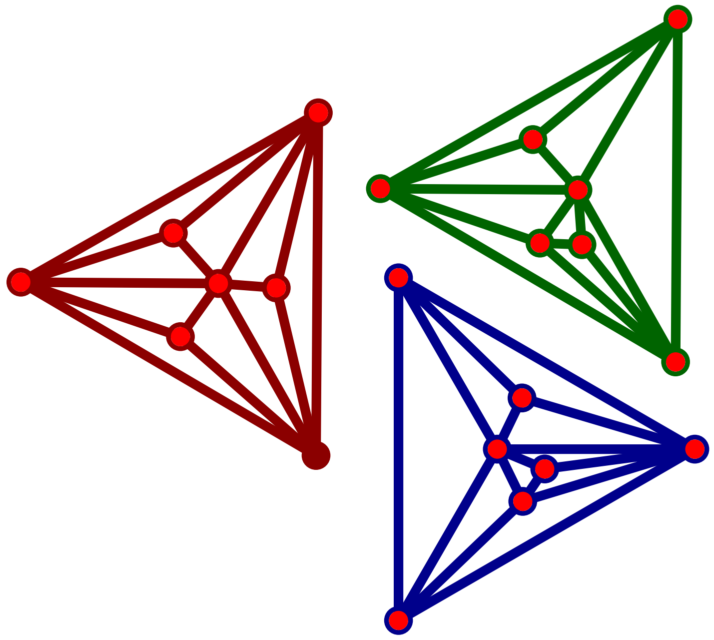
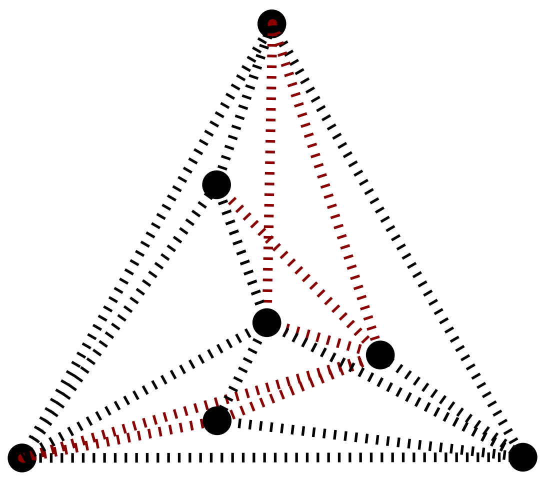
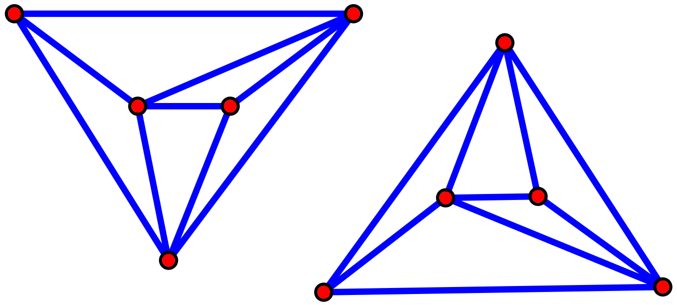
Universality in Geometric Graph Theory

Csaba D. Tóth
Cal State Northridge
University of Calgary
Tufts University





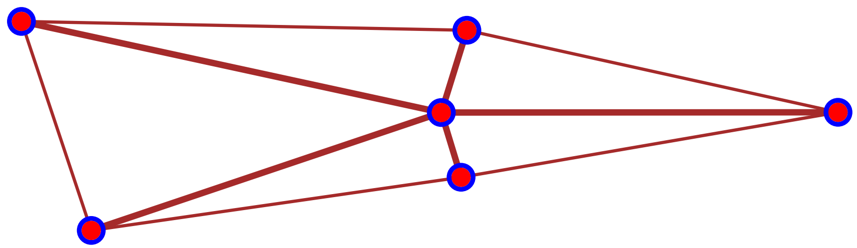
Outline

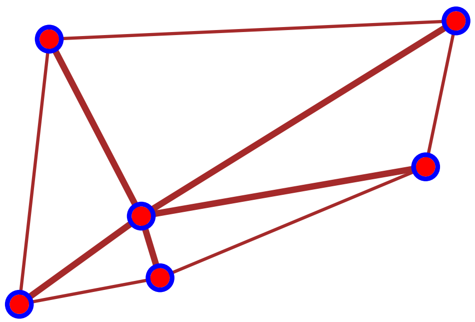


- Introduction: Geometric Graphs
- Counting Problems on n Points
  - Labeled Plane Graphs
  - Unlabeled Plane Graphs
- Universality
  - Configurations Compatible with Many Graphs

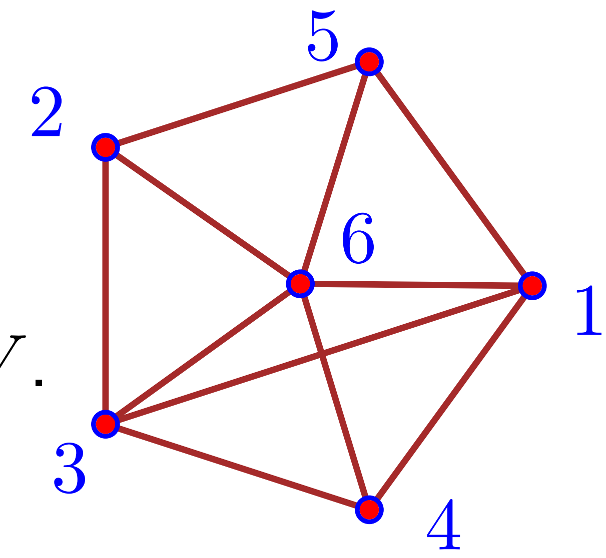
    Universal Point Sets, Universal Slope Sets, etc.
  - Graphs Compatible with Many Parameters

    Globally Rigid Graphs, Length Universal Graphs, etc.
- Open Problems





Geometric Graphs



A *geometric graph* is $G = (V, E)$,
V =set of points in the plane,
E =set of line segments between points in V.

Applications:

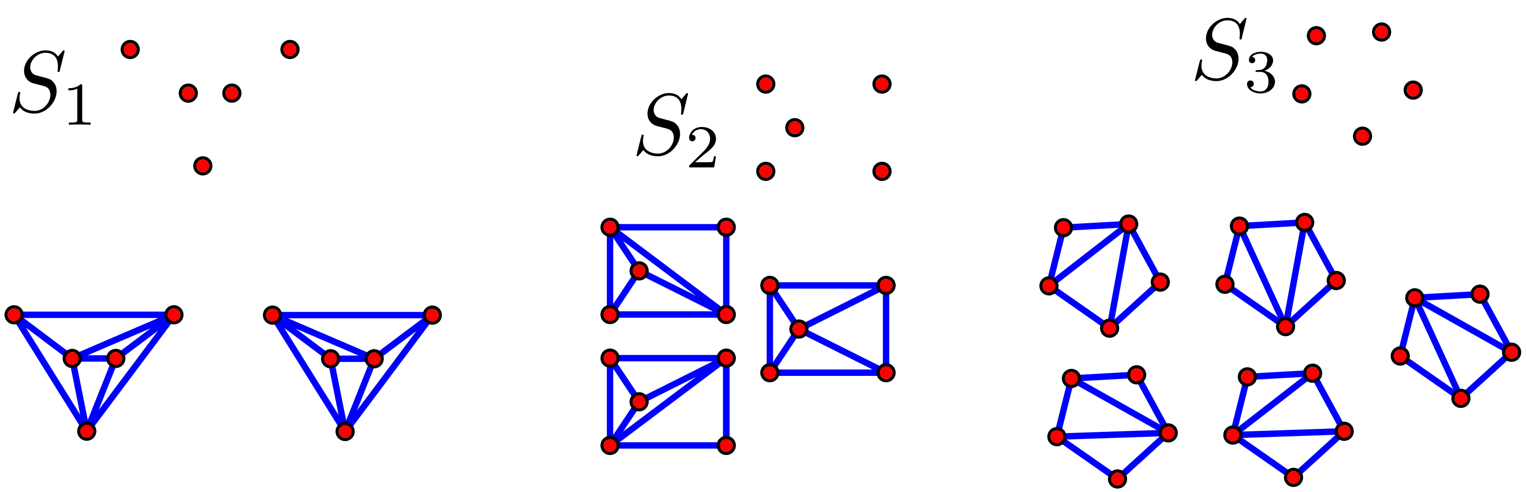
- Cartography (GIS, Navigation, etc.)
- Networks (VLSI Design, Optimization, etc.)
- Combinatorial Geometry (Incidences, Unit Distances, etc.)
- Rigidity (Robot arms, etc.)

Counting labeled plane graphs

Giménez and Noy (2009): The asymptotic number of (labeled) planar graphs on n vertices is $g \cdot n^{-7/2} \gamma^n n!$ where $\gamma \approx 27.22688$ and $g \approx 4.26 \cdot 10^{-6}$.

Fáry (1957): Every planar graph has an embedding in the plane as a geometric graph.

Ajtai, Chvátal, Newborn, & Szemerédi (1982): On any n points in $\mathbb{R}^2$, at most c^n labeled planar graphs can be embedded, where $c < 10^{13}$. Hoffmann et al. (2010): $c < 207.85$.



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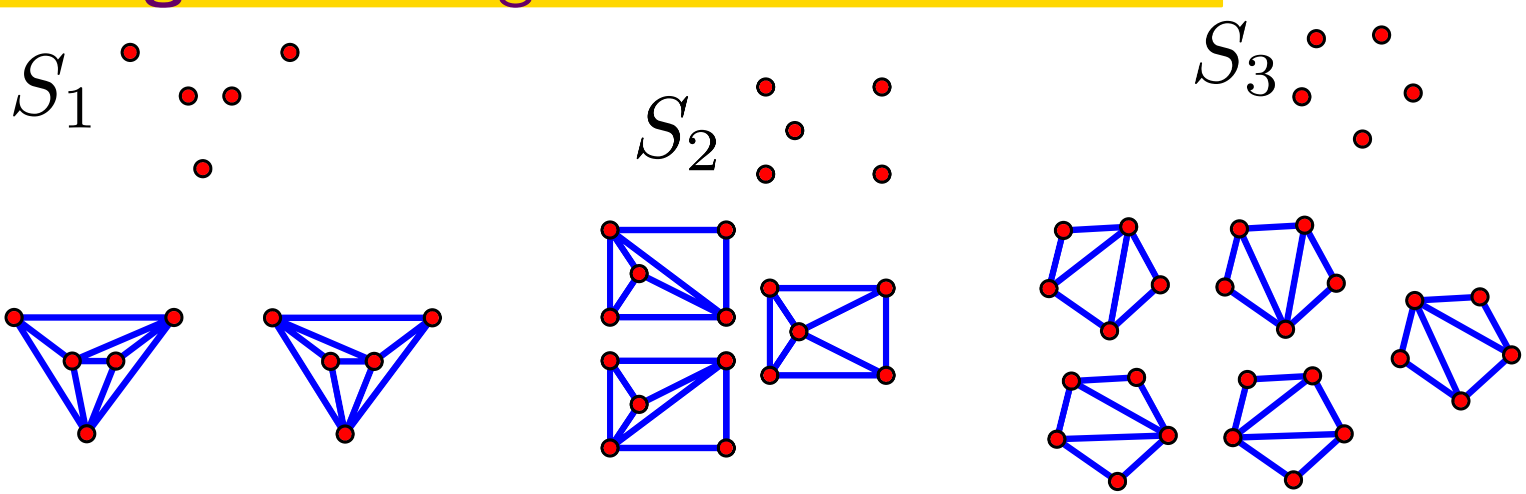
Requiring **straight-line** edges is a real restriction.



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- Allow the edges to bend
- Allow graph isomorphisms

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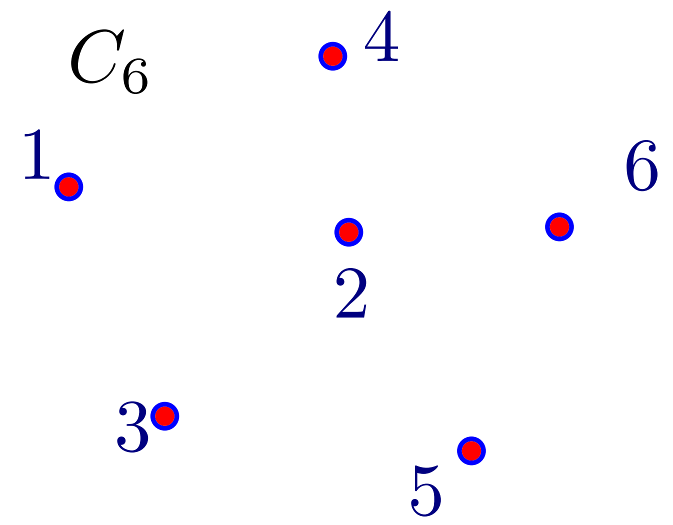

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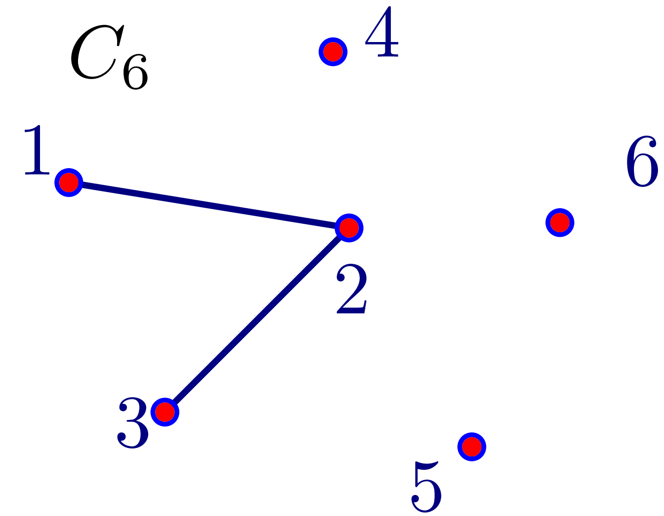

