Realizations of Symmetric Sets

Peter McMullen
University College London

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Short bibilography

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Symmetric sets

A symmetric set is a pair $(\mathcal{V}, \boldsymbol{G})$, with $\mathcal{V}$ a finite set on which a subgroup $\boldsymbol{G}$ of its permutations acts transitively. We often call $\mathcal{V}$ itself a symmetric set; then $\boldsymbol{G}$ is the automorphism group of $\mathcal{V}$. We always think of the $n := \operatorname{card} \mathcal{V}$ points of $\mathcal{V}$ as ordered in some way.

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Pick $\boldsymbol{e} \in \mathcal{V}$ any element, and let $\boldsymbol{H}$ be the stabilizer of $\boldsymbol{e}$ in $\mathcal{V}$. Thus we may identify $\mathcal{V}$ with the family of (right) cosets $\boldsymbol{Hx}$ of $\boldsymbol{H}$ in $\boldsymbol{G}$, and write $\boldsymbol{x}$ for the corresponding element of $\mathcal{V}$. However, it is helpful to retain $\mathcal{V}$ as a separate entity.

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Pick $e \in \mathcal{V}$ any element, and let H be the stabilizer of e in $\mathcal{V}$. Thus we may identify $\mathcal{V}$ with the family of (right) cosets Hx of H in G, and write x for the corresponding element of $\mathcal{V}$. However, it is helpful to retain $\mathcal{V}$ as a separate entity.

Remark

In particular, we can identify e with the identity of G.

Diagonal classes and layers

A diagonal in $\mathcal{V}$ is an unordered pair $\{\boldsymbol{x}, \boldsymbol{y}\}$ of elements of $\mathcal{V}$. A diagonal class consists of a family of diagonals equivalent under $\boldsymbol{G}$. We label the diagonal classes $\mathcal{D}_0, \ldots, \mathcal{D}_r$, with the trivial class $\mathcal{D}_0 := \{x, x\}$ always first.

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Similarly, the points of $\mathcal{V}$ fall into layers $\mathcal{L}_k$ from the initial point $\boldsymbol{e}$:

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$$\mathcal{L}_k := \{\boldsymbol{x} \in \mathcal{V} \mid \{\boldsymbol{e}, \boldsymbol{x}\} \in \mathcal{D}_k\}.$$

If $\ell_k := \mathrm{card}\, \mathcal{L}_k$, so that $\ell_0 + \cdots + \ell_r = \boldsymbol{n}$ (and $\ell_0 = 1$), then we define

$$\Lambda := (\ell_0, \ldots, \ell_r)$$

to be the layer vector of $(\mathcal{V}, \boldsymbol{G})$ (or of $\mathcal{V}$).

Realizations

A realization of $(\mathcal{V}, \boldsymbol{G})$ is a mapping $\Psi\colon \mathcal{V} \times \boldsymbol{G} \to \mathbb{E} \times \mathbf{O}$, with $\mathbb{E}$ a euclidean space and $\mathbf{O}$ its orthogonal group, such that

$$(\boldsymbol{x}\boldsymbol{g})\Psi = (\boldsymbol{x}\Psi)(\boldsymbol{g}\Psi)$$

for all $\boldsymbol{x} \in \mathcal{V}$ and $\boldsymbol{g} \in \boldsymbol{G}$. In other words, Ψ is compatible with the group action; in particular, Ψ induces a homomorphism on $\boldsymbol{G}$.

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Write $\mathbf{G} := \boldsymbol{G}\Psi$ and $V := \mathcal{V}\Psi$. Thus $\mathbf{G}$ is a finite orthogonal group acting transitively on V. We often identify Ψ with the image set V.

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The dimension of V is $\dim V := \dim \operatorname{lin} V$.

Wythoff space

The Wythoff space of a realization Ψ is the set of points W of $\mathbb{E}$ fixed by $\mathbf{H} := \boldsymbol{H}\Psi$, namely,

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We thus call the image $v := \boldsymbol{e}\Psi$ of the initial point $\boldsymbol{e} \in \mathcal{V}$ the initial point of the realization. Observe that some representations of $\boldsymbol{G}$ may have trivial Wythoff spaces $W = \{o\}$, and so yield trivial realizations.

