Equal Sum Sequences and Imbalance Sets of Tournaments

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Imbalance

The **imbalance** $t(v)$ of a vertex v in a digraph equals its outdegree minus the indegree.

The **imbalance sequence** of a digraph is formed by listing the imbalances in nonincreasing order.

The **imbalance set** is simply the set of vertex imbalances of a digraph.

A **tournament** is a complete simple digraph.

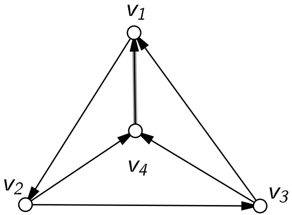


Figure : A tournament of order 4 with imbalance sequence $1, 1, -1, -1$ and imbalance set $\{1, -1\}$

Imbalance Sequences

Theorem (Mubayi, Will, West 2001)

A sequence $[t_i]_1^n$ of integers in nonincreasing order is the imbalance sequence of a simple digraph if and only if

$$\sum_{i=1}^{j} t_i \leq j(n-j), \tag{1}$$

for $1 \leq j \leq n$, with equality when $j = n$.

Theorem (Koh, Ree 2003)

A sequence $[t_i]_1^n$ of integers is the imbalance sequence of a tournament if and only if conditions (1) are satisfied and $n-1, t_1, \ldots, t_n$ have the same parity.

Imbalance Sets of Tournaments

Theorem (Pirzada 2008)

A set of integers is the imbalance set of a simple digraph if and only if it is the set $\{0\}$ or contains at least one positive and at least one negative integer.

QUESTION

Which sets of integers are imbalance sets of tournaments?

- Important due to its connection with **Reid's score set theorem** (any set of non-negative integers is the score set of some tournament).
- Generating tournaments with desired properties.
- Connections with the NP-hard **Equal Sum Sets Problem** and its variants.

Necessary Conditions

The set $\{0\}$ is the imbalance set of any **regular tournament**.

Theorem

If a finite nonempty set Z of integers is the imbalance set of a tournament of order n then all the elements of Z have the same parity as $n-1$ and it either contains only a single element 0 or contains at least one positive and at least one negative integer.

Are the Necessary Conditions Sufficient?

NO!

Example

Let $Z = \{6, -10\}$*. Then* Z *satisfies the necessary conditions. However, any sequence with elements chosen from* Z *can sum to zero only if it consists of an even number of elements* (*e.g.*, 6, 6, 6, 6, 6, -10, -10, -10)*. Thus the parity condition for tournament imbalance sequences can never be satisfied.*

Surprisingly, if Z consists of odd integers then the neccessary conditions are also sufficient.

Some Notations

Given a set Z of integers, let $X = \{x_1, \ldots, x_\ell\}$ be the set of non-negative and $Y = \{-y_1, \ldots, -y_m\}$ be the set of negative integers in Z.

$$x_1 > \cdots > x_\ell$$

$$-y_1 > \cdots > -y_m$$

$$L = \sum_{i=1}^{\ell} x_i$$

$$M = \sum_{i=1}^{m} y_i$$

$$n = \ell M + mL$$

Let $x^{(p)}$ denote that x is appearing in p consecutive terms of a sequence.

Odd Imbalance Sets

Theorem

Let $Z = X \cup Y$ be a set of odd integers, then there exists a tournament of order n with imbalance set Z if and only if X and Y are nonempty.

Proof. (Sketch) Form the sequence

$$[t_i]_1^n = {x_1}^{(M)}, \ldots, {x_\ell}^{(M)}, -{y_1}^{(L)}, \ldots, -{y_m}^{(L)},$$

then the terms of $[t_i]_1^n$ have the same parity as $n-1$. Verify inequality (1) for

$$j = M, 2M, \ldots, \ell M, \ell M + L, \ell M + 2L, \ldots, \ell M + mL (= n).$$

Show that if some $j_0 \neq M, \ldots, \ell M, \ell M + L, \ldots, n$ violates (1), then $j_0 + 1$ violates (1). $\square$

The Case of Even Imbalances

Recall, $\{0\}$ is the imbalance set of any regular tournament.