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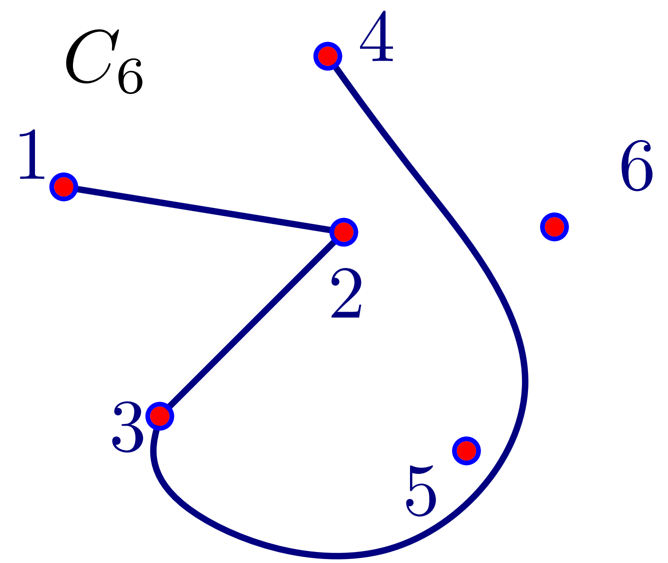

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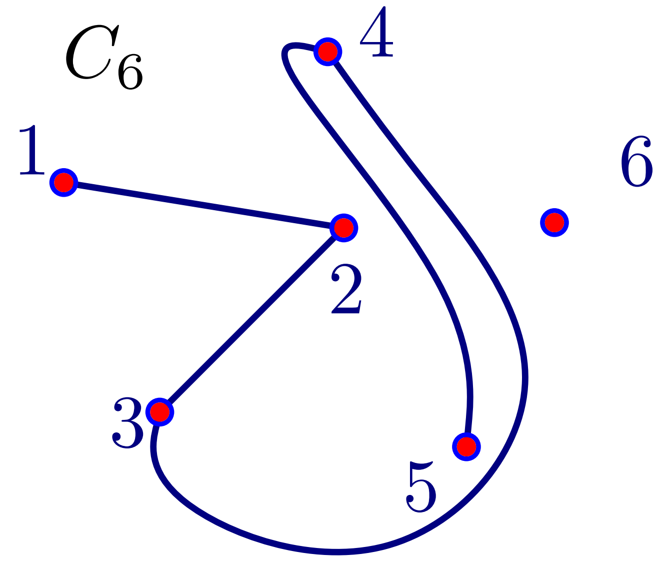

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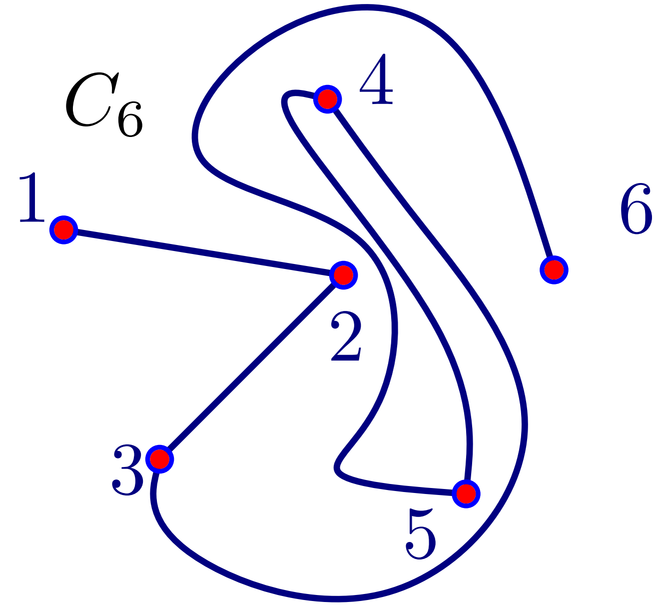

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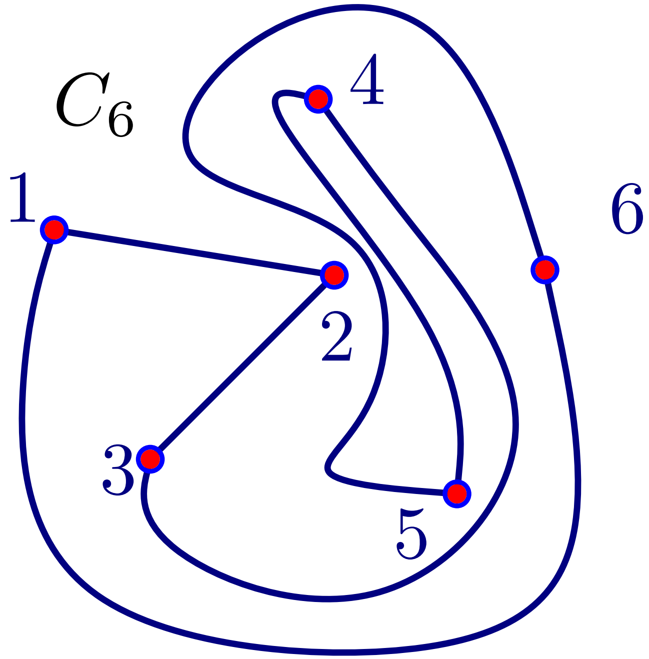

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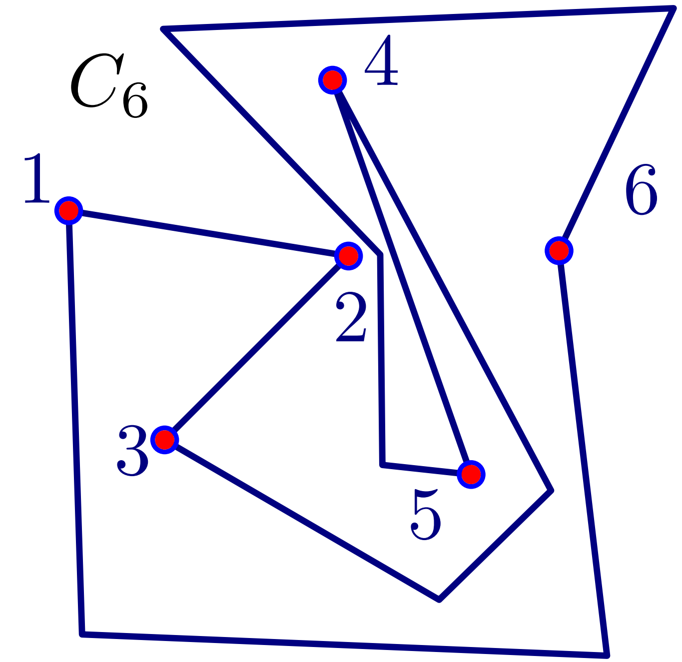

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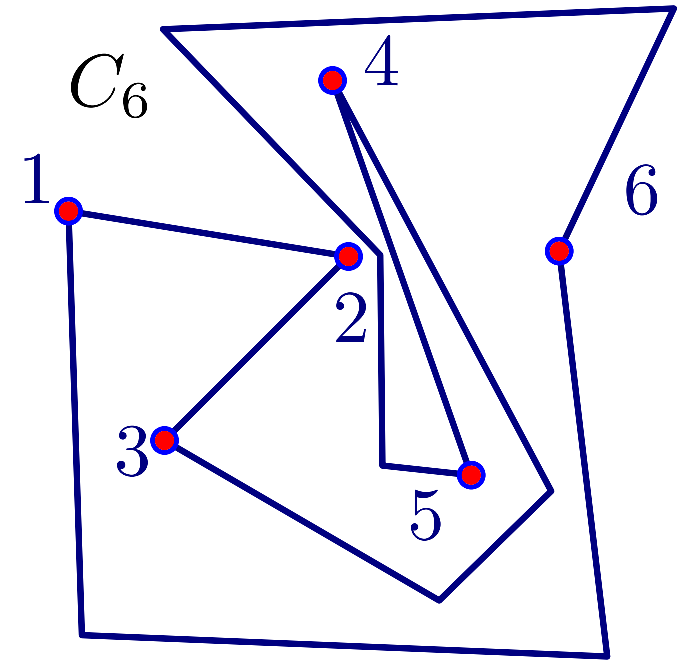

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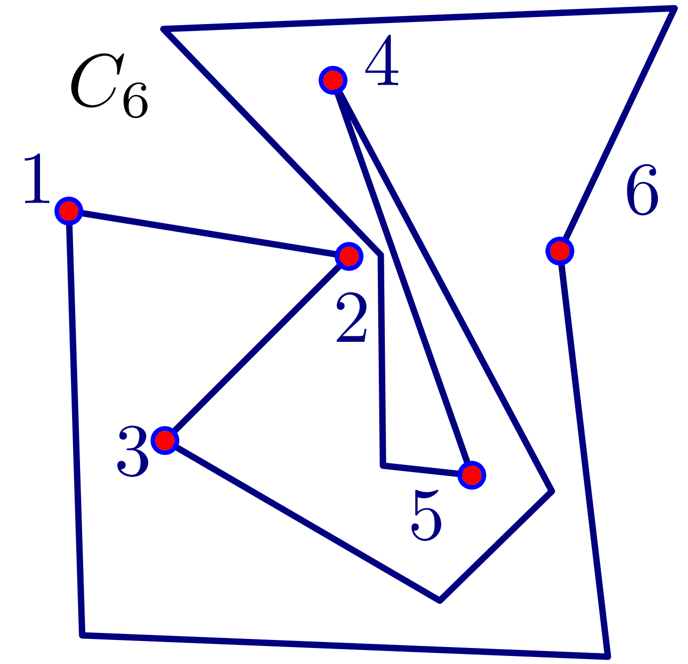

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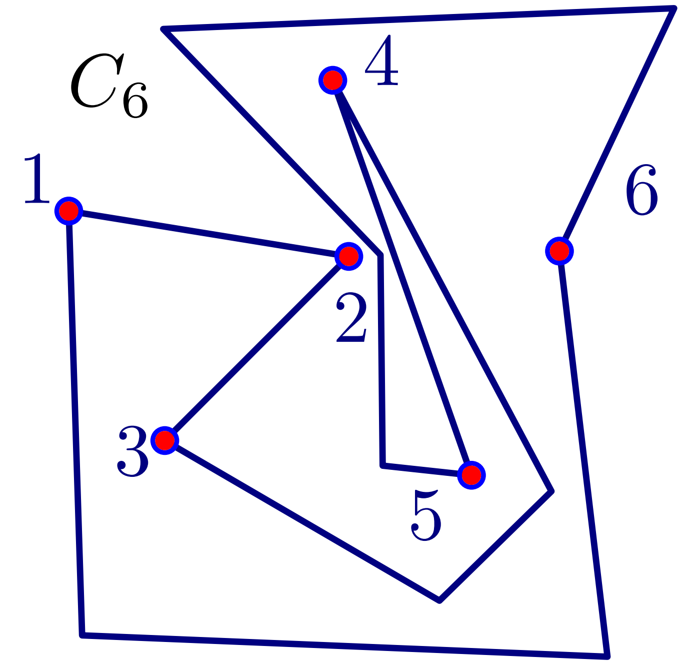

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Problem: Can this bound be improved to $2^{O(n \log k)}$?

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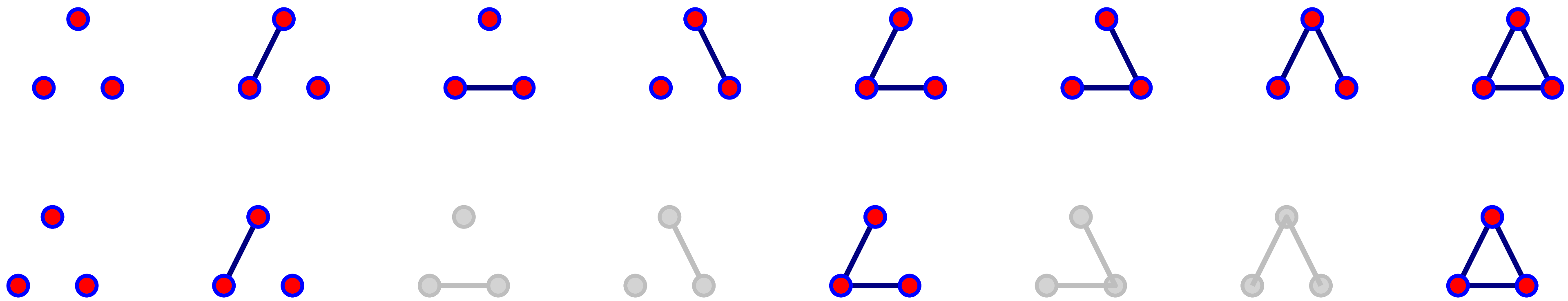

Every n-vertex planar graph has a straight line embedding, but not all of them can be embedded on an arbitrary set of n points.

- C_4 can be embedded on any 4 points in the plane.
- K_4 cannot be embedded on 4 points in convex position.