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Henceforth, we demand that $V \neq \{o\}$ (and hence $W \neq \{o\}$), so that V is a subset of some sphere centred at the origin o.

Operations on realizations

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Inner-product vectors

Let $\{\boldsymbol{x}, \boldsymbol{y}\}$ represent the kth diagonal class $\mathcal{D}_k$ of $\mathcal{V}$. If Ψ is a realization of $\mathcal{V}$, write

$$\sigma_k = \sigma_k(\Psi) := \langle \boldsymbol{x}\Psi, \boldsymbol{y}\Psi \rangle.$$

Then $\Sigma = \Sigma(\Psi) := (\sigma_0, \ldots \sigma_r)$ is the inner-product vector of Ψ.

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We scale inner-product vectors and add them in the usual way. We further define a (term-by-term) product ab of two vectors $a = (\alpha_0, \ldots, \alpha_r)$ and $b = (\beta_0, \ldots, \beta_r)$ by

$$ab := (\alpha_0\beta_0, \ldots, \alpha_r\beta_r).$$

Realization cone

The effects of the operations on realizations are captured in

Theorem

If Ψ and Ω are two realizations of $\mathcal{V}$ and $\lambda \in \mathbb{R}$, then

$$\Sigma(\lambda\Psi) = \lambda^2 \Sigma(\Psi),$$
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We identify two realizations Ψ and Ω if the corresponding images $\mathcal{V}\Psi$ and $\mathcal{V}\Omega$ are congruent, and henceforth use $\mathcal{V}$ to mean the family of congruence classes of realizations. In this sense, we have

Corollary

The family $\mathcal{V}$ has the structure of an $(r+1)$-dimensional closed convex cone, called the realization cone.

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Last, we also have

Theorem
The multiplicative unity Ψ_0 *is given by* $\boldsymbol{x}\Psi_0 = 1 \in \mathbb{R}$ *for all* $\boldsymbol{x} \in \mathcal{V}$.

Purity

Since we are only concerned with congruence classes, we see that

$$\lambda V \# \mu V = \nu V$$

with $\nu^2 = \lambda^2 + \mu^2$. In particular, V always admits trivial expressions $V = \lambda V \# \mu V$ as a blend, with $\lambda^2 + \mu^2 = 1$.

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Remark

It should be clear that pure realizations V correspond to irreducible representations Ψ of the group $\boldsymbol{G}$.

Realization domain

Identifying a realization Ψ with its image $V = \mathcal{V}\Psi$, we shall write λV, $U \# V$, $U \otimes V$, and so on. In this sense, the unity Ψ_0 is identified with $\{1\}$, and is called the henogon.

A realization V is normalized if V is a subset of the unit sphere. The realization domain $\mathcal{N}$ of $(\mathcal{V}, \boldsymbol{G})$ consists of the normalized realizations. Observe that $\{1\} \in \mathcal{N}$.

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General (non-negative) linear combinations in $\mathcal{V}$ are replaced by convex ones in $\mathcal{N}$. More specifically, we are restricted to scaling and blending combinations $\lambda U \# \mu V$, where $\lambda^2 + \mu^2 = 1$.

Then we have

Theorem

The realization domain $\mathcal{N}$ has the structure of an r-dimensional compact convex set. It is a pyramid with apex $\{1\}$.

Cosine vectors

The cosine vector $\Gamma = \Gamma(\Psi) = (\gamma_0, \gamma_1, \ldots, \gamma_r)$ of a realization Ψ is given in terms of its inner-product vector $\Sigma = (\sigma_0, \ldots, \sigma_r)$ by

$$\Gamma := \sigma_0^{-1}\Sigma;$$

the cosine vector is the inner-product vector of the normalization of Ψ (recall that $\sigma_0 > 0$ by assumption). Note that $\gamma_0 := 1$ represents the trivial diagonal class $\{x, x\}$.