What about other sets of even integers?

The sequence

$$[t_i]_1^n = {x_1}^{(M)}, \ldots, {x_\ell}^{(M)}, -{y_1}^{(L)}, \ldots, -{y_m}^{(L)},$$

gives a digraph but not a tournament.

If we cannot guarantee a complete digraph, how close can we get?

Lemma (Mubayi, Will, West 2000)

Let D be a simple digraph with maximum number of arcs realizing the imbalance sequence $[t_i]_1^n$. Then any vertex in D has at most one non-neighbour and the number of arcs in D equals $\sum_{i=1}^{n} \left\lfloor \frac{n-1+t_i}{2} \right\rfloor$.

The Case of Even Imbalances

Since all the imbalances t_i are even while $n-1$ is odd,

$$\sum_{i=1}^{n} \left\lfloor \frac{n-1+t_i}{2} \right\rfloor = \sum_{i=1}^{n} \frac{n-2+t_i}{2} = \frac{n(n-2)}{2},$$

which is $\frac{n}{2}$ less than the number of arcs of a tournament of order n. Therefore, every vertex of D has exactly one non-neighbour. We say that D is a **near tournament**.

Let us call the $O(n^2)$ algorithm that generates D, Max Arcs.

Theorem

Let $Z \neq \{0\}$ be a set of even integers and $Z = X \cup Y$, with X and Y being nonempty. Then there exists a near tournament of order n with imbalance set Z.

Sufficient Conditions for Even Imbalance Sets

Theorem

Let X, Y, ℓ, m, L, M and n be as before. The set $X \cup Y$ is the imbalance set of a tournament if any one of the following conditions is satisfied:

(I) $0 \in X \cup Y$,

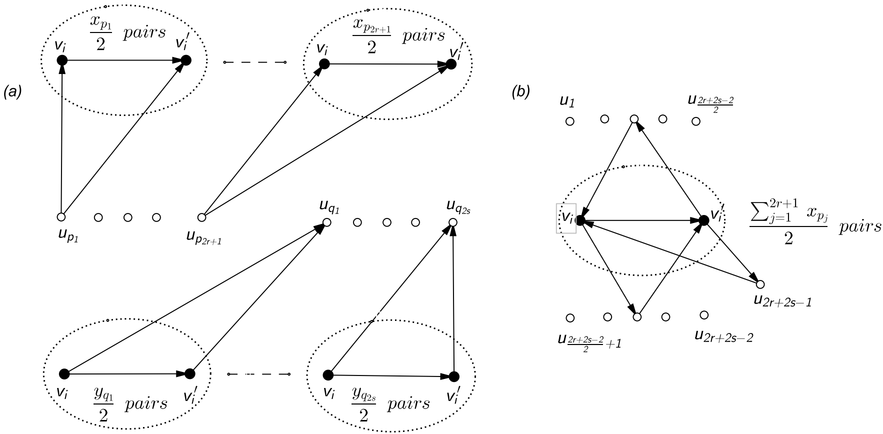
(II) there exist an odd number of (not necessarily distinct) $x_{p_1}, \ldots, x_{p_{2r+1}} \in X$ *and an even number of (not necessarily distinct)* $-y_{q_1}, \ldots, -y_{q_{2s}} \in Y$ *such that* $\sum_{j=1}^{2r+1} x_{p_j} = \sum_{j=1}^{2s} y_{q_j}$,

(III) there exist an odd number of (not necessarily distinct) $-y_{p_1}, \ldots, -y_{p_{2r+1}} \in Y$ *and an even number of (not necessarily distinct)* $x_{q_1}, \ldots, x_{q_{2s}} \in X$ *such that* $\sum_{j=1}^{2r+1} y_{p_j} = \sum_{j=1}^{2s} x_{q_j}$.