Counting unlabeled planar geometric graphs

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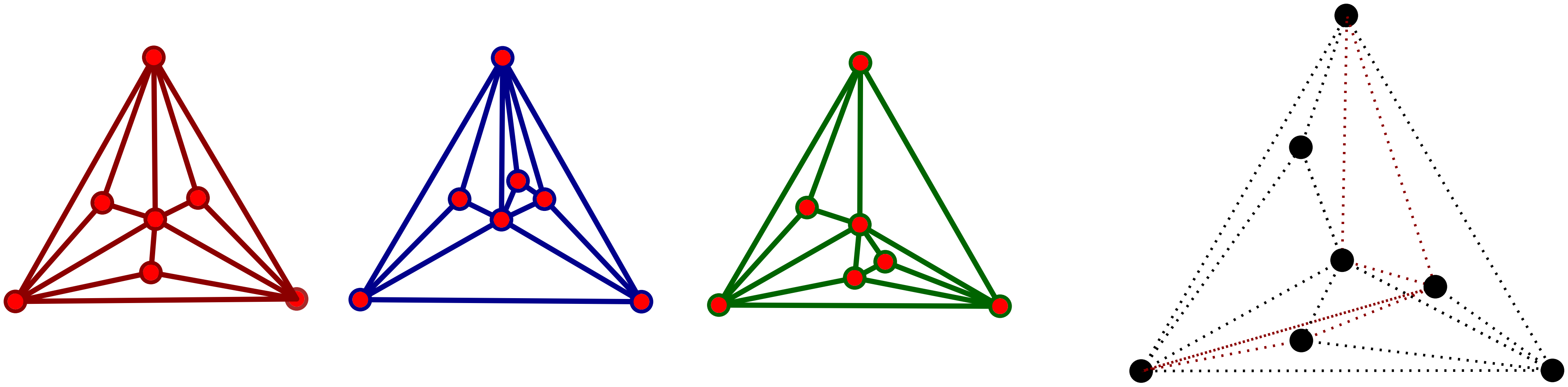

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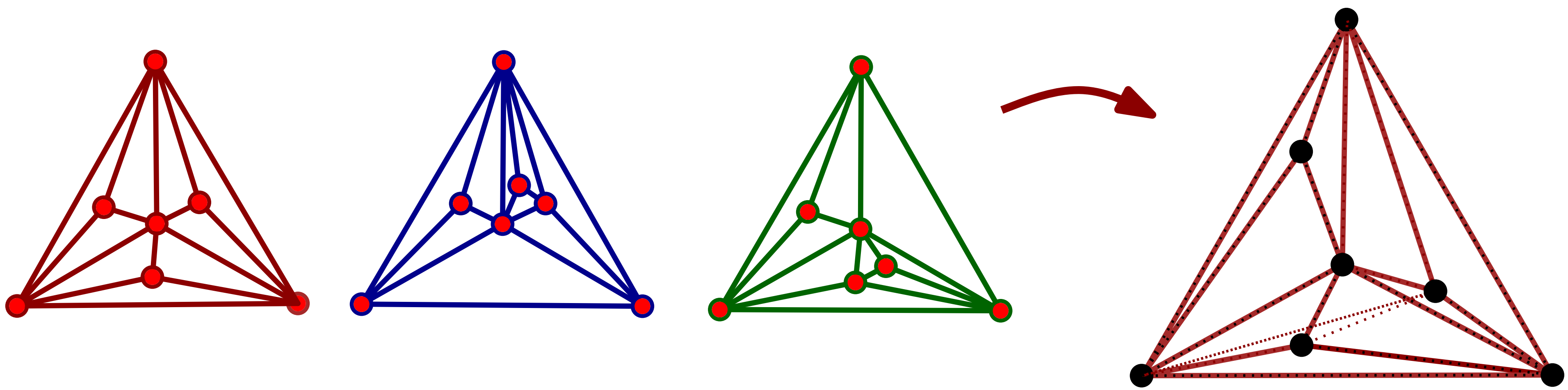

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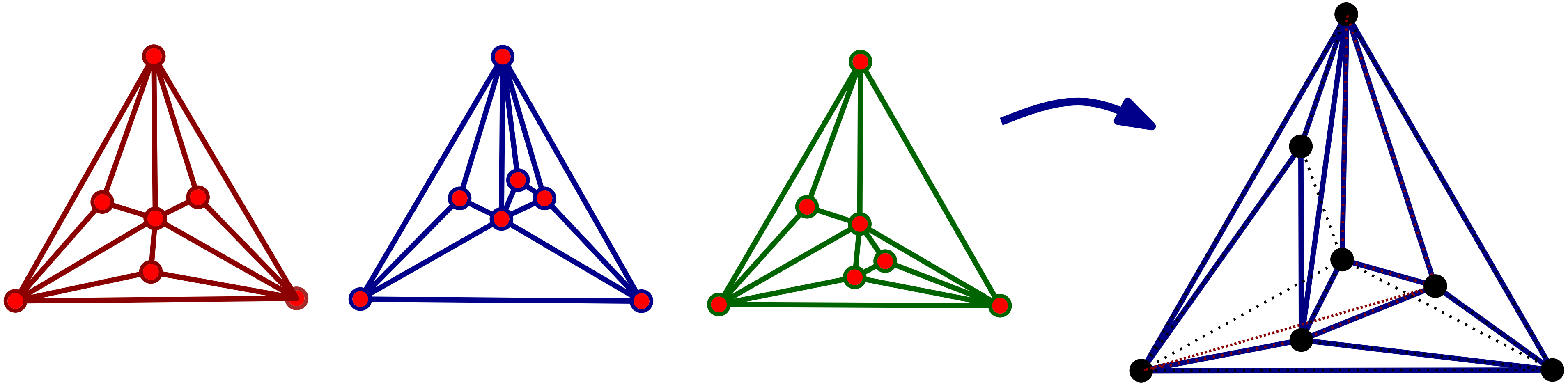

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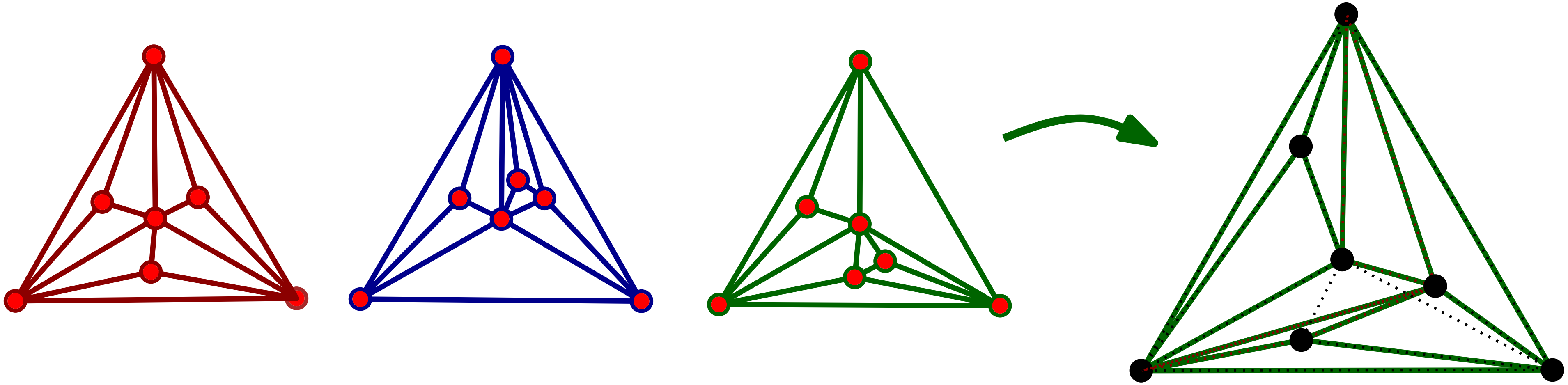

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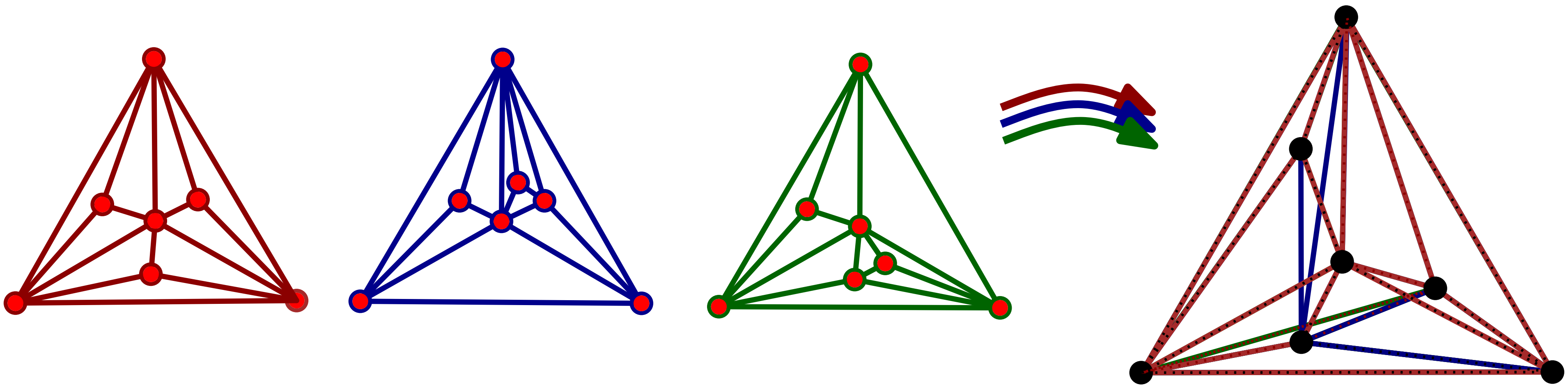

A point set $S \subset \mathbb{R}^2$ is n-**universal** if every n-vertex planar graph has an embedding such that the vertices map into S.

Cardinal, Hoffmann, & Kusters (2013):

- For $n = 1, \ldots, 10$, there is an n-element point set that can host all n-vertex planar graphs (by exhaustive search).
- For $n \geq 15$, there is no n-element point set that can accommodate all n-vertex planar graphs (by counting argument).

Universal point sets

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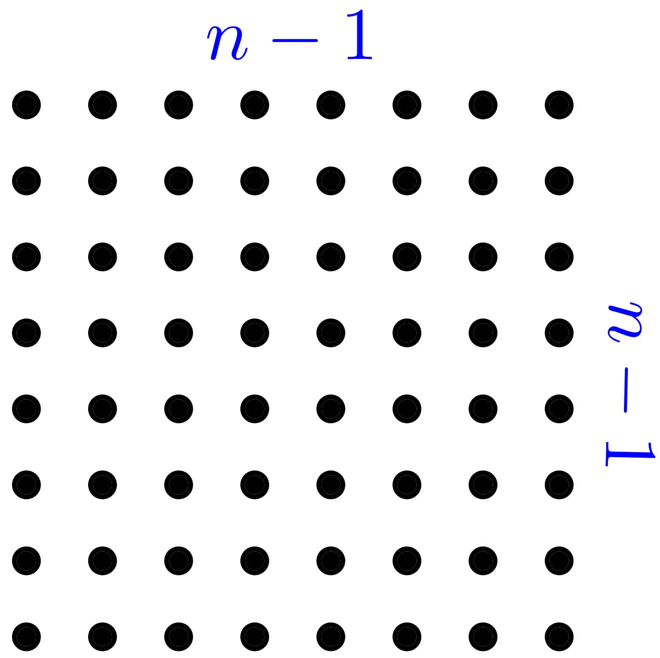
De Fraysseix, Pach, & Pollack (1990) and Schnyder (1990):
An $(n-1) \times (n-1)$ section of the integer lattice is n-universal.

Methods:

- partial orders defined on the vertices
- three Schnyder trees (Schnyder wood)

One method is an incremental algorithm, the other embedding all vertices at once. They have turned out to be equivalent...



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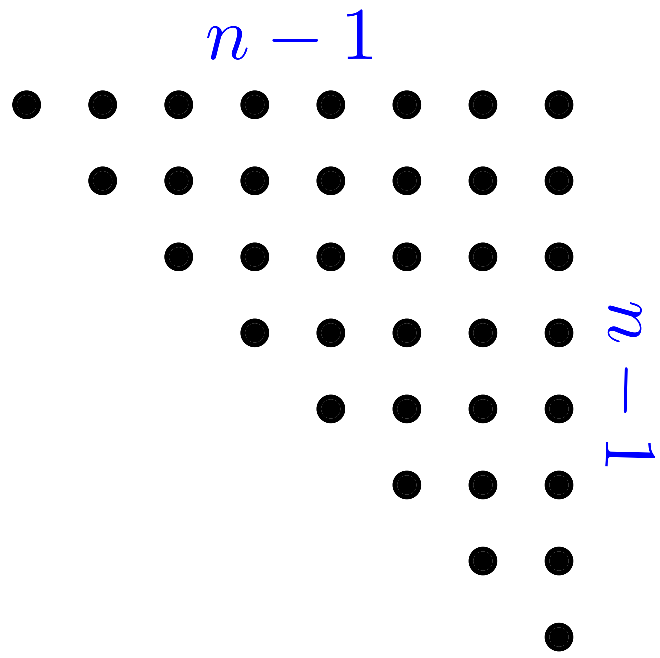
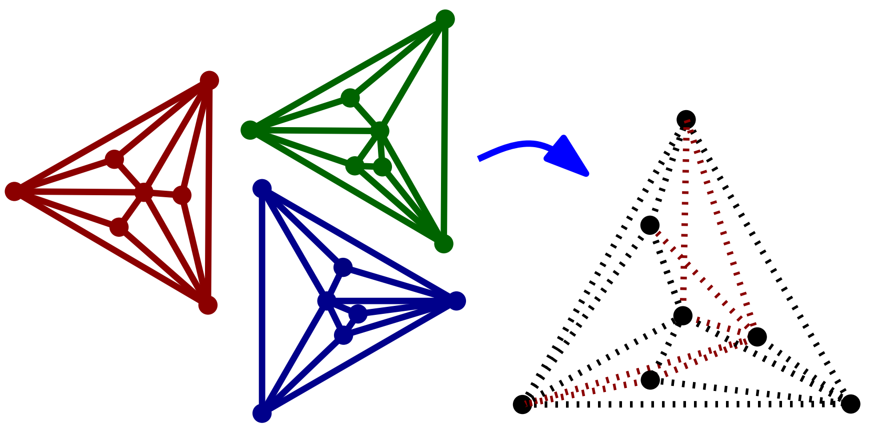


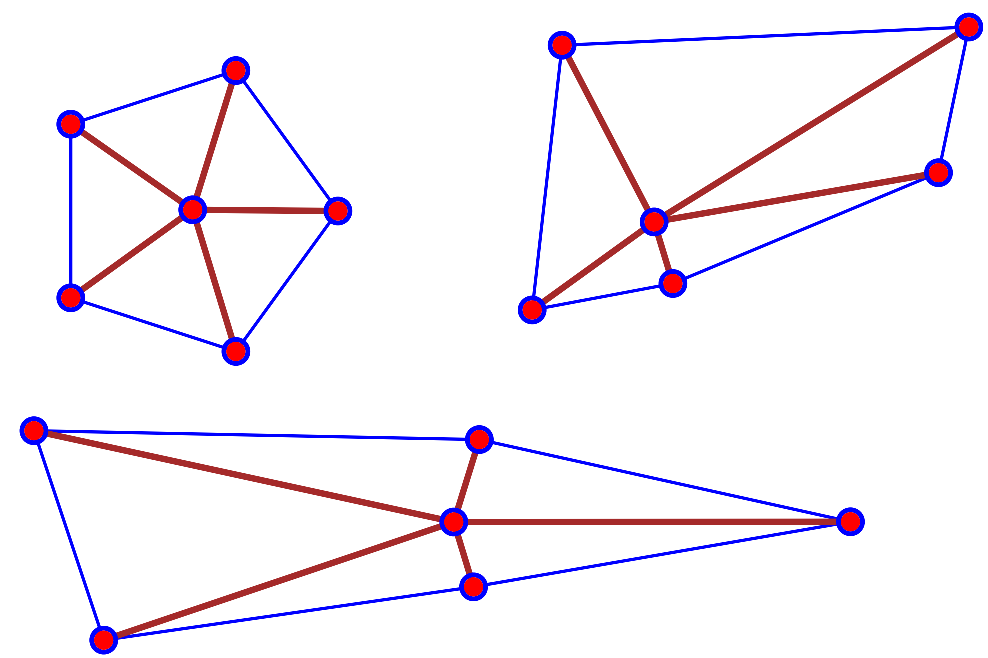
$\frac{n^2}{2}$ points suffice if we do not insist on a rectangular lattice.