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Theorem

The product $\Psi \otimes \Omega$ of realizations has cosine vector

$$\Gamma(\Psi \otimes \Omega) = \Gamma(\Psi)\Gamma(\Omega).$$

Layer inequality

The cosine vector Γ of a centred realization (that is, the centroid of its points is the origin o) must satisfy the layer equation

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(take the inner product of $\sum_{x \in V} x = o$ with the initial point v).

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More generally, a cosine vector $\varGamma$ must satisfy the layer inequality $\langle \varLambda, \varGamma \rangle \geqslant 0$. If $\varGamma = \alpha_0 \varGamma_0 + \alpha_1 \varGamma_1$, a convex combination, with $\varGamma_1$ corresponding to the centred component, then $\alpha_0 = \langle \varLambda, \varGamma \rangle / n$.

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The meaning of this is

Lemma

If λ is the distance from o to the centroid of V, then

$$\lambda^2 = \langle \Lambda, \Gamma \rangle / n.$$

Λ-inner-product

Define the (positive definite) Λ-inner-product $\langle \cdot, \cdot \rangle_\Lambda$ by

$$\langle a, b \rangle_\Lambda := \langle ab, \Lambda \rangle / n,$$

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The product of realizations is another realization. Moreover, if $\{u_1, \dots, u_d\}$ is an orthonormal basis of $\mathbb{E}^d$ and $x \in \mathbb{E}^d$, then $\langle x \otimes x, u_1 \otimes u_1 + \cdots + u_d \otimes u_d \rangle = \|x\|^2$. There follows

Lemma

- *If Γ_1, Γ_2 are cosine vectors of realizations of $\mathcal{V}$, then $\langle \Gamma_1, \Gamma_2 \rangle_\Lambda \geqslant 0$.*
- *If the realization V has cosine vector Γ, then*

$$\|\Gamma\|_\Lambda^2 := \langle \Gamma, \Gamma \rangle_\Lambda \geqslant \frac{1}{\dim V}.$$

Dimension equation

The simplex realization $T \in \mathcal{N}$ of $\mathcal{V}$ is the ordered orthonormal basis $(e_1, \ldots, e_n)$ of $\mathbb{E}^n$; its cosine vector is thus $\Gamma(T) = (1, 0^r)$.

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Theorem

If the simplex realization T of $\mathcal{V}$ is decomposed into components $V_1, \ldots, V_s$ in orthogonal subspaces, where V_j has dimension d_j and cosine vector Γ_j for $j = 1, \ldots, s$, then

$$\sum_{j=1}^{s} d_j \Gamma_j = n(1, 0^r).$$

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This (linear) dimension equation follows from the fact that the radius ρ_j of V_j satisfies $\rho_j^2 = d_j/n$.

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If the centrally symmetric set $\mathcal{V}$ has $n = 2m$ points, then it has a cross-polytope realization X, whose points are those of an ordered orthonormal basis $(e_1, \ldots, e_m)$ of $\mathbb{E}^m$, together with their opposites $(-e_1, \ldots, -e_m)$.

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Morever, these points can then be identified in opposite pairs, to give the small simplex realization S, whose points are those of an ordered orthonormal basis $(e_1, \ldots, e_m)$ of $\mathbb{E}^m$.

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There are natural analogues of the dimension equation for X and S. Observe that a pure realization of $\mathcal{V}$ is (up to scaling) a component either of X or of S.

Simplex and cross-polytope

The (vertex-set of the) d-simplex has layer vector $\Lambda = (1, d)$, and two pure realizations with cosine vectors

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The d-cross-polytope has layer vector $\Lambda = (1, 2(d-1), 1)$, and three pure realizations with cosine vectors

$$\Gamma_0 = (1, 1, 1),$$
$$\Gamma_1 = (1, -\tfrac{1}{d-1}, 1),$$
$$\Gamma_2 = (1, 0, -1).$$

Λ-orthogonality

The Λ-orthogonality theorem is a fundamental relationship governing realizations.

