Retrospective Workshop
Fields Institute
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On the number of distinct solutions generated by the simplex method for LP

Tomonari Kitahara and Shinji Mizuno

Tokyo Institute of Technology

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Contents

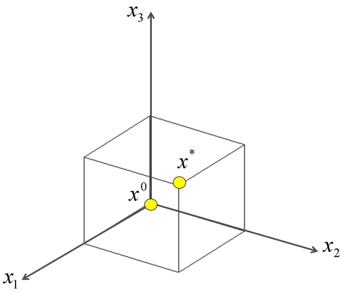
New Section

The simplex method and our results

- The simplex method for LP was originally developed by G. Dantzig in 1947.
- The simplex method needs an exponential number $\mathbf{(2^{n/2} - 1)}$ of iterations for Klee-Minty's LP.
- We get new bounds for the number of distinct solutions generated by the simplex method with Dantzig's rule and with any rule.

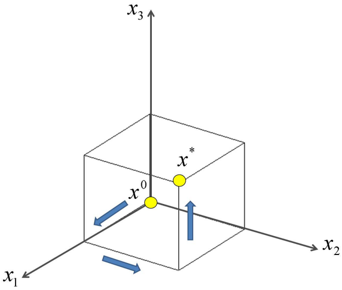
A simple example of LP on a cube

min $-(x_1 + x_2 + x_3)$, subject to $0 \leq x_1, x_2, x_3 \leq 1$



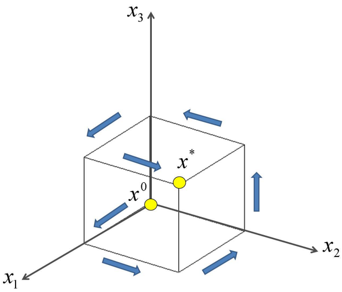
The initial point is $x^0 = (0, 0, 0)^T$ and the optimal solution is $x^* = (1, 1, 1)^T$.

The shortest path



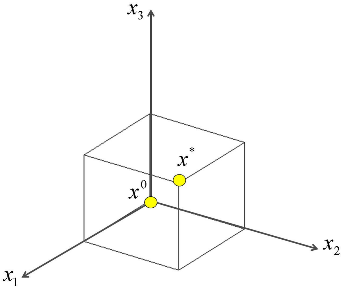
The length (number of edges) of the shortest path from $\boldsymbol{x}^{\mathbf{0}}$ to $\boldsymbol{x}^*$ is equal to the dimension $\boldsymbol{m} = \mathbf{3}$.

The longest path



The length of the shortest path is $\boldsymbol{m = 3}$.
The length of the longest path is $\mathbf{2^m - 1 = 7}$.

The simplex method on the cube

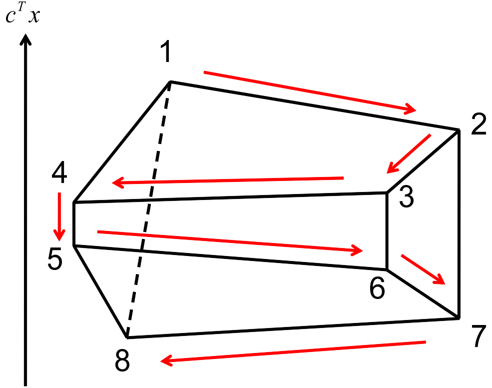


m $\leq$ the number of vertices (or BFS) generated by the simplex method $\leq$ $\mathbf{2^m - 1}$.

Klee-Minty's LP

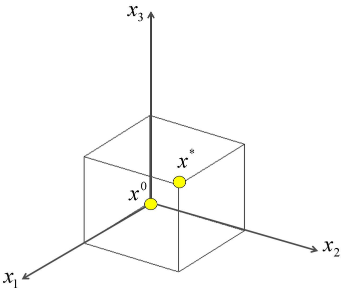
Klee and Minty showed that the simplex method generates an exponential number $(2^m - 1)$ of vertices for a special LP on a perturbed cube, where $n = 2m$.

Klee-Minty's LP (image)



Number of vertices (or BFS) generated is $\mathbf{2^m - 1 = 7}$.

The simplex method on the cube (2)



The length of any monotone path (objective value is strictly decreasing) between $\boldsymbol{x}^{\mathbf{0}}$ and $\boldsymbol{x}^*$ is at most $\boldsymbol{m}$. Hence the number of iterations of the primal simplex method is at most $\boldsymbol{m}$.