Universality in Geometric Graphs

1. A structute is **universal** if it is “compatible” with every geometric graph from a certain family (e.g., universal point sets, universal slopes, etc.)

2. An abstract graph is **universal** if it has a geometric realization for any possible choice of certain parameters (e.g., globally rigid graphs, length-universal graphs, area universal floorplans).





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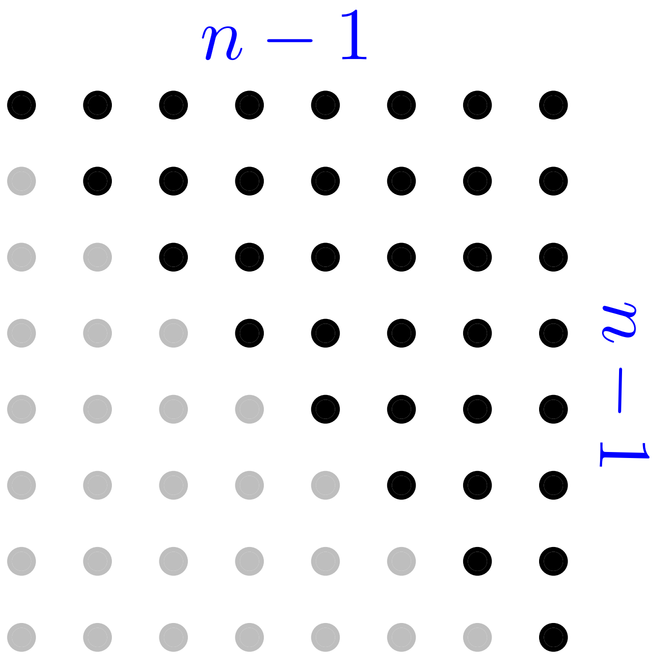

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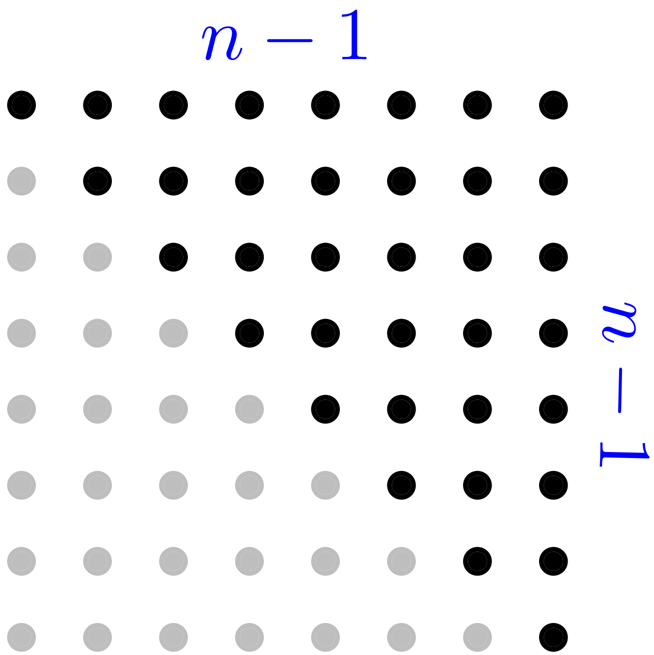

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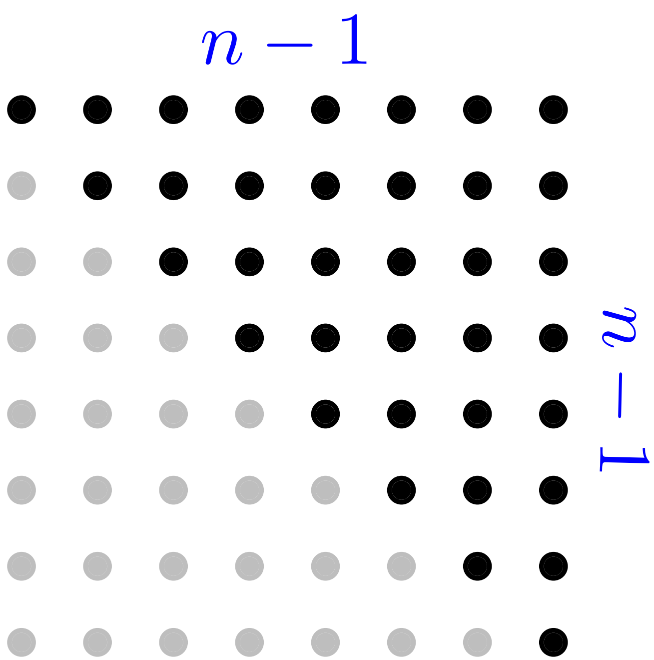

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Open Problem: Find n-universal point sets of size $o(n^2)$.

Universal point sets

Bannister et al. (2013) there is an n-universal point set of size $n^2/4 + \Theta(n)$ for all $n \in \mathbb{N}$. (not a lattice section)

Kurowski (2004): The size of an n-univeral set is at least $1.235n - o(n)$.

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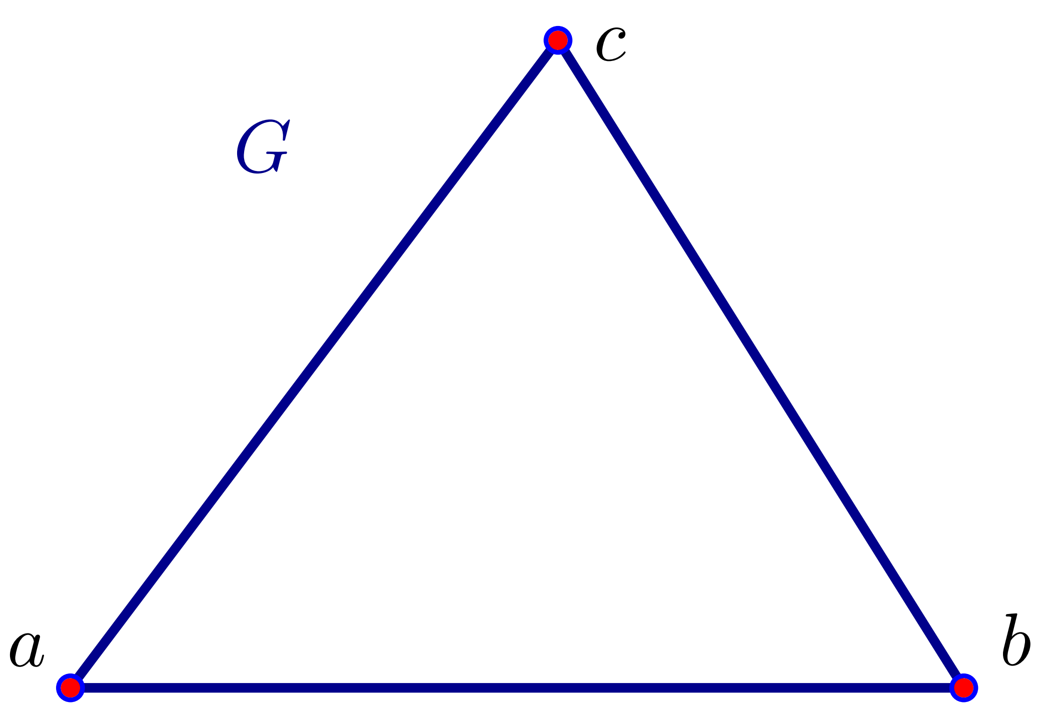

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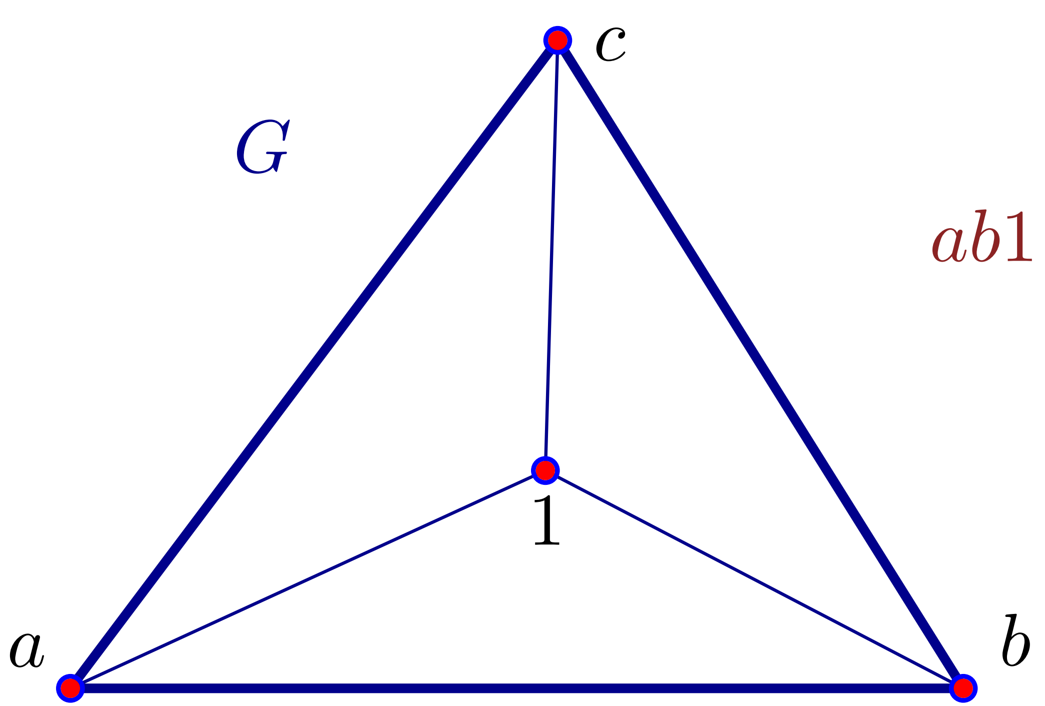

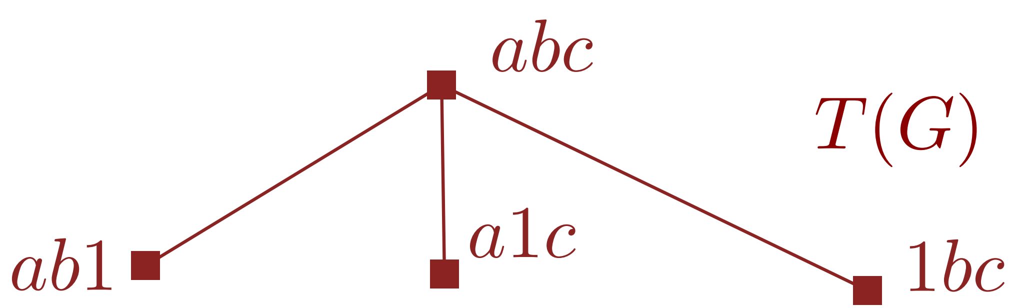

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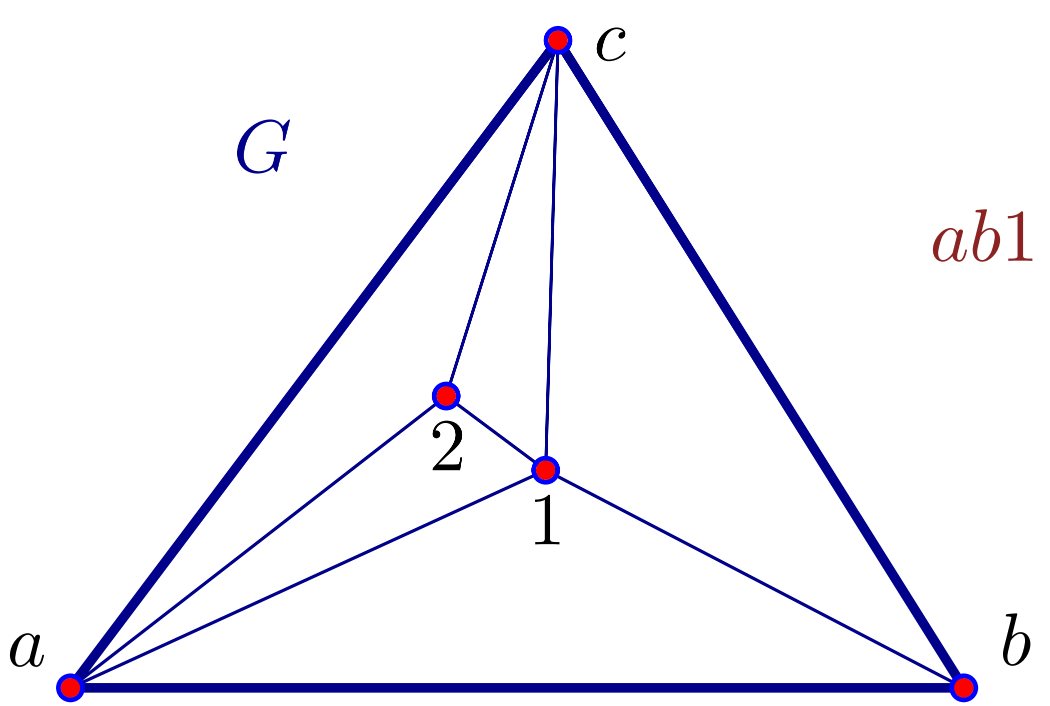