Theorem

If the simplex realization T of $\mathcal{V}$ is decomposed into components $V_1, \ldots, V_s$ in orthogonal subspaces, where V_j has dimension d_j and cosine vector Γ_j for $j = 1, \ldots, s$, then

$$\langle \Gamma_j, \Gamma_k \rangle_\Lambda = \frac{\delta_{jk}}{d_k};$$

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For this, take the Λ-inner-product of the dimension equation with Γ_k, and use the fact that $\langle \Gamma_j, \Gamma_k \rangle_\Lambda \geqslant 0$ and $\|\Gamma_k\|_\Lambda^2 \geqslant 1/d_k$.

Comments

We have already seen the Λ-norm, given by $\|\Gamma\|_\Lambda^2 := \langle \Gamma, \Gamma \rangle_\Lambda$; we can also talk about Λ-orthogonality. Among other things, the last theorem says:

- the cosine vector Γ of a d-dimensional pure realization satisfies

$$\|\Gamma\|_\Lambda^2 = 1/d;$$

- if V_1, V_2 are two pure realizations of different dimensions, then $V_1 \otimes V_2$ is centred.

Note also something that is useful for calculations: if $\Gamma_1, \Gamma_2, \Gamma_3$ are any cosine vectors, then

$$\langle \Gamma_1 \Gamma_2, \Gamma_3 \rangle_\Lambda = \langle \Gamma_1, \Gamma_2 \Gamma_3 \rangle_\Lambda.$$

Wythoff space

Let W be the Wythoff space of a subfamily of realizations with a given symmetry group $\mathbf{G}$. Different realizations $V(x)$ will usually arise from different choices of $x \in W$. If $x, y \in W$, then we write

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which we call their sum; more generally, with scaling as well we can form linear combinations of realizations.

Linear combinations and blends interact as follows.

Lemma

If U, V are realizations with symmetry group $\mathbf{G}$, then

$$U \# V = (\lambda U + \mu V) \# (\mu U - \lambda V)$$

whenever $\lambda, \mu \in \mathbb{R}$ are such that $\lambda^2 + \mu^2 = 1$.

Essential Wythoff space

Write $\mathcal{V}_{\mathbf{G}}$ for the subcone of $\mathcal{V}$ of all realizations which are blends of ones with a fixed irreducible symmetry group $\mathbf{G}$.

If $\mathbf{G}$ has a non-trivial centralizer in $\mathbf{O}$ (that is, other than $\{\pm I\}$), then it will be isomorphic to the complex numbers of unit modulus or the unit quaternions. We pass to an essential Wythoff space W^*, transverse to the action of the centralizer, whose dimension w^* will be $w/2$ or $w/4$, as appropriate.

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Cases when $w > 1$ are associated with asymmetric diagonal classes $\{\boldsymbol{e}, \boldsymbol{x}\}$, such that $(\boldsymbol{e}, \boldsymbol{x}) \not\equiv (\boldsymbol{x}, \boldsymbol{e})$ under $\boldsymbol{G}$. We have

Lemma

The diagonal class containing $\{\boldsymbol{e}, \boldsymbol{x}\}$ is symmetric if and only if

$$\boldsymbol{x}^{-1} \in \boldsymbol{H}\boldsymbol{x}\boldsymbol{H}.$$

Coefficient matrix

For a fixed orthonormal basis $E = (e_1, \ldots, e_{w^*})$ of W^*, there are $\Gamma_{jk} = \Gamma_{kj}$ (depending only on E) such that the realization $V(x)$ with initial point $x = \xi_1 e_1 + \cdots + \xi_{w^*} e_{w^*} \in W^*$ has inner-product vector

$$\Sigma(x) = \sum_{j,k=1}^{w^*} \xi_j \xi_k \Gamma_{jk}.$$

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$$\mathit{\Sigma}(x) = \sum_{j,k=1}^{w^*} \xi_j \xi_k \mathit{\Gamma}_{jk}.$$

A general member $V \in \mathcal{V}_{\mathbf{G}}$ thus has inner-product vector of the form $\mathit{\Sigma}(A) = \sum_{j,k} \alpha_{jk} \mathit{\Gamma}_{jk}$, with $A = (\alpha_{jk})$ a symmetric $w^* \times w^*$ matrix; A is the coefficient matrix of $\mathit{\Sigma}(A)$ or of $V(A) := V$.