Motivation of our research

Although

if

then

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the simplex method for an LP on a perturbed cube may generates exponential number $2^m - 1$ of vertices,

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the number of vertices (BFS) generated is bounded by m.

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the simplex method for an LP on a perturbed cube may generates exponential number $\mathbf{2^{m} - 1}$ of vertices,

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the number of vertices (BFS) generated is bounded by $\boldsymbol{m}$.

Question: Is it possible to get a good upper bound for general LP, which is small for LP on the cube (but must be big for Klee-Minty's LP)?

Standard form of LP

The standard form of LP is

$$\begin{array}{ll} \min & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1, \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \\ & (x_1, x_2, \cdots, x_n)^T \geq 0. \end{array}$$

or

$$\begin{array}{ll} \min & c^T x, \\ \text{subject to} & Ax = b, \; x \geq 0 \end{array}$$

by using vectors and a matrix.

- n is the number of variables.
- m is the number of equality constraints.

Upper Bound 1

- The number of distinct BFSs (basic feasible solutions) generated by the simplex method with Dantzig's rule (the most negative pivoting rule) is bounded by

$$nm\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta}),$$

where δ and γ are the minimum and the maximum values of all the positive elements of primal BFSs.

- When the primal problem is nondegenerate, it becomes a bound for the number of iterations.
- The bound is almost tight in the sense that there exists an LP instance for which the number of iterations is $\frac{\gamma}{\delta}$ where $\frac{\gamma}{\delta} = 2^m - 1$.

Ye's result for MDP

- Our work is influenced by Ye (2010), in which he shows that the simplex method is strongly polynomial for the Markov Decision Problem (MDP).
- We extend his analysis for MDP to general LPs.
- Our results include his result for MDP.

Upper Bound 2

- The number of distinct BFSs (basic feasible solutions) generated by the primal simplex method with <u>any pivoting rule</u> is bounded by

$$m\frac{\gamma\gamma'_D}{\delta\delta'_D}$$

where δ'_D and γ'_D are the minimum and the maximum absolute values of all the negative elements of dual BFSs for primal feasible bases.

- The bound is tight in the sense that there exists an LP instance for which the number of iterations is $m\frac{\gamma\gamma'_D}{\delta\delta'_D}$.

The bounds are small for special LPs

- We can show that the upper bounds are small for some special LPs, including network problems, LP with a totally unimodular matrix, MDP, and LP on the cube.
- When $\boldsymbol{A}$ is totally unimodular and $\boldsymbol{b}$ and $\boldsymbol{c}$ are integral, the upper bounds become

$$\begin{array}{ll} nm\|\boldsymbol{b}\|_1 \log(m\|\boldsymbol{b}\|_1) & \text{(Dantzig's rule)}, \\ m\|\boldsymbol{b}\|_1 \, \|\boldsymbol{c}\|_1 & \text{(any pivoting rule)}. \end{array}$$

New Section

LP and its dual

The standard form of LP is

$$\begin{array}{ll} \textbf{min} & \boldsymbol{c}^T\boldsymbol{x}, \\ \textbf{subject to} & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \boldsymbol{x} \geq \boldsymbol{0}. \end{array}$$

The dual problem is

$$\begin{array}{ll} \textbf{max} & \boldsymbol{b}^T\boldsymbol{y}, \\ \textbf{subject to} & \boldsymbol{A}^T\boldsymbol{y} + \boldsymbol{s} = \boldsymbol{c},\ \boldsymbol{s} \geq \boldsymbol{0}. \end{array}$$

Assumptions and notations

Assume only that

- $\mathbf{rank}(A) = m$,
- the primal problem has an optimal solution,
- an initial BFS x^0 is available.

Let

- x^*: an optimal BFS of the primal problem,
- (y^*, s^*): an optimal solution of the dual problem,
- z^*: the optimal value.

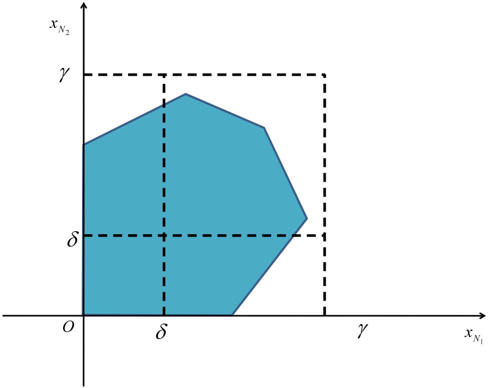
δ and γ

· Let δ and γ be the minimum and the maximum values of all the positive elements of BFSs, i. e., we have

$$\delta \leq \hat{x}_j \leq \gamma \text{ if } \hat{x}_j \neq 0$$

for any BFS $\hat{\boldsymbol{x}}$ and any $j \in \{1, 2, \ldots, n\}$.
· The values of δ and γ depend only on $\boldsymbol{A}$ and $\boldsymbol{b}$ (feasible region), but not on $\boldsymbol{c}$ (objective function).