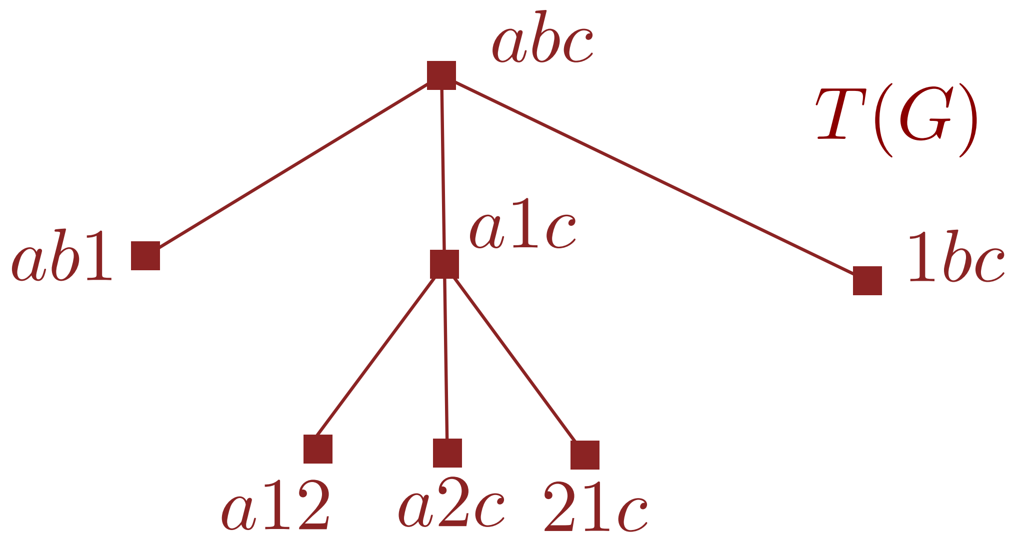

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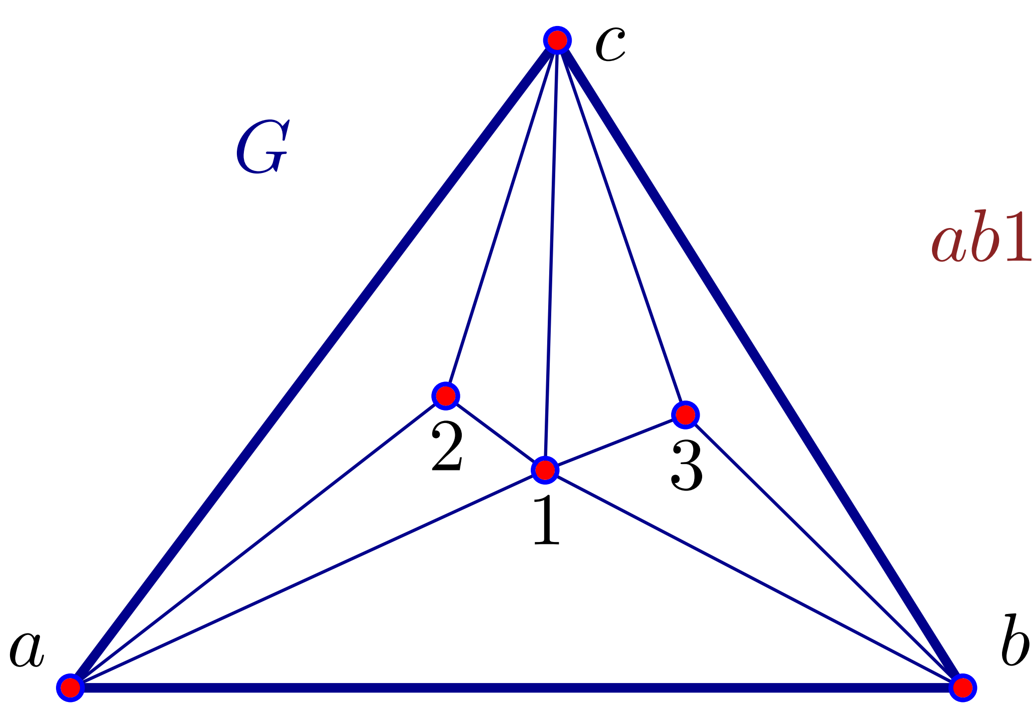

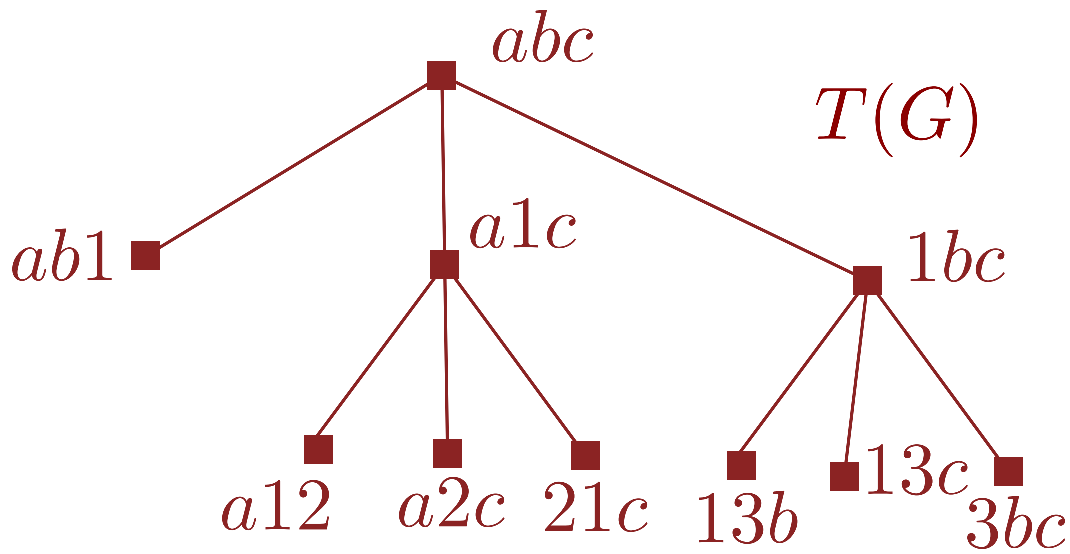

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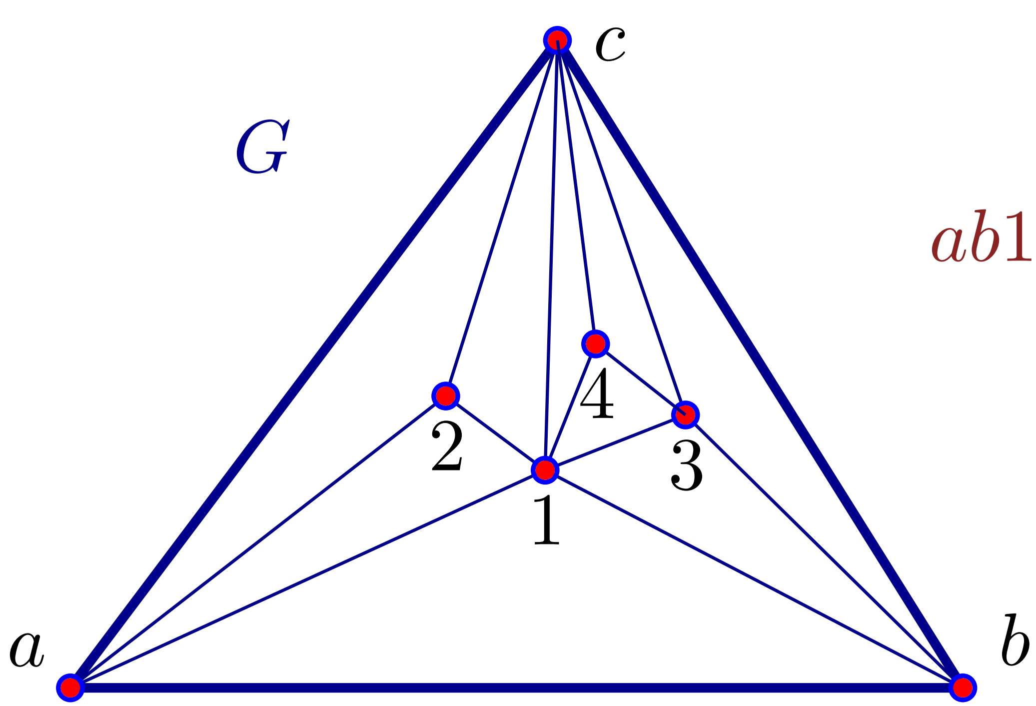
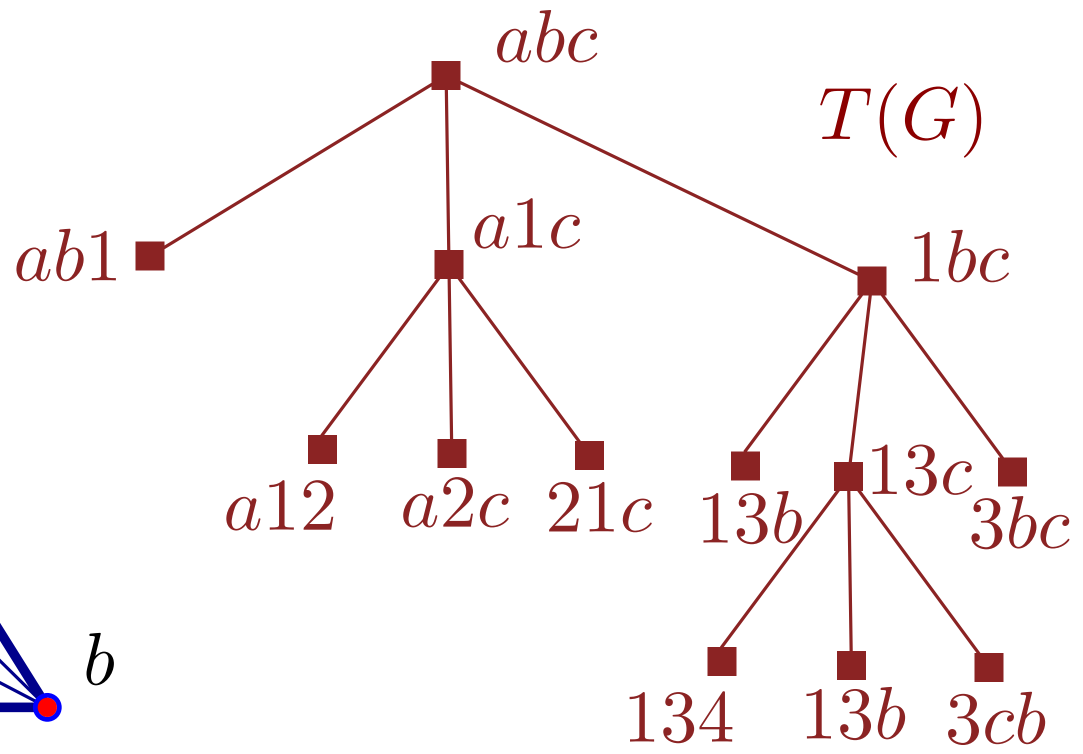
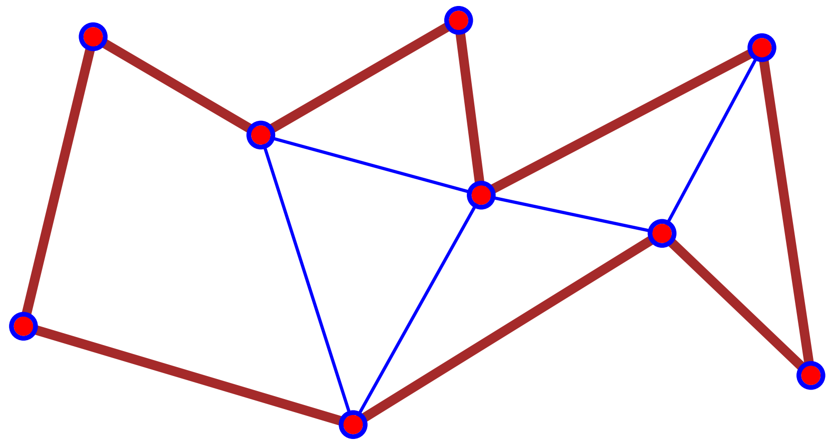
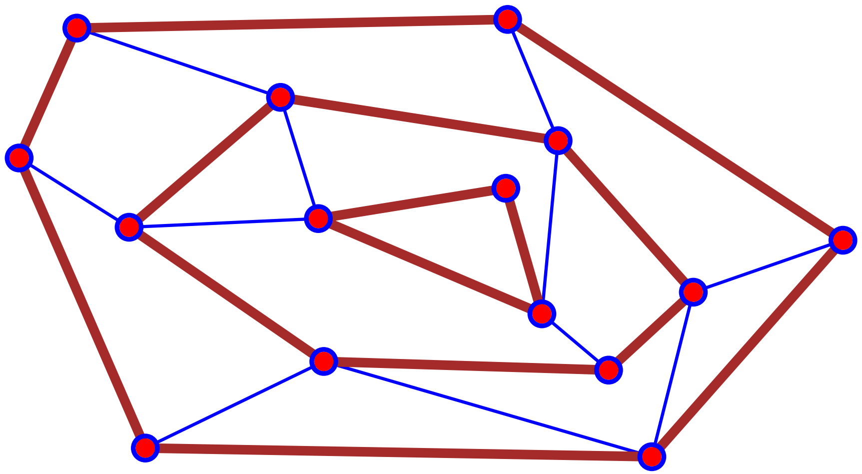




Universal point sets in special classes

Gritzman et al. (1991): Every n-element point set in general position is n-universal **for outerplanar graphs**





Angelini et al (2011): There is an n-univrsal point set of size $O(n(\log n / \log\log n)^2)$ **for simply nested planar graphs**.