Proof. (Sketch) (I) Add a vertex v to T in such a way that for every pair of non-adjacent vertices v_i and v_i' insert the arcs (v_i, v_i'), (v_i', v) and (v, v_i).

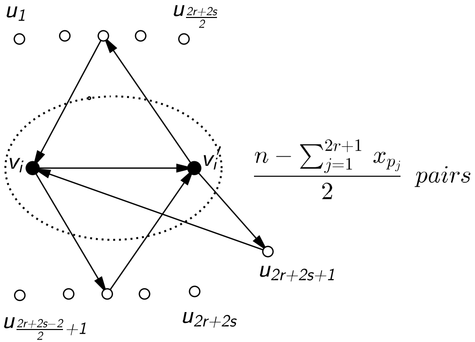
Sufficient Conditions for Even Imbalance Sets

(II) Consider $\frac{\sum_{j=1}^{2r+1} x_{p_j}}{2}$ pairs of non-adjacent vertices.



Sufficient Conditions for Even Imbalance Sets

(II) For the remaining $\frac{n-\sum_{j=1}^{2r+1} x_{p_j}}{2}$ pairs of non-adjacent vertices.



Let us call this procedure ADD ARCS.

Sufficient Conditions are Necessary

Theorem

Let $Z = X \cup Y$ be a finite nonempty set of even integers. Then Z is the imbalance set of a tournament if and only if either $Z = \{0\}$ or both X and Y are nonempty and satisfy one of the conditions (I), (II) or (III).

Proof. (Sketch) Let $0 \notin X \cup Y$ and $X \cup Y$ be the imbalance set of a tournament of order k. We can form a sequence $[t_i]_1^k$ consisting of an odd number of not necessarily distinct terms from the elements of $X \cup Y$ that sums to zero. Since k is odd, either the number of terms from X is odd or the number of terms from Y is odd, but not both. □

Algorithmic Aspects

Imbalance Set Problem (Decision Version)

Given a set of integers, decide whether it is the imbalance set of some tournament.

Imbalance Set Problem (Search Version)

Given a tournament imbalance set, construct a tournament realizing that imbalance set.

An Algorithm for ISP

Procedure: IMBALANCE

1. If Z contains both odd and even integers, return 'No'.
2. If $X = \varnothing$ or $Y = \varnothing$, return 'No'.
3. Form the sequence $[t_i]_1^n = {x_1}^{(M)}, \ldots, {x_\ell}^{(M)}, -{y_1}^{(L)}, \ldots, -{y_m}^{(L)}$.
4. Call MAX ARCS to to realize $[t_i]_1^n$ as a simple digraph D with maximum number of arcs.
5. If Z consists of odd integers, D is a tournament. Return D.
6. If Z consists of even integers, search for sequences $[x]_1^a$ and $[-y]_1^b$, where a and b have different parity and $\sum x = \sum y$. If no such sequences exist, return 'No'.
7. Call ADD ARCS to add $a + b$ vertices and arcs to D to form a tournament T. Return T.

Equal Sum Sequences

Equal Sum Sets (ESS) Problem

Given two sets of non-negative integers, find their subsets with equal sum.

- Dynamic programming algorithm by Bazgan, Santha and Tuza (2002).
- $O(|Input| \times Sum^2)$ running time (pseudopolynomial).
- ESS is weakly NP-hard.

However, for step 6 we want to find equal sum sequences.

Equal Sum Sequences (ESSeq) Problem

Given sets X and Y of non-negative integers and a positive integer k, find nonempty finite sequences $[x]$ and $[y]$ of elements from X and Y, with each element allowed to repeat at the most k times, such that $\sum x = \sum y$.

The Bounding Theorem

The ESS algorithm can be adapted to solve ESSeq (use the multisets $X^{(k)}$ and $Y^{(k)}$ as input, with each element repeated k times).

Let us call the resulting algorithm EQUAL SEQ.

We can call EQUAL SEQ to find equal sum sequences in step 6 of IMBALANCE. provided we can determine a bound on k that works.

Theorem

Let X, Y, ℓ, m, L, M and n be as defined before. If $k = p + q$ is the minimum odd number such that there exists a p-term sequence from X and a q-term sequence from Y having the same sum, then $k < n$.

ISP Algorithm Revisited

Procedure: IMBALANCE

1. If Z contains both odd and even integers, return 'No'.
2. If $X = \varnothing$ or $Y = \varnothing$, return 'No'.
3. Form the sequence $[t_i]_1^n = {x_1}^{(M)}, \ldots, {x_\ell}^{(M)}, -{y_1}^{(L)}, \ldots, -{y_m}^{(L)}$.
4. Call MAX ARCS to to realize $[t_i]_1^n$ as a simple digraph D with maximum number of arcs.
5. If Z consists of odd integers, D is a tournament. Return D.
6. Call EQUAL SEQ with the input $(X^{(n)}, Y^{(n)}, n)$ to find sequences $[x]_1^a$ and $[-y]_1^b$, with a and b having different parity and $\sum x = \sum y$. If no such sequences exist, return 'No'.
7. Call ADD ARCS to add $a + b$ vertices and arcs to D to form a tournament T. Return T.

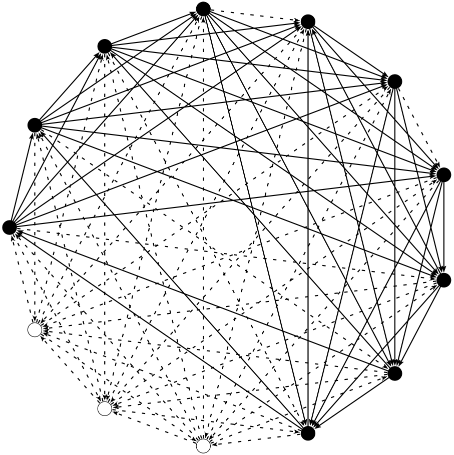


Figure : A tournament with imbalance set $\{4, 2, -2\}$.

- Max Arcs gives a digraph D (black vertices and solid arcs) with imbalance sequence $[4^{(2)}, 2^{(2)}, -2^{(6)}]$.
- Equal Seq gives 4 from X, $-2, -2$ from Y with $4 = 2 + 2$.
- Add Arcs inserts white vertices with imbalances $4, -2, -2$ and dashed arcs to form a tournament with imbalance sequence $[4^{(3)}, 2^{(2)}, -2^{(8)}]$.

Complexity of ISP

Theorem

The ISP decision problem is NP-complete and can be solved in $O(n^3(\ell + m)(L + M)^2)$ time.

Proof. (Sketch) The NP-completeness follows by reduction from Equal Sum Sets Problem. The running time of the algorithm IMBALANCE is dominated by step 6 which takes $O(|Input| \times Sum^2)$ time. $\square$

Minimal Order of the ISP Output

Theorem

Let Z be a tournament imbalance set and $ord(Z)$ denote the minimum order of a tournament realizing Z.

(a) If Z consists of odd integers then

$$ord(Z) \leq n.$$

(b) If Z consists of even integers and $0 \in Z$ then

$$ord(Z) \leq n + 1.$$

(c) If Z consists of even integers and $0 \notin Z$ then

$$ord(Z) < 2n.$$

Open Problems

1. Given a set Z of integers, construct a tournament of minimal order realizing Z as its imbalance set. Can we express this minimal order as a function of elements of Z and its cardinality?
2. Investigate the Equal Sum Sequences problem and its variants in more detail.
3. Use the constructions given here to obtain a constructive proof of Reid's theorem.
4. Generalization to hypertournaments.

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Thank you!