Figure of δ, γ, and BFSs (vertices)



Pivoting

- At k-th iterate (BFS) x^k of the simplex method, if all the reduced costs are nonnegative ($\bar{c}_N \geq \mathbf{0}$), x^k is optimal.
- Otherwise we conduct a pivot. We always choose a nonbasic variable x_j whose reduced cost $\bar{c}_j$ is negative.
- Under Dantzig's rule, we choose a nonbasic variable x_j whose reduced cost is minimum, i.e.,

$$j = \arg\min_{j \in N} \bar{c}_j.$$

New Section

Constant reduction rate of the gap (1)

- Let $\{x^k\}$ be a sequence of BFSs generated by the simplex method.
- If there exists a $\lambda > 0$ such that

$$c^T x^{k+1} - z^* \le (1 - \frac{1}{\lambda})(c^T x^k - z^*)$$

whenever $x^{k+1} \ne x^k$, the number of distinct BFSs generated by the simplex method is bounded by

$$\lambda \log \frac{c^T x^0 - z^*}{c^T \bar{x} - z^*} \text{ or simply } \lambda L$$

where $\bar{x}$ is the second optimal solution, z^* is the optimal value, and L is the size of LP.

Constant reduction rate of the gap (2)

- When we use Dantzig's rule, we have

$$c^T x^{k+1} - z^* \leq \left(1 - \frac{1}{\lambda}\right)(c^T x^k - z^*)$$

for $\lambda = m\frac{\gamma}{\delta}$ whenever $x^{k+1} \neq x^k$. Hence the number of distinct BFSs is bounded by

$$m\frac{\gamma}{\delta} \log \frac{c^T x^0 - z^*}{c^T \bar{x} - z^*}.$$

- Note that the upper bound depends on c (the objective function).

Reduction of a variable

- If x^p is not optimal, there exists a (current basic) variable x_j such that $x_j^p > 0$ and

$$x_j^k \leq m\gamma \frac{c^T x^k - z^*}{c^T x^p - z^*}$$

for any (basic) feasible solution x^k.

- Suppose that we use Dantzig's rule. The value of variable x_j becomes 0 and stays 0 if we generate more than

$$M = m\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta})$$

distinct BFSs after p-th iterate.

Number of BFSs (Dantzig's rule)

The number of distinct BFSs generated by the simplex method with Dantzig's rule is bounded by

$$nM = nm\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta}).$$

Constant reduction of the objective function (1)

- Let $\{\boldsymbol{x}^k\}$ be a sequence of BFSs generated by the simplex method.
- If there exists a constant $K > 0$ such that

$$\boldsymbol{c}^T\boldsymbol{x}^k - \boldsymbol{c}^T\boldsymbol{x}^{k+1} \geq K$$

whenever $\boldsymbol{x}^{k+1} \neq \boldsymbol{x}^k$, the number of distinct BFSs generated by the simplex method is bounded by

$$\frac{\boldsymbol{c}^T\boldsymbol{x}^0 - z^*}{K}.$$

Number of BFSs (any rule)

- For any pivoting rule, we have that

$$\boldsymbol{c}^T\boldsymbol{x}^k - \boldsymbol{c}^T\boldsymbol{x}^{k+1} \geq \delta\delta'_D$$

whenever $\boldsymbol{x}^{k+1} \neq \boldsymbol{x}^k$. We also see that

$$\boldsymbol{c}^T\boldsymbol{x}^0 - z^* \leq m\gamma\gamma'_D.$$

(Here δ'_D and γ'_D are the minimum and the maximum absolute values of all the negative elements of dual BFSs for primal feasible bases.)