Bannister et al. (2013): There is an n-universal point set of size $O(n \log n)$ **for simply nested planar graphs**, and of size $O(n \operatorname{polylog} n)$ **for planar graphs of bounded pathwidth.**

Thm. (2013): There is an n-universal point set of size $O(n^{3/2} \log n)$ **for planar 3-trees.**

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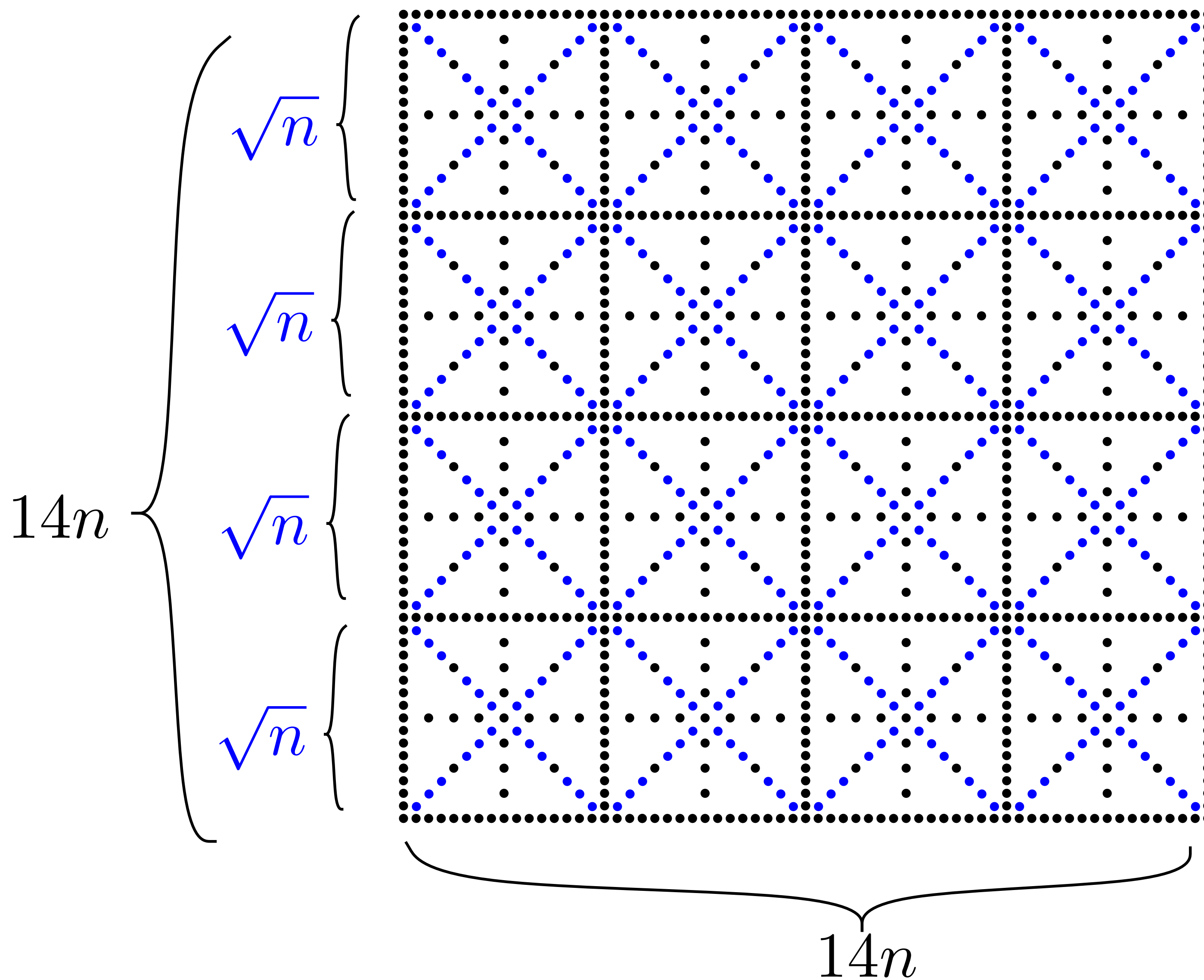

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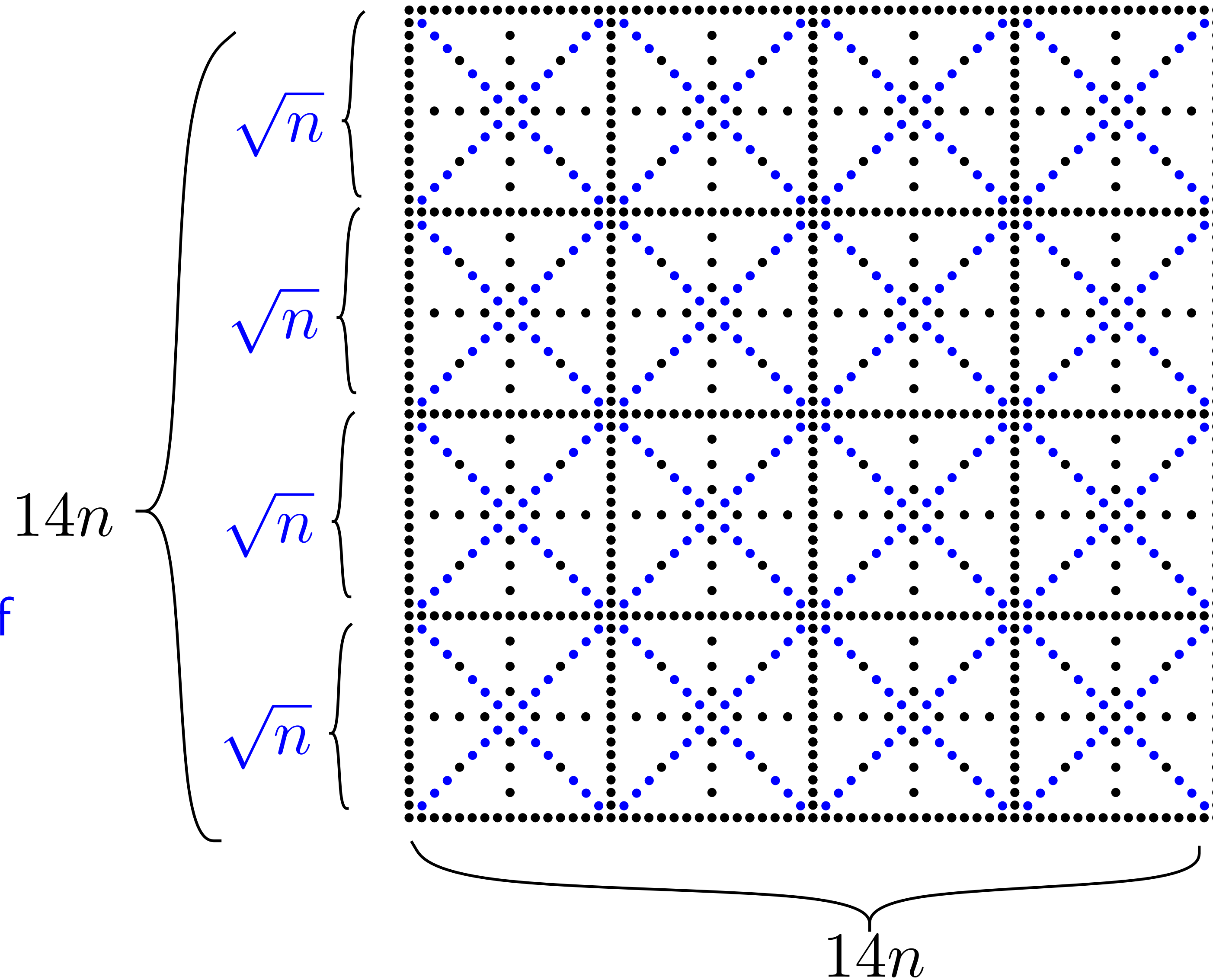

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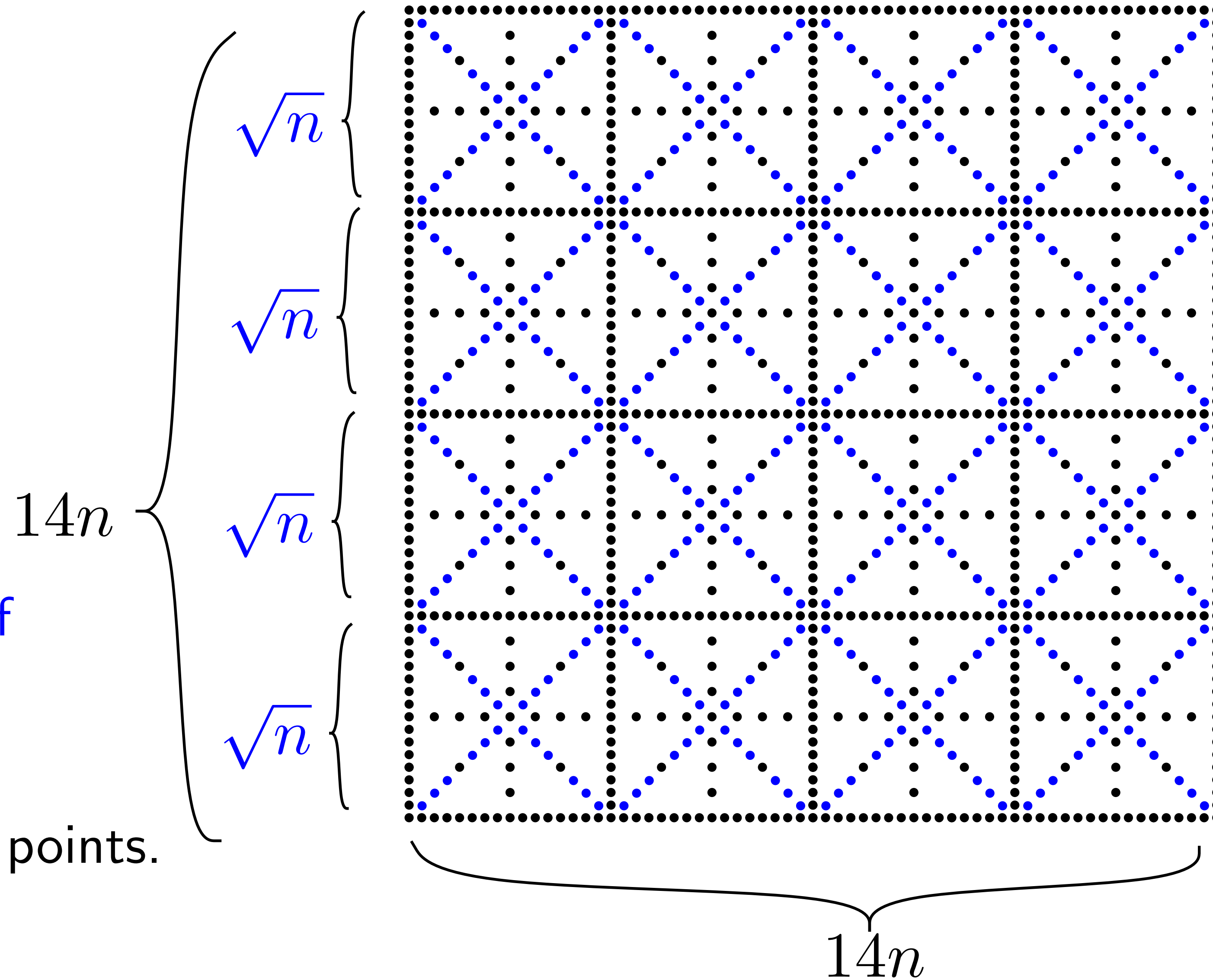

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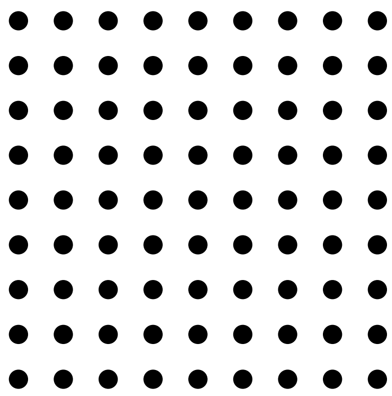

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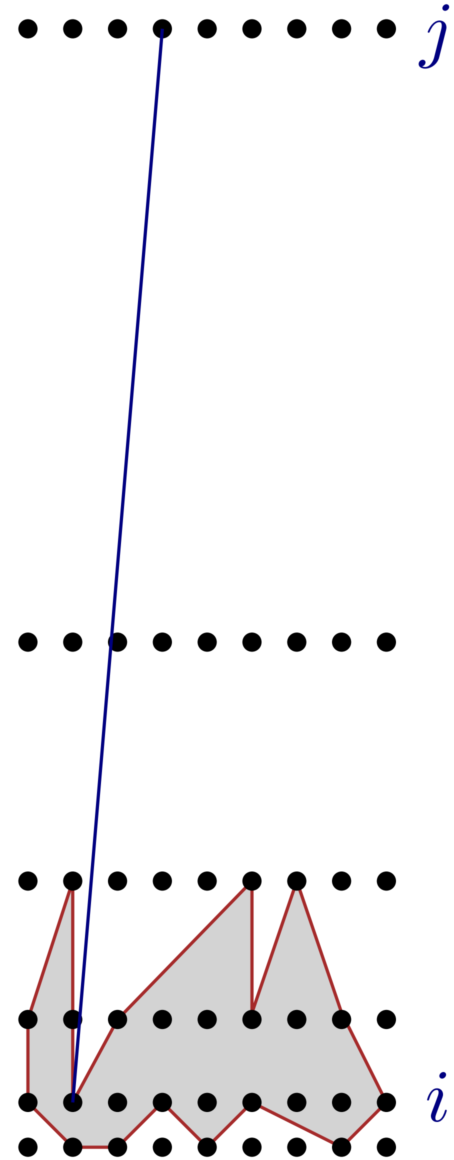

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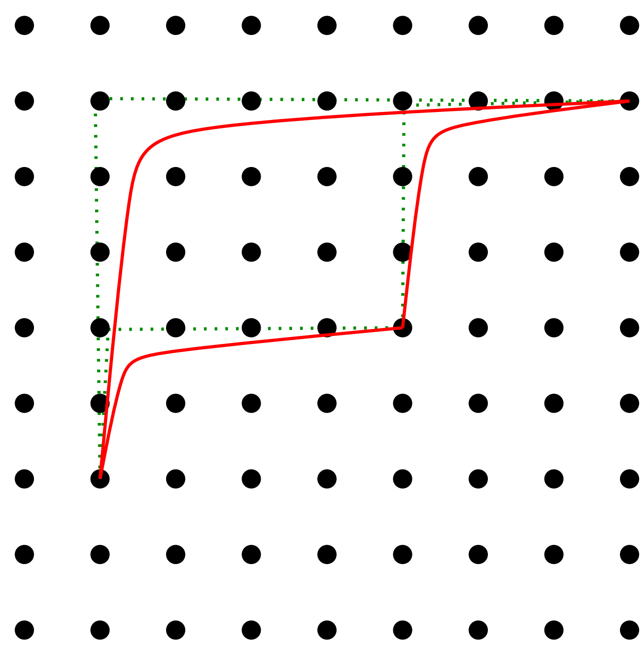