Theorem

The symmetric matrix A is the coefficient matrix of a realization $V(A) \in \mathcal{N}_{\mathbf{G}}$ if and only if A is positive semi-definite with trace 1.

Λ-orthogonal basis

It should come as no surprise that the Γ_{jk} form a Λ-orthogonal basis for the inner-product vectors in $\mathcal{V}_{\mathbf{G}}$. More exactly, we have

Theorem

If the irreducible representation $\mathbf{G}$ *of* G *has degree* d *and the* Γ_{jk} *are defined as before with respect to a fixed orthonormal basis* E *of an essential Wythoff space of dimension* w^**, then*

- *distinct* Γ_{jk} *are* Λ*-orthogonal,*
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Observe that the theorem assigns a 'notional' dimension $2d$ to those Γ_{jk} with $j \neq k$.

Numerical relationships

We can now put together all the Γ_{jk} arising from different irreducible representations to obtain a Λ-orthogonal basis of the whole of $\mathcal{V}$. Counting the various contributions, we then arrive at

Theorem

With the previous notation,

$$\sum_{\Psi} w^*(\Psi) d(\Psi) = n,$$

$$\sum_{\Psi} \tfrac{1}{2} w^*(\Psi)\big(w^*(\Psi)+1\big) = r+1.$$

In each case, the sum runs over all irreducible representations Ψ of the automorphism group $\boldsymbol{G}$.

Cosine matrix

A cosine matrix of $\mathcal{V}$ is obtained by listing, in some order, the Γ_{jk} for each irreducible representation $\mathbf{G}$. Bear in mind that, when $w^*(\mathbf{G}) > 1$, these depend on a choice of basis of W^*.

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When $w^* = 2$, a useful alternative expression for the general cosine vector of a pure realization in $\mathcal{N}_{\mathbf{G}}$ is

$$\Gamma(\vartheta) = \Gamma_m + \cos(2\vartheta)\Gamma_c + \sin(2\vartheta)\Gamma_s,$$

where

$$\Gamma_m = \tfrac{1}{2}(\Gamma_{11} + \Gamma_{22}),\ \Gamma_c = \tfrac{1}{2}(\Gamma_{11} - \Gamma_{22}),\ \Gamma_s = \Gamma_{12}.$$

Each of $\Gamma_m, \Gamma_c, \Gamma_s$ has square Λ-norm $1/2d$; only Γ_m is a genuine cosine vector.

Induced cosine vectors

Henceforth, we just consider polytopes with a lot of symmetry (such as regular or uniform ones), although much of what we say generalizes.

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A sufficiently symmetric section $\mathcal{Q}$ of an abstract polytope $\mathcal{P}$ will itself have a layer vector $\Lambda_{\mathcal{Q}}$, and a cosine vector Γ of $\mathcal{P}$ will give a corresponding induced cosine vector $\Gamma_{\mathcal{Q}}$ of $\mathcal{Q}$, whose entries will be a subset of those of Γ.

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However, particularly in the case that $\mathcal{Q}$ is the vertex-figure or facet of a regular polytope $\mathcal{P}$, induced cosine vectors can also help to find cosine vectors of the latter.

Vertex-figure inequality

If $\mathcal{Q}$ is the vertex-figure of the regular polytope $\mathcal{P}$ and P is a realization of $\mathcal{P}$ with cosine vector Γ, then we write

$$\eta_{\mathbf{v}}(P) := \langle \Lambda_{\mathcal{Q}}, \Gamma_{\mathcal{Q}}(P) \rangle / m,$$

where $\mathcal{Q}$ has m vertices. Now $\eta_{\mathbf{f}}(P)$ is the squared distance of the centroid of the corresponding realization Q of $\mathcal{Q}$ from o. Taking $\mathcal{Q}$ to form layer $\mathcal{L}_1$, we therefore have

Theorem

For each realization P of $\mathcal{P}$,

$$\eta_{\mathbf{v}}(P) \geqslant \big(\gamma_1(P)\big)^2,$$

with equality if $w(P) = 1$.