- Hence the number of distinct BFSs is bounded by

$$m\frac{\gamma\gamma'_D}{\delta\delta'_D}.$$

New Section

0-1 vertices

Assume that all the elements of BFSs (such as an assignment problem) is **0** or **1**, that is, $\delta = \gamma = 1$. Then the number of distinct BFSs generated by the simplex method with Dantzig's rule is bounded by

$$nm \log m.$$

Shortest path problem

$$
\begin{aligned}
\min \quad & \textstyle\sum_{(i,j)\in E} c_{ij} x_{ij}, \\
\text{s.t.} \quad & \textstyle\sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ij} = \begin{cases} |V| - 1 & \text{for source} \\ -1 & \text{other nodes} \end{cases} \\
& x \geq 0.
\end{aligned}
$$

Since the shortest path problem is nondegenerate, $n = |E|$, $m = |V|$, $\gamma \leq |V| - 1$, and $\delta \geq 1$, the number of iterations of the simplex method with Dantzig's rule is bounded by

$$|E||V|^2 \log |V|^2.$$

Minimum cost flow problem

$$\begin{array}{ll} \min & \sum_{(i,j)\in E} c_{ij}x_{ij}, \\ \text{s.t.} & \sum_{j:(i,j)\in E} x_{ij} - \sum_{j:(j,i)\in E} x_{ij} = b_i \text{ for } i \in V \\ & 0 \le x \le u. \end{array}$$

Assume that the capacities u_{ij} and the supplies b_i are integral. Since $n = |E|$, $m = |V|$, $\gamma \le U = \max_{(i,j)\in E} u_{ij}$, and $\delta \ge 1$, the number of distinct solutions generated by the simplex method with Dantzig's rule is bounded by

$$|E|^2 U \log |E| U.$$

Minimum cost flow problem (continue)

It is known that if we perturb the minimum cost flow problem by adding $-(|V|-1)/|V|$ to b_i for the root node and $1/|V|$ for the other nodes, then the problem is nondegenerate and we can solve the original problem by solving this perturbed problem. Hence the number of iterations of the simplex method with Dantzig's rule for solving a minimum cost flow problem is bounded by

$$|E|^2|V|U \log |E||V|U.$$

LP with a totally unimodular matrix

When a constraint matrix $\boldsymbol{A}$ is totally unimodular and constant vectors $\boldsymbol{b}$ and $\boldsymbol{c}$ are integral, the number of distinct solutions generated by the simplex method is at most

$$nm||\boldsymbol{b}||_1 \log(m||\boldsymbol{b}||_1)$$

for Dantzig's rule and

$$m||\boldsymbol{b}||_1 \, ||\boldsymbol{c}||_1$$

for any pivoting rule.

MDP

- The Markov Decision Problem (MDP):

$$\begin{array}{ll} \min & c_1^T x_1 + c_2^T x_2, \\ \text{subject to} & (I - \theta P_1)x_1 + (I - \theta P_2)x_2 = e, \\ & x_1, x_2 \geq 0. \end{array}$$

- (Y. Ye) The simplex method with Dantzig's rule for solving MDP finds an optimal solution in at most

$$n \frac{m^2}{1-\theta} \log \frac{m^2}{1-\theta}$$

iterations, where $n = 2m$.

New Section

Outline of this section

- Klee-Minty's LP requires an exponential number of iterations $(2^m - 1)$ by Dantzig's simplex method. Therefore the ratio γ/δ for Klee-Minty's LP must be big. In fact, it is about 100^m.
- We construct a variant of Klee-Minty's LP, for which the number of iterations (Dantzig's rule) is equal to $\frac{\gamma}{\delta}$ where $\frac{\gamma}{\delta} = 2^m - 1$.
- We also present a simple LP on a cube for which the number of iterations (any rule) is equal to $m\frac{\gamma\gamma'_D}{\delta\delta'_D}$.

A variant of Klee-Minty's LP

- The variant of Klee-Minty's LP is represented as

$$\begin{array}{ll} \max & \sum_{i=1}^{m} x_i, \\ \text{s. t.} & 2\sum_{i=1}^{k-1} x_i + x_k \leq 2^k - 1 \ (k = 1, 2, \cdots, m), \\ & x \geq 0. \end{array}$$

(Only b has exponential size).