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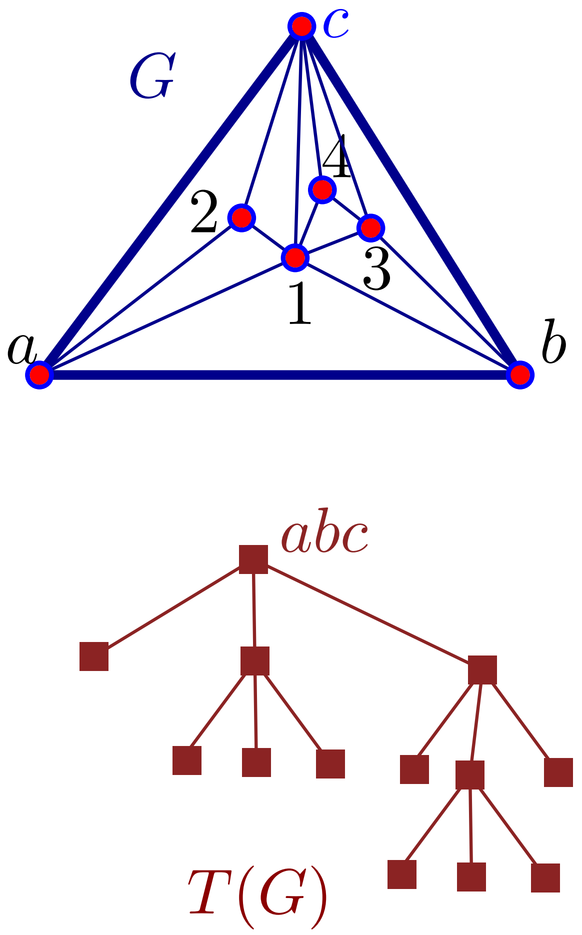
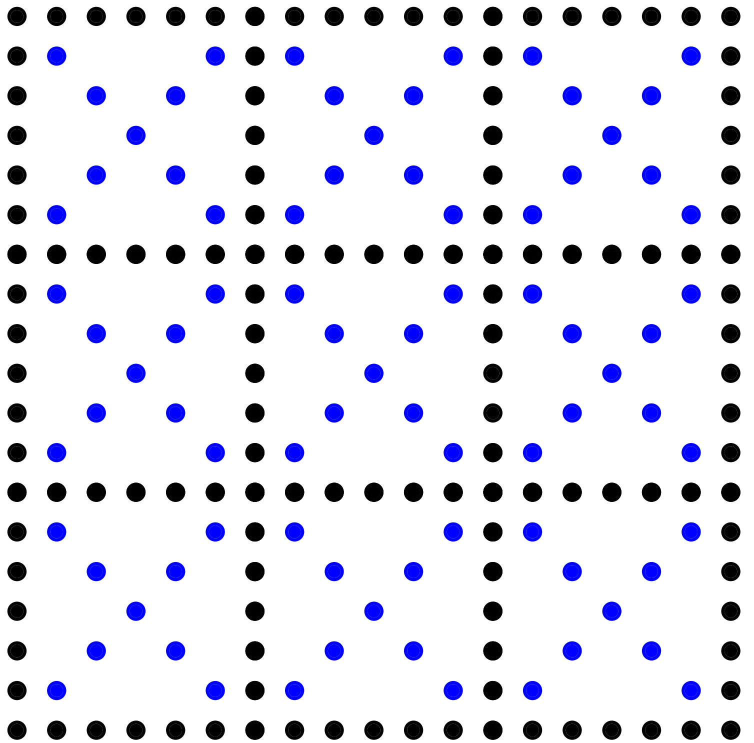
When we pull back the stretched grid to the integer grid, the straight-line edges become Γ-shaped curves.

Embedding algorithm

Every n-vertex planar 3-tree can be embedded such that the vertices arremapped into our point set.





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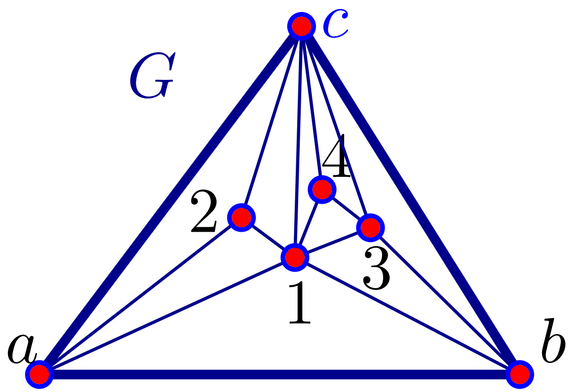

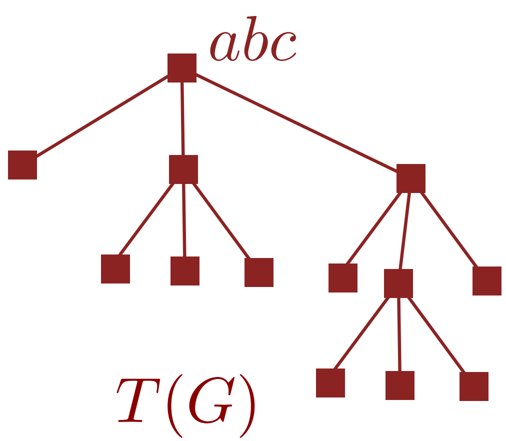

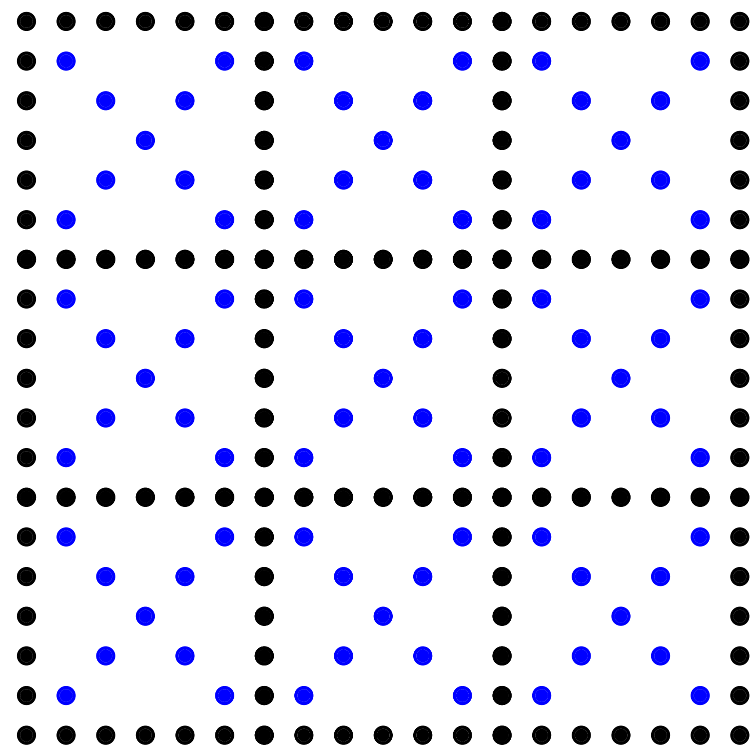

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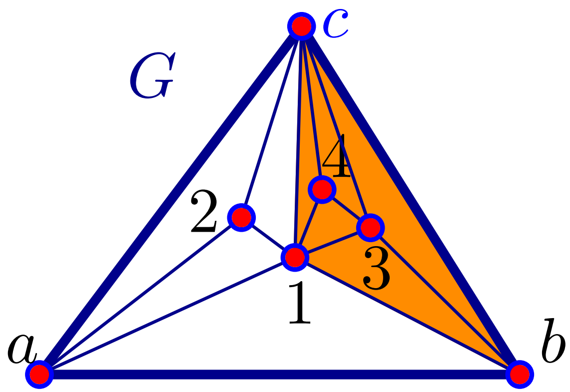

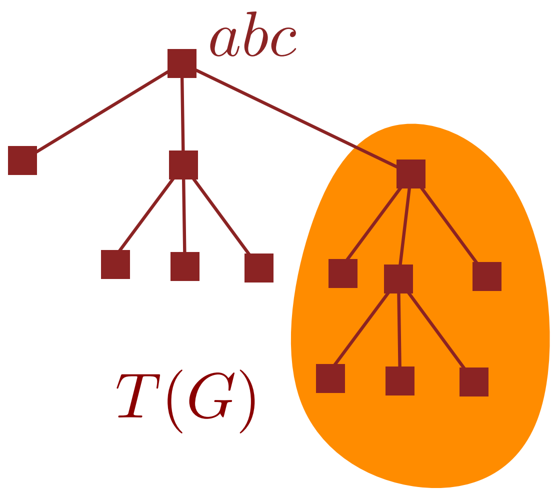

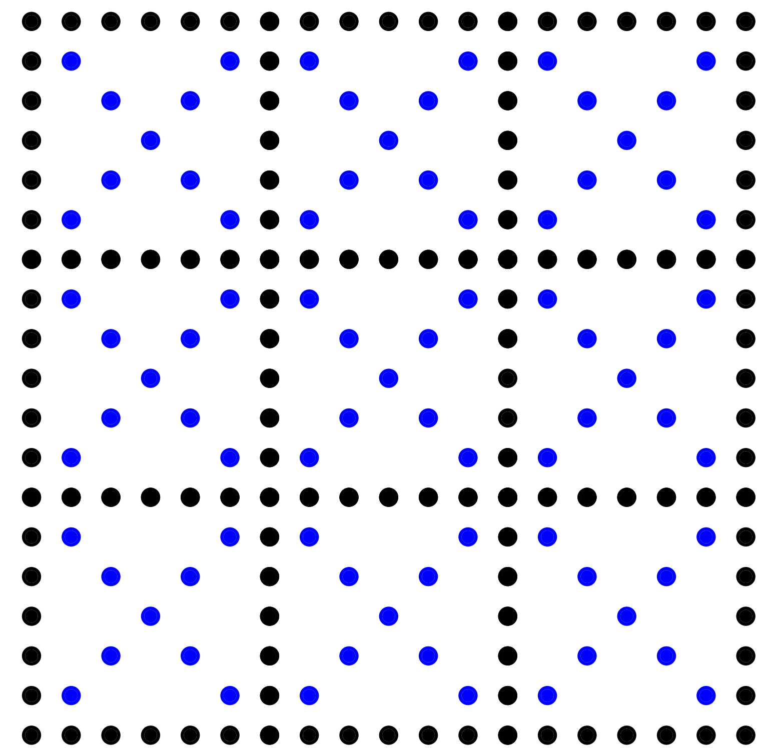

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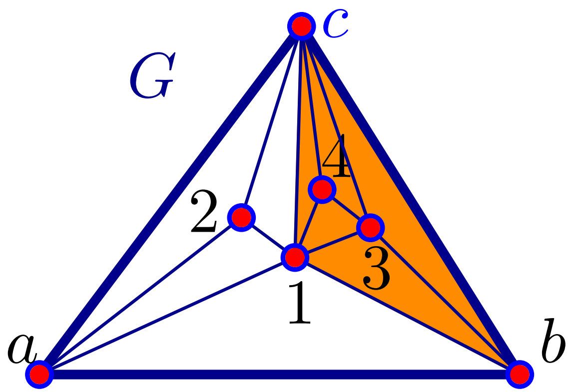

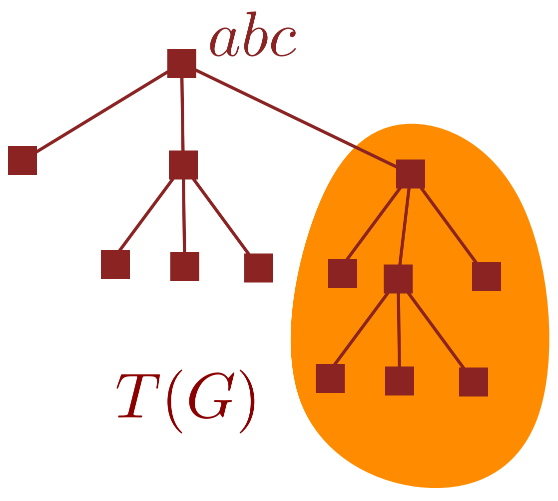

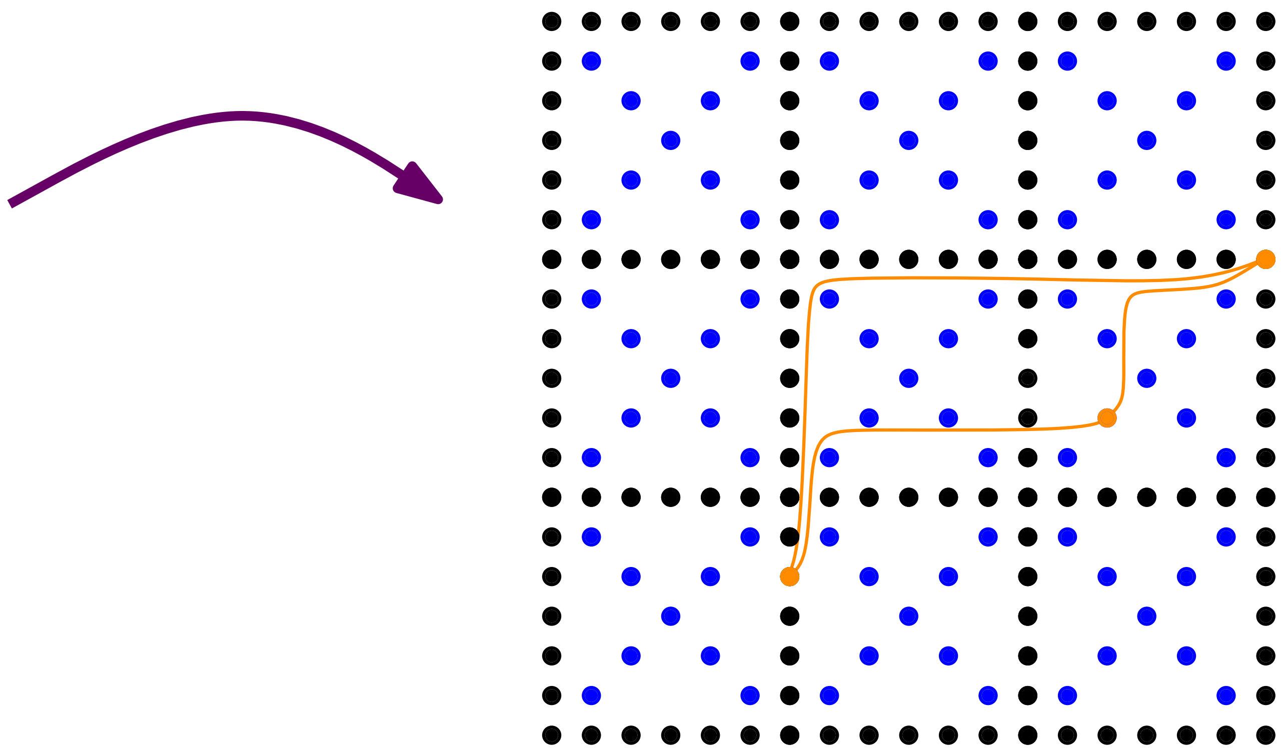

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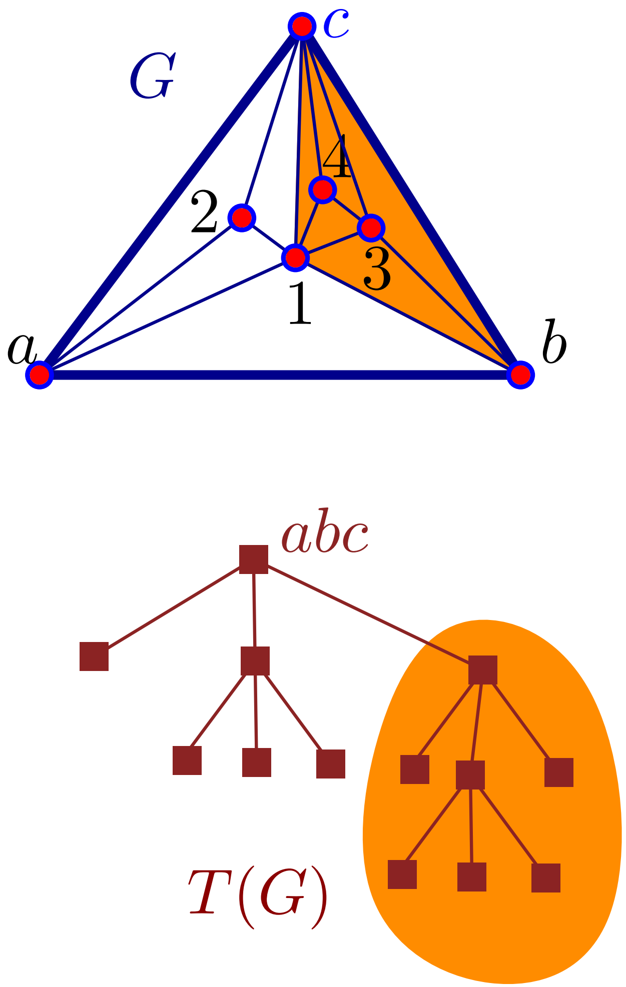