Icosahedron

Treating the icosahedron $\{3, 5\}$ just as a symmetric map shows that it is centrally symmetric with layer vector $\Lambda = (1, 5, 5, 1)$, and so with 12 vertices. Since each diagonal is symmetric, $w(P) = 1$ for each pure realization P.

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For the components of the cross-polytope realization X, from the layer equation and Λ-orthogonality with respect to Γ_1 we see that the cosine vectors are of the form $(1, \alpha, -\alpha, -1)$ for some α. The induced layer and cosine vectors of the vertex-figure $\mathcal{Q} = \{5\}$ are $\Lambda_{\mathcal{Q}} = (1, 2, 2)$ and $\Gamma_{\mathcal{Q}} = (1, \gamma_1, \gamma_2)$. We thus have

$$\alpha^2 = \eta_{\mathbf{v}} = \frac{1}{5}\big(1 + 2\alpha + 2(-\alpha)\big) = \frac{1}{5} \Longrightarrow \alpha = \pm\frac{1}{\sqrt{5}}.$$

Dimensions

We have not assumed anything about the dimensions of these last two realizations. With $\alpha = \pm\frac{1}{\sqrt{5}}$, the Λ-orthogonality theorem tells us that their common dimension d is given by

$$\frac{1}{d} = \frac{1}{12}\big(1 + 5\alpha^2 + 5(-\alpha)^2 + 1\big) = \frac{1}{3} \Longrightarrow d = 3.$$

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We have thus obtained the cosine matrix of $\{3, 5\}$, namely,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -\frac{1}{5} & -\frac{1}{5} & 1 \\ 1 & \frac{1}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & -1 \\ 1 & -\frac{1}{\sqrt{5}} & \frac{1}{\sqrt{5}} & -1 \end{bmatrix}.$$

Hemi-icosidodecahedron

Abstractly, this is the 15 diameters of the icosidodecahedron acted on by the icosahedral group $[3, 5]^+$. Then $\Lambda = (1, 4, 4, 4, 2^*)$, where an asterisk indicates an asymmetric diagonal class (only 3-fold rotations permute three mutually orthogonal diameters).

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The five sets of mutually orthogonal diameters can be identified in threes, giving cosine vector $(1, -\frac{1}{4}, -\frac{1}{4}, -\frac{1}{4}, 1)$ and dimension 4.

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The layer equation and Λ-orthogonality show that the remaining cosine vectors are of the form $(1,\alpha,\beta,\gamma,-\frac{1}{2})$, with $\alpha+\beta+\gamma=0$. With dimension $15-1-4=10$ but 3 diagonal classes to account for, it follows that $d=5$ and $w^*=2$ is the only possibility. Applying the Λ-orthogonality theorem for the dimension leads to the 2-parameter family $\alpha^2+\beta^2+\gamma^2=\frac{3}{8}$.

Facet inequality

If now $\mathcal{Q}$ is the facet of a regular polytope $\mathcal{P}$ and P is a realization of $\mathcal{P}$, then we write

$$\eta_{\mathbf{f}}(P) := \langle \Lambda_{\mathcal{Q}}, \Gamma_{\mathcal{Q}}(P) \rangle / m,$$

where m is the number of vertices of $\mathcal{Q}$. In this situation, we have

Theorem

If P is a pure realization of $\mathcal{P}$ such that $\eta_{\mathbf{f}}(P) > 0$, then the dual $\mathcal{P}^{\delta}$ of $\mathcal{P}$ has a pure realization P^{δ} with the same symmetry group. Moreover, if $w(P) = 1$, then there is such a dual P^{δ} for which

$$\eta_{\mathbf{f}}(P^{\delta}) = \eta_{\mathbf{f}}(P).$$

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$$\eta_{\mathbf{f}}(P^{\delta}) = \eta_{\mathbf{f}}(P).$$

Here, the vertices of P^{δ} are the scaled centroids of the facets of P.