- The standard form is

$$\begin{array}{ll} \max & \sum_{i=1}^{m} x_i, \\ \text{s. t.} & 2\sum_{i=1}^{k-1} x_i + x_k + y_k = 2^k - 1 \ (k = 1, 2, \cdots, m), \\ & x \geq 0, \ y \geq 0. \end{array}$$

Properties of the variant

The variant has the following properties

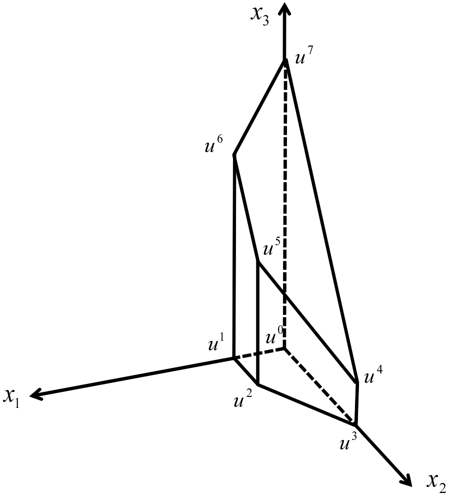
- for each $i \in \{1, 2, \cdots, m\}$ at any BFS, exactly one of x_i and y_i is a basic variable,
- the problem has 2^m BFSs,
- each component of any BFS is an integer,
- the problem is nondegenerate,
- The optimal BFS is $x^* = (0, 0, \cdots, 0, 2^m - 1)^T$, $y^* = (1, 2^2 - 1, \cdots, 2^{m-1} - 1, 0)^T$, and the optimal value is $(2^m - 1)$.
- $\delta = 1$ and $\gamma = (2^m - 1)$.

Properties of the variant (2)

When we generate a sequence of BFSs by Dantzig's simplex method for the variant from an initial BFS where $\boldsymbol{x} = \mathbf{0}$,

- any reduced cost of every dictionary is $\mathbf{1}$ or $\mathbf{-1}$, which implies $\delta'_{\boldsymbol{D}} = \mathbf{1}$ and $\gamma'_{\boldsymbol{D}} = \mathbf{1}$,
- the number of iterations is $(\mathbf{2}^{\boldsymbol{m}} - \mathbf{1})$, which is equal to $\frac{\gamma}{\delta}$ and $\frac{\gamma\gamma'_{\boldsymbol{D}}}{\delta\delta'_{\boldsymbol{D}}}$,
- the objective function value increases by $\mathbf{1}$ at each iteration, so there exists exactly one BFS whose objective function value is $\boldsymbol{k}$ for each integer $\boldsymbol{k} \in [\mathbf{0}, \mathbf{2}^{\boldsymbol{m}} - \mathbf{1}]$.

Vertices generated by Dantzig's simplex method ($\boldsymbol{m = 3}$)



An LP on a cube

- The standard form of LP on a cube

$$\begin{array}{ll} \max & \sum_{i=1}^{m} x_i, \\ \text{s. t.} & x_k + y_k = 1, x_k \geq 0, y_k \geq 0 \ (k = 1, 2, \cdots, m). \end{array}$$

- We see that

$$\delta = \gamma = 1 \text{ and } \delta'_D = \gamma'_D = 1.$$

- When the initial solution is $x = 0$, the number of iterations is exactly m, which is equal to

$$m \frac{\gamma \gamma'_D}{\delta \delta'_D}.$$

New Section

Problems, Pivoting, and Assumptions

- Problems:
  - The standard form of LP and its dual.
- Pivoting:
  - Dantzig's rule
  - Any rule which chooses a nonbasic variable whose reduced cost is negative.
- Assumptions:
  - $\mathbf{rank}(A) = m$.
  - The primal problem has an optimal solution.
  - An initial BFS is available.

Results

1. The number of BFSs is bounded by

$$nm\frac{\gamma}{\delta}\log(m\frac{\gamma}{\delta}) \text{ or } m\frac{\gamma\gamma'_D}{\delta\delta'_D}.$$

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2. Totally unimodular case:

$$nm\|b\|_1\log\left(m\|b\|_1\right) \text{ or } m\|b\|_1\|c\|_1.$$

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2. Totally unimodular case:

$$nm||b||_1\log(m||b||_1) \text{ or } m||b||_1||c||_1.$$

3. There exists an LP (a variant of Klee-Minty's LP) for which the number of iterations is $\frac{\gamma}{\delta}$ and $\frac{\gamma\gamma'_D}{\delta\delta'_D}$ where $\frac{\gamma}{\delta} = 2^m - 1$ and $\frac{\gamma'_D}{\delta'_D} = 1$.

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4. There exists an LP (on a cube) for which the number of iterations is $m\frac{\gamma}{\delta}$ and $m\frac{\gamma\gamma'_D}{\delta\delta'_D}$.

Announcement

ICCOPT V 2016 TOKYO

(The 5th International Conference on Continuous Optimization of the Mathematical Optimization Society)

Place: Roppongi, Tokyo, JAPAN

Dates: Aug. 6 (Sat) - 11 (Thu), 2016

Venue: National Graduate Institute for Policy Studies (GRIPS)