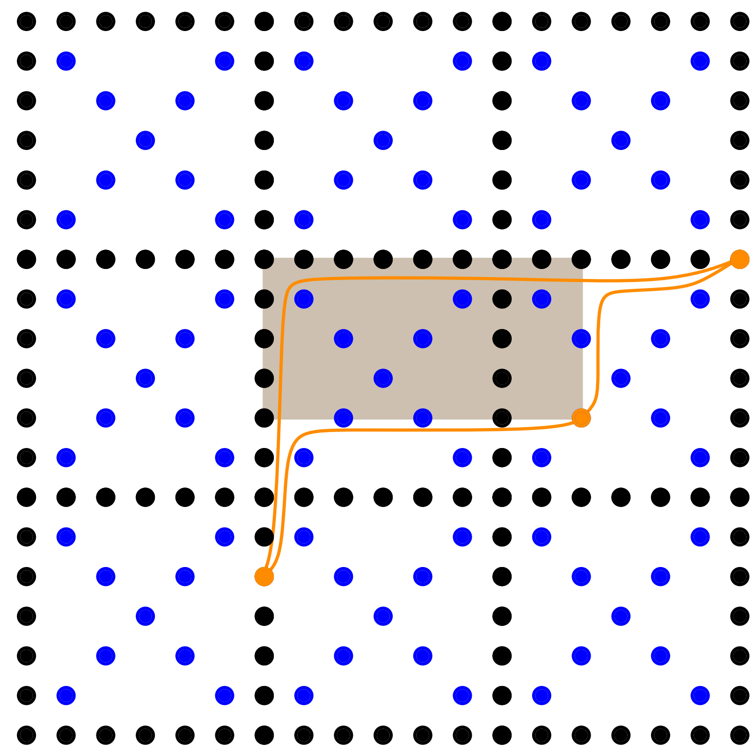

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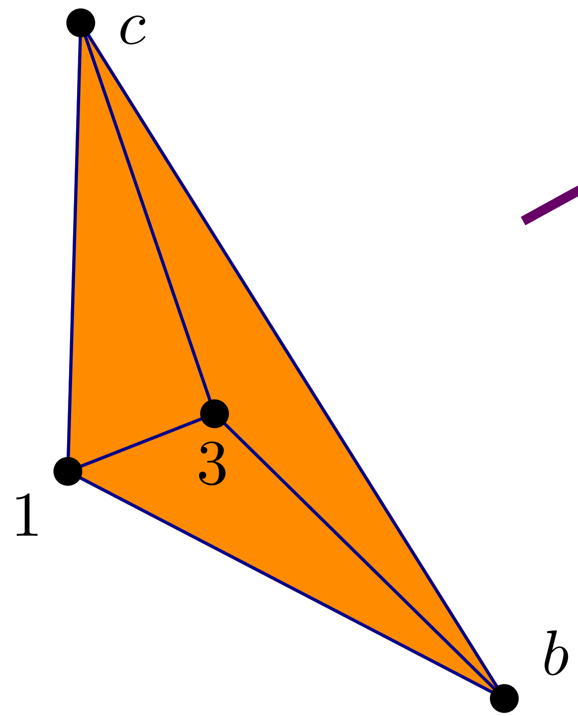

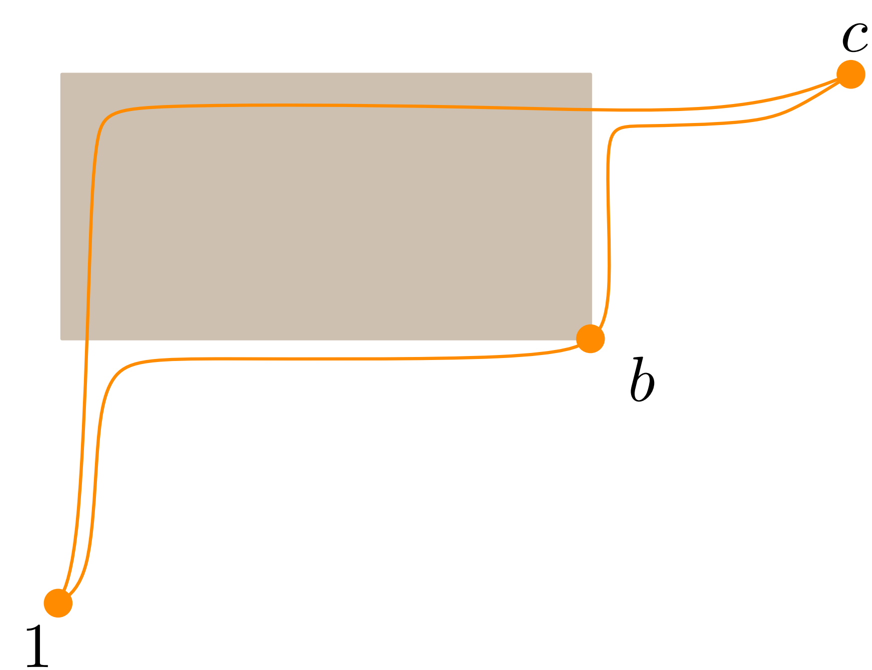

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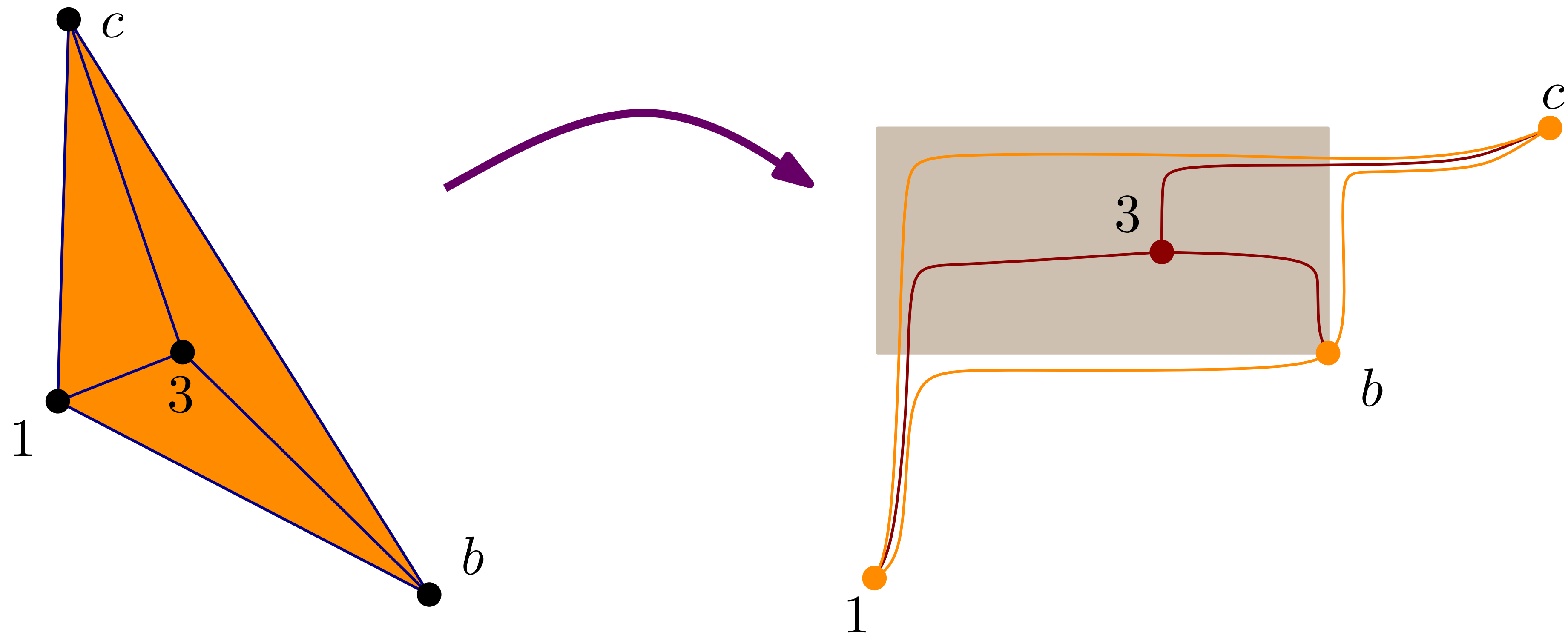

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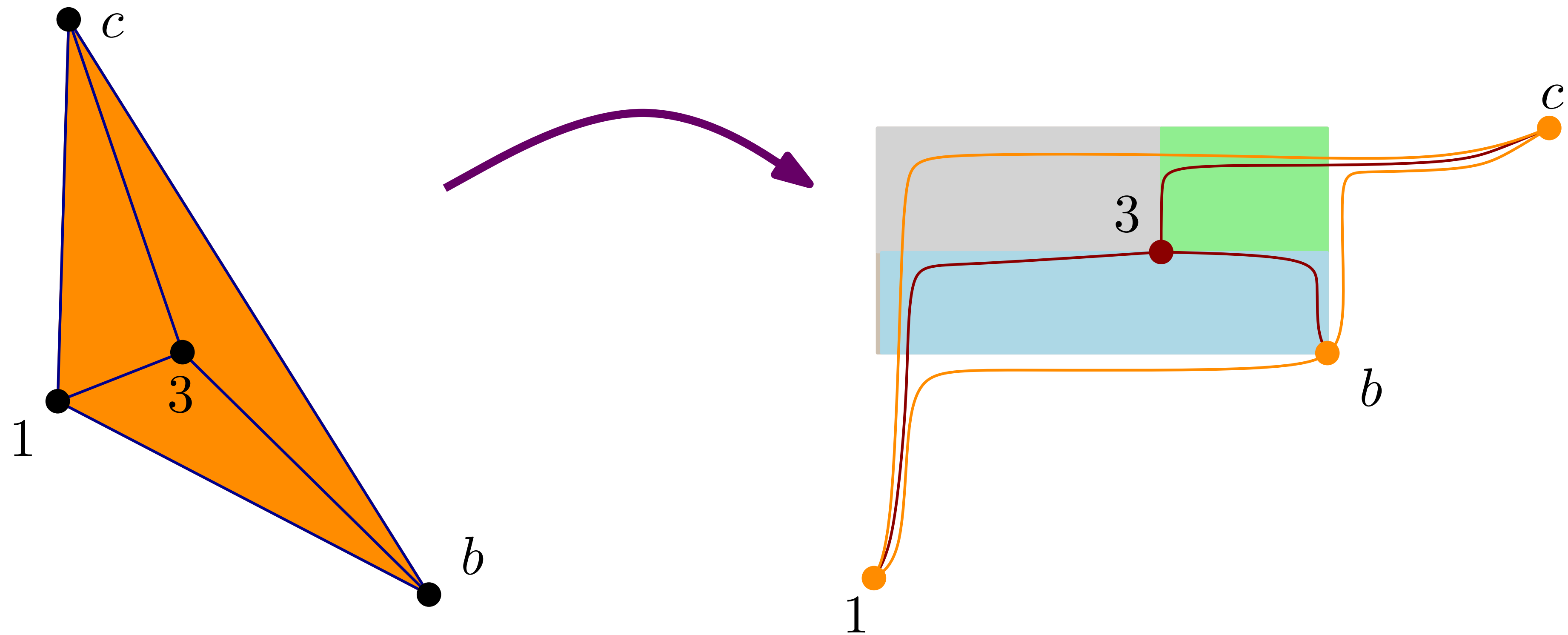
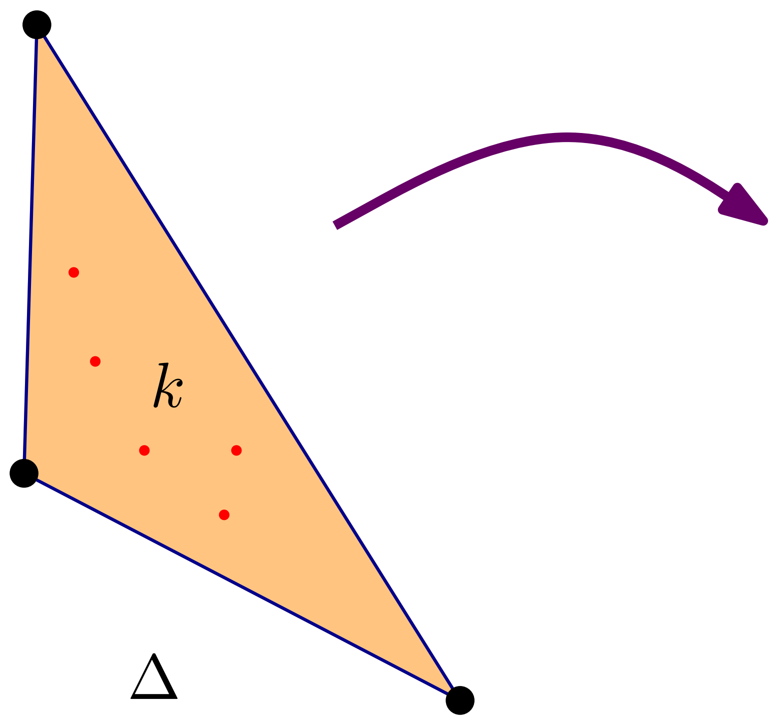
When a new vertex is inserted, the rectangle is subdivided into four rectangles: **left**, **right**, and **bottom** rectangles.