Dodecahedron

The dodecahedron $\{5,3\}$has layer vector $\Lambda = (1,3,6,6,3,1)$; its facet $\mathcal{Q} = \{5\}$ has induced cosine vector $\Gamma_{\mathcal{Q}} = (1,\gamma_1,\gamma_2)$. Since the dual $\{3,5\}$ has trigonal facets, each of its pure realizations has $\eta_{\boldsymbol{f}} > 0$; these give rise to pure realizations of $\{5,3\}$ of the same dimensions $1,5,3,3$.

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However, we have to find a further pure component of each of S and X, both of dimension 4. The first, identifying opposite vertices, will have cosine vector of the form $\Gamma(P) = (1,\alpha,\beta,\beta,\alpha,1)$, with $1+3\alpha+6\beta = 0$ from the layer equation. But $\eta_{\boldsymbol{f}}(P) = 0$, because P cannot give rise to a geometric dual; therefore the induced realization of the facet $\{5\}$ must be centred. Hence we also have $1+2\alpha+2\beta = 0$, from which follows

$$\Gamma(P) = (1,-\tfrac{2}{3},\tfrac{1}{6},\tfrac{1}{6},-\tfrac{2}{3},1).$$

Dodecahedron (continued)

We cannot perform the same trick for the component of X, because the layer equation tells us nothing in the centrally symmetric case, with cosine vectors of the form $(1, \alpha, \beta, -\beta, -\alpha, -1)$. Nevertheless, since the induced cosine vector for the vertex-figure $\{3\}$ is $(1, \gamma_2)$, and all diagonals of the dodecahedron are symmetric, we can apply the vertex-figure criterion, and solve

$$\alpha^2 = \tfrac{1}{3}(1 + 2\beta), \quad 1 + 2\alpha + 2\beta = 0,$$

to obtain $\alpha = -\frac{2}{3}$ or 0. We recognize the first as that of the component of S which we have already found (our calculation made no distinction between S and X), and so the second gives the cosine vector we are looking for, namely,

$$\Gamma = (1, 0, -\tfrac{1}{2}, \tfrac{1}{2}, 0, -1).$$

Dodecahedron (continued)

Our approach is not the most efficient; it is designed to illustrate various techniques. The cosine matrix of $\{5, 3\}$ is

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -\frac{2}{3} & \frac{1}{6} & \frac{1}{6} & -\frac{2}{3} & 1 \\ 1 & \frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & 1 \\ 1 & \frac{\sqrt{5}}{3} & \frac{1}{3} & -\frac{1}{3} & -\frac{\sqrt{5}}{3} & -1 \\ 1 & -\frac{\sqrt{5}}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{\sqrt{5}}{3} & -1 \\ 1 & 0 & -\frac{1}{2} & \frac{1}{2} & 0 & 1 \end{bmatrix}.$$

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The dimension vector (listing them) is $D = (1,4,5,3,3,4)$.

Final remarks

We only have time just to mention a case using the product. If Γ_5, Γ_6 are the cosine vectors of the realizations $\{3,3,5\}, \{3,3,\frac{5}{2}\}$ of the abstract 600-cell $\{3,3,5\}$, then $\Gamma_1, \Gamma_2, \Gamma_3$, given by

$$\begin{aligned} \Gamma_5^2 &= \tfrac{1}{4}\Gamma_0 + \tfrac{3}{4}\Gamma_1, \\ \Gamma_6^2 &= \tfrac{1}{4}\Gamma_0 + \tfrac{3}{4}\Gamma_2, \\ \Gamma_5\Gamma_6 &= \Gamma_3, \end{aligned}$$

are mutually Λ-orthogonal cosine vectors of realizations of $\{3,3,5\}/2$. They must be pure; their dimensions are $9, 9, 16$.

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are mutually Λ-orthogonal cosine vectors of realizations of $\{3,3,5\}/2$. They must be pure; their dimensions are $9, 9, 16$.

From the dimension equation for the small simplex realization S, the final cosine vector Γ_4 of $\{3,3,5\}/2$ is given by

$$\Gamma_0 + 9\Gamma_1 + 9\Gamma_2 + 16\Gamma_3 + 25\Gamma_4 = 60\Gamma(S).$$