2. *for each node i, the set $\{p^i\} \cup \{p^j : \{i,j\} \in B \cup C^* \cup S^*\}$ affinely span $\mathbb{R}^r$.*

Then the "no conic at infinity" condition holds. Consequently, dimensional rigidity implies universal rigidity.

Corollary (A. and V-T Nguyen '13)

Let (G, p) be an r-dimensional tensegrity in $\mathbb{R}^r$. If the following conditions hold:

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3. Thus the universal rigidity problem becomes amenable to semi-definite programming.

Theorem (A. and V-T Nguyen '13)

Let (G, p) be an r-dimensional tensegrity in $\mathbb{R}^r$ and let Ω be a proper positive semidefinite stress matrix of (G, p). Then Ω is a proper stress matrix for all tensegrities (G, p') dominated by (G, p).

Gale Matrices

- A Gale matrix of r-dimensional tensegrity (G, p) in $\mathbb{R}^r$ is any $n \times (n - r - 1)$ matrix Z such that the columns of Z form a basis of the null space of :
$$\begin{bmatrix} p^1 & p^2 & \cdots & p^n \\ 1 & 1 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} P^T \\ e^T \end{bmatrix}.$$

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- In Polytope theory, the rows of Z ($z^1, \ldots, z^n$ in $\mathbb{R}^{n-r-1}$) are called Gale transforms of $p^1, \ldots, p^n$.
- The Gale matrix Z encodes the affine dependencies among the points $p^1, \ldots, p^n$.

Gale Matrix Z and Stress Matrix Ω

Theorem (A '07)

Let Ω and Z be, respectively, a stress matrix and a Gale matrix of (G, p). Then

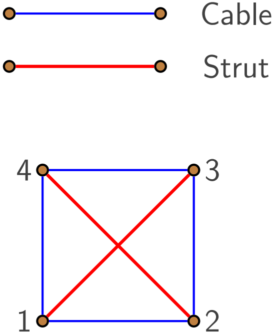
$$\Omega = Z \Psi Z^T \text{ for some symmetric matrix } \Psi.$$

On the other hand, let Ψ' be any symmetric matrix such that

$$z^{i\,T} \Psi' z^j = 0 \text{ for all } \{i, j\} \notin E,$$

where z^i is the ith row of Z. Then $Z \Psi' Z^T$ is a stress matrix of (G, p).

Example



Gale matrix is

$$Z = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}.$$

and stress matrix $\Omega = ZZ^T$.

Properties of Gale Transform

Lemma

Let (G, p) be an r-dimensional tensegrity in $\mathbb{R}^r$ and let $z^1, \ldots, z^n$ be, respectively, Gale transforms of $p^1, \ldots, p^n$. Let $J \subseteq \{1, \ldots, n\}$ and assume that the set of vectors $\{p^i : i \in J\}$ affinely span $\mathbb{R}^r$. Then the set $\{z^i : i \in \overline{J}\}$ is linearly independent, where $\overline{J} = \{1, \ldots, n\} \backslash J$.

Affine-Domination

Let $F^{ij} = (e^i - e^j)(e^i - e^j)^T$, e^i is the ith standard unit vector in $\mathbb{R}^n$. Recall that the configuration matrix $P^T = [p^1 \cdots p^n]$.

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Lemma

Let (G, p) be an r-dimensional tensegrity in $\mathbb{R}^r$. Then the "no conic at infinity" holds iff there does not exist a nonzero symmetric matrix Φ such that:

$$\begin{aligned} trace(F^{ij}(P\Phi P^T)) &= 0 \text{ for all } \{i,j\} \in B. \\ trace(F^{ij}(P\Phi P^T)) &\leq 0 \text{ for all } \{i,j\} \in C. \\ trace(F^{ij}(P\Phi P^T)) &\geq 0 \text{ for all } \{i,j\} \in S. \end{aligned}$$

Affine-Domination

E^{ij} is the matrix with 1s in the ijth and jith entries and 0's elsewhere.

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Lemma

Let (G, p) be an r-dimensional tensegrity in $\mathbb{R}^r$ and let Z be a Gale matrix of (G, p). Then the "no conic at infinity" holds iff there does not exist a nonzero $y = (y_{ij}) \in \mathbb{R}^{|\bar{E}|+|C|+|S|}$ and $\xi = (\xi_i) \in \mathbb{R}^{n-r-1}$ where $y_{ij} \geq 0$ for all $\{i, j\} \in C$ and $y_{ij} \leq 0$ for all $\{i, j\} \in S$ such that:

$$\mathcal{E}(y)Z = e\xi^T,$$

where $\mathcal{E}(y) = \sum_{\{i,j\}\in\bar{E}\cup C\cup S} y_{ij}E^{ij}$.

Affine-Domination when a proper Ω is Known

The following are equivalent:

1. the 'no conic at infinity" holds.
2. (Whiteley unpublished) $\nexists$ symmetric $\Phi \neq 0$ such that:

$$\begin{aligned} \text{trace}(F^{ij}(P\Phi P^T)) &= 0 \text{ for all } \{i,j\} \in B \cup C^* \cup S^*. \\ \text{trace}(F^{ij}(P\Phi P^T)) &\leq 0 \text{ for all } \{i,j\} \in C^0. \\ \text{trace}(F^{ij}(P\Phi P^T)) &\geq 0 \text{ for all } \{i,j\} \in S^0. \end{aligned}$$

3. $\nexists y = (y_{ij}) \neq 0 \in \mathbb{R}^{|\bar{E}|+|C^0|+|S^0|}$ and $\xi = (\xi_i) \in \mathbb{R}^{n-r-1}$ where $y_{ij} \geq 0 \;\; \forall \;\; \{i,j\} \in C^0$ and $y_{ij} \leq 0 \;\; \forall \;\; \{i,j\} \in S^0$ such that:

$$\mathcal{E}^0(y) Z = e\xi^T,$$

where $\mathcal{E}^0(y) = \sum_{\{i,j\} \in \bar{E} \cup C^0 \cup S^0} y_{ij} E^{ij}$.

Affine-Domination, Ω is proper of rank $n-r-1$

Lemma

Assume that $\Omega = Z\Psi Z^T$ is a proper stress matrix of (G,p) of rank $n-r-1$. Then the following are equivalent:

1. *the "no conic at infinity" holds*
2. $\nexists$ $y = (y_{ij}) \neq 0 \in \mathbb{R}^{|\bar{E}|+|C^0|+|S^0|}$ *and* $\xi = (\xi_i) \in \mathbb{R}^{n-r-1}$ *where* $y_{ij} \geq 0$ *for all* $\{i,j\} \in C^0$ *and* $y_{ij} \leq 0$ *for all* $\{i,j\} \in S^0$ *such that:*

$$\mathcal{E}^0(y)Z = 0,$$

where $\mathcal{E}^0(y) = \sum_{\{i,j\}\in \bar{E}\cup C^0\cup S^0} y_{ij}E^{ij}$.

Outline of the Proof of the main Theorem

It suffices to prove that under the theorem assumptions, the only solution of

$$\mathcal{E}^0(y)Z = 0 \qquad (1)$$

is the trivial solution $y = 0$. Hence, the "no conic at infinity" condition holds.

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Equation (1) can be written as

$$\sum_{j=1}^{n} (\mathcal{E}^0(y))_{ij} z^i = 0 \text{ for all } i = 1, \ldots, n.$$

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Thus the result follows from the linear independence of $\{z^i : \{i,j\} \in \overline{E} \cup C^0 \cup S^0\}$.

Thank You