Tight simplices and codes in compact spaces

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Joint work with Henry Cohn and Gregory Minton

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Outline

- Codes and LP bounds.
- Tight codes and tight simplices.
- Previously known examples.
- Our results.
- Method of proof.

Coding theory

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In this talk, we'll mostly be concerned with special codes on compact metric spaces, such as Hamming space $\mathbb{F}_2^d$, spheres $S^{d-1} \subset \mathbb{R}^d$, projective spaces such as $\mathbb{RP}^{d-1}$, Grassmannians $\mathbb{G}(m, n)$, etc.

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Basic problem of coding theory: distribute N points on X in a "good" way (e.g. spaced far apart).

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Codes in Grassmannians: multiple-antenna communication.

Linear programming bounds

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Linear programming bounds were originally used by Delsarte for coding theory/association schemes. But later applied to spherical codes, sphere packings, energy minimization, etc.

Example: simplices in S^{d-1}

Proposition

Let N be a collection of $d+1$ unit vectors $v_1, \ldots, v_{d+1}$ in S^{d-1}, with $\langle v_i, v_j \rangle \leq \alpha$ for $i \neq j$. Then $\alpha \geq -1/d$, with equality iff they form a regular simplex.

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Proof.

$$\begin{aligned} 0 &\leq \langle \sum v_i, \sum v_i \rangle \\ &= d + 1 + \sum_{i \neq j} \langle v_i, v_j \rangle \\ &\leq d + 1 + (d+1)d\alpha. \end{aligned}$$

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This is the linear programming bound with the positive definite kernel $f(x) = x$.

Positive definite kernels

More generally, for a 2-point homogeneous space with isometry group G, the decomposition of $L^2(X)$ into unitary reps of G gives rise to zonal spherical functions C_k, which happen to be positive definite kernels.

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For projective spaces $K\mathbb{P}^{d-1}$, where $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ or $\mathbb{O}$ they are Jacobi polynomials $P_k^{(\alpha,\beta)}$, where $\alpha = (d-1)(\dim_{\mathbb{R}} K)/2 - 1$ and $\beta = (\dim_{\mathbb{R}} K)/2 - 1$. They can be computed easily. Normalize $C_0 = 1$.

General LP bounds

Proposition

Let $\theta \in [0, \pi]$*, and suppose the polynomial*

$$f(z) = \sum_{k=0}^{n} f_k C_k(z)$$

satisfies $f_0 > 0$*,* $f_k \geq 0$ *for* $1 \leq k \leq n$*, and* $f(z) \leq 0$ *for* $-1 \leq z \leq \cos\theta$*. Then every code in* X *with minimal geodesic distance at least* θ *has size at most* $f(1)/f_0$*.*

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It's called an LP bound, because typically we take $f_0 = 1$, then $f_1, \ldots, f_n$ are the variables. The constraints and objective function (i.e. the bound) are linear in these.

Proof of LP bound

Proof.

Let $\mathcal{C}$ be such a code. Then

$$\sum_{x,y\in\mathcal{C}} f(\cos\vartheta(x,y)) = \sum_{x,y\in\mathcal{C}} \sum_{k\geq 0} f_k C_k(\cos\vartheta(x,y))$$
$$= \sum_{k\geq 0} f_k \sum_{x,y\in\mathcal{C}} C_k(\cos\vartheta(x,y)) \geq f_0|\mathcal{C}|^2,$$

On the other hand, $f(\cos\vartheta(x,y)) \leq 0$ whenever $\vartheta(x,y) \geq \theta$, and hence

$$\sum_{x,y\in\mathcal{C}} f(\cos\vartheta(x,y)) \leq |\mathcal{C}| f(1)$$

because only the diagonal terms contribute positively. It follows that $f_0|\mathcal{C}|^2 \leq f(1)|\mathcal{C}|$, as desired. □

Tight codes

Definition

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In general a tight design is a tight code, but not vice versa.

Many examples of tight codes on spheres and real and complex projective spaces were described by Levenshtein (some newer examples by others, too). Cohn-Kumar studied these from the perspective of energy minimization and universal optimality.

Projective spaces

Our results pertain to families of tight simplices in projective spaces over $\mathbb{H}$ and $\mathbb{O}$. A regular simplex in a metric space X is a code for which all the pairwise distances between distinct points are equal.

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Recall: projective space $K\mathbb{P}^{d-1}$ over $K = \mathbb{R}, \mathbb{C}, \mathbb{H}$ can be thought of as lines in K^d: identify x and $x\alpha$ for $x \in K^d \setminus \{0\}$ and $\alpha \in K^\times$. For $\mathbb{OP}^2$ the description is more complicated (need the coordinates of x to associate).

The LP bounds for projective spaces

Points in projective space may be considered as projectors of rank 1 (namely $xx^\dagger$). The inner product can then be defined as $\langle A, B\rangle = \mathrm{Re}\,\mathrm{Tr}(AB)$, and the chordal distance as $\sqrt{2 - 2\langle A, B\rangle}$.

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Let e be the dimension of the algebra K over $\mathbb{R}$, i.e. $1, 2, 4, 8$ for $K = \mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$ respectively.

Proposition (Lemmens-Seidel)

A regular simplex in $K\mathbb{P}^{d-1}$ can have at most $d + e(d^2 - d)/2$ points. If it has N points, then the maximal inner product α satisfies

$$\alpha \geq \frac{N - d}{d(N - 1)}.$$

LP bounds for projective spaces II

Proof idea.

The bound on N comes from considering the dimension of the space of Hermitian $d \times d$ matrices over K. It can be shown that the projectors corresponding to the points of a regular simplex are linearly independent. The bound on α follows from a linear programming bound on N, using a linear positive definite kernel. □

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Note: this definition is a priori stronger than saying that it's a simplex which is a tight code, since the strongest LP bound could be attained by a non-linear function.

Tight simplices in real projective spaces

Dimension	# points	Name/origin
d	d	cross-polytope
d	$d+1$	Euclidean simplex
d	$2d$	symm. conf. matrix of order $2d$ $(\star)$
6	16	Clebsch
7	28	equiangular lines
23	276	equiangular lines
$\frac{v(v-1)}{k(k-1)}$	$v\left(1+\frac{v-1}{k-1}\right)$	Steiner construction $(\star)$
d	N	strongly regular graph with parameters $(N-1, k, (3k-N)/2, k/2)$, where $k = \frac{N}{2} - 1 + \left(1 - \frac{N}{2d}\right)\sqrt{\frac{d(N-1)}{N-d}}$ $(\star)$

$\star$: The code may exist only for certain parameter settings.

Tight simplices in complex projective spaces

Dimension	# points	Name/origin
d	$2d$	skew-symm. conf. matrix of order $2d$ $(\star)$
d	d^2	SIC-POVMs $(\star)$
$2k-1$	$4k-1$	skew-Hadamard matrix of order $4k$ $(\star)$
$2k$	$4k-1$	skew-Hadamard matrix of order $4k$ $(\star)$
$\frac{v(v-1)}{k(k-1)}$	$v\left(1+\frac{v-1}{k-1}\right)$	Steiner construction $(\star)$
$\lvert S\rvert$	$\lvert G\rvert$	difference set S in abelian group G $(\star)$

$\star$: The code may exist only for certain parameter settings.

Previously known families of tight codes in $\mathbb{HP}^{d-1}$ and $\mathbb{OP}^2$

Space	# points	$\max \lvert\langle x, y\rangle\rvert^2$	Name/origin
$\mathbb{HP}^{d-1}$	$d(2d+1)$	$1/d$	$2d+1$ mutually unbiased bases $(\star)$
$\mathbb{HP}^4$	165	$1/4$	quaternionic reflection group
$\mathbb{OP}^2$	819	$1/2$	generalized hexagon of order $(2, 8)$

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For brevity we omit the tight simplices from $\mathbb{RP}^{d-1}$ and $\mathbb{CP}^{d-1}$.

Our results for $\mathbb{HP}^{d-1}$ and $\mathbb{OP}^2$

For $\mathbb{HP}^2$ through $\mathbb{HP}^5$, we find positive dimensional families of tight simplices for many values of N in the allowed range.

Similarly for $\mathbb{OP}^2$.

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Finding the codes: We find an *approximate* tight simplices in one of two ways:

- through energy minimization (gradient descent).
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Proving existence: through an effective version of the implicit function theorem. This also allows us to find the local dimension of the space of solutions to the system of equations, and therefore a lower bound for the dimension of the space of tight simplices.

Some highlights

- For $\mathbb{HP}^2$, we find tight simplices for all the allowed values of N (i.e. between 1 and 15) except for $N = 14$. We conjecture that there does not exist a tight simplex of 14 points.

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- For $\mathbb{OP}^2$, we find tight simplices for all the allowed values of N (i.e. between 1 and 27) except for $N = 26$. We conjecture that there does not exist a tight simplex of 26 points.

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- For $\mathbb{OP}^2$, we find tight simplices for all the allowed values of N (i.e. between 1 and 27) except for $N = 26$. We conjecture that there does not exist a tight simplex of 26 points.
- In particular, we settle the existence of a tight 27-point simplex (equivalently, a tight 2-design) in $\mathbb{OP}^2$ (conjectured not to exist by Hoggar).

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- For $\mathbb{OP}^2$, we find tight simplices for all the allowed values of N (i.e. between 1 and 27) except for $N = 26$. We conjecture that there does not exist a tight simplex of 26 points.
- In particular, we settle the existence of a tight 27-point simplex (equivalently, a tight 2-design) in $\mathbb{OP}^2$ (conjectured not to exist by Hoggar).
- We also rigorously show the existence of many tight simplices in Grassmannians $G(m, n, \mathbb{R})$, which were reported by Conway, Hardin and Sloane, based on numerical evidence.

Tight simplices discovered for $\mathbb{HP}^2$

N	$r(3, N, \mathbb{H})$	N	$r(3, N, \mathbb{H})$
5	0	10	10
6	4	11	9
7	7	12	2
8	9	13	2
9	10	15	14

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Of these, 5 through 11 were found and proved through the same schema of equations. For tight simplices of 12, 13 and 15 points, we have to do some extra work (e.g. impose symmetry for 12 and 13) so their local dimensions do not fit the general pattern.

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- Start from a random code of N points and do gradient descent for potential energy for the potential function $1/r$, where r is the chordal distance. Stop when the simulation stabilizes to a high accuracy. Likely to get a tight simplex.

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- Start from a random code of N points and do gradient descent for potential energy for the potential function $1/r$, where r is the chordal distance. Stop when the simulation stabilizes to a high accuracy. Likely to get a tight simplex.
- Try to directly solve the system of equations below, using gradient descent and then Newton's method. Software package written by G. Minton called `QNewton` is very useful for this purpose.

System of equations

Naively, we may try to define a tight simplex of N points in $K\mathbb{P}^{d-1}$ by the following set of equations:

$$|\langle x_i, x_j \rangle|^2 = \frac{N-d}{d(N-1)} \text{ for } 1 \leq i < j \leq N.$$

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The problem is that this set of nonlinear constraints does not have surjective Jacobian at a tight simplex. So cannot apply implicit function theorem.

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It's not at all obvious if a subset of constraints will give us a surjective Jacobian while still giving a tight simplex.

System of equations

We use the equations corresponding to the following conditions.

Proposition

Suppose $x_1, \ldots, x_N \in \mathbb{H}^d$ $(d > 1)$ *and* $w_1, \ldots, w_N \in \mathbb{R}$ *satisfy the following conditions:*

- $|x_i|^2 = 1$ *for all* $i = 1, \ldots, N$*;*
- $|\langle x_i, x_j \rangle|^2 = |\langle x_{i'}, x_{j'} \rangle|^2$ *for all* $1 \leq i < j \leq N$ *and* $1 \leq i' < j' \leq N$*; and*
- $\sum_{i=1}^{N} w_i x_i x_i^\dagger = I$.

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- $\sum_{i=1}^{N} w_i x_i x_i^\dagger = I$.

Then $w_1 = \cdots = w_N = \frac{d}{N}$ *and* $\{x_1, \ldots, x_N\}$ *is a tight simplex.*

Proof idea.

The first condition is just to normalize the scaling, since we're in projective space. The second condition cuts out a regular simplex. The final condition, which is the key, says that being tight is equivalent to being a 1-design. From it we can recover the value of the inner product. □

Effective implicit function theorem

Definition

For a polynomial $p\colon \mathbb{R}^m \to \mathbb{R}$ given by $p(x) = \sum_I c_I x^I$, define $|p| = \sum_I |c_I|$. Given a polynomial map $p = (p_1, \ldots, p_n)\colon \mathbb{R}^m \to \mathbb{R}^n$, define $|p| = \max |p_i|$.

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Theorem

Let $m \geq n$, $\varepsilon > 0$, and $x_0 \in \mathbb{R}^m$. Suppose $f : \mathbb{R}^m \to \mathbb{R}^n$ is a polynomial function of total degree s, and let $\mathbb{R}^m$ and $\mathbb{R}^n$ carry the ℓ_∞ norm. Set $\eta = \max(1, |x_0| + \varepsilon)$. If there exists a linear operator $T\colon \mathbb{R}^n \to \mathbb{R}^m$ such that

$$||Df(x_0) \circ T - \mathrm{id}_{\mathbb{R}^n}|| + \varepsilon |f| s(s-1)\eta^{s-2} ||T|| < 1 - \frac{||T|| \cdot |f(x_0)|}{\varepsilon},$$

then there exists $x_* \in B(x_0, \varepsilon)$ such that $f(x_*) = 0$, and the zero locus $f^{-1}(0)$ is locally a manifold of dimension $m - n$.

Demonstrating existence

To use the theorem, we find an x_0 (as described earlier) and let T be a right inverse for $Df(x_0)$. These are computed using floating point numbers. But then we can round them to rational numbers with denominator 10^9 say, and everything is completely rigorously verified using integer arithmetic.

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Using the theorem, we get (for instance) existence for N-point tight simplices in $\mathbb{HP}^2$, for $3 \leq N \leq 11$.

Additional tweaks for special cases

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For 15 points, need something quite a bit more sophisticated.

Tight simplices in $\mathbb{OP}^2$

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No 26-point tight simplex seems to exist.

We also produce, by explicit algebraic construction, 39 points in $\mathbb{OP}^2$ forming thirteen triples of mutually orthogonal vectors such that the inner product between points in distinct triples is $1/3$. These form a maximal system of mutually unbiased bases in $\mathbb{OP}^2$, which was previously only conjectured to exist.

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Using our techniques and suitable adaptations of the proposition giving a good set of constraints, we are able to prove most of their conjectured examples of simplices. (Again, we get positive dimensional families.)

Conclusion and open questions

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- Likely lots of examples of tight simplices in complex Grassmannians too. We haven't explored these.
- Tight codes in groups? We found a tight simplex of 17 points and a tight cross-polytope of 32 points in $\mathrm{SO}(4)$.

Reference: H. Cohn, A. Kumar, and G. Minton, *Simplices and optimal codes in projective spaces*, `arXiv.org:1308.3188`.

Thank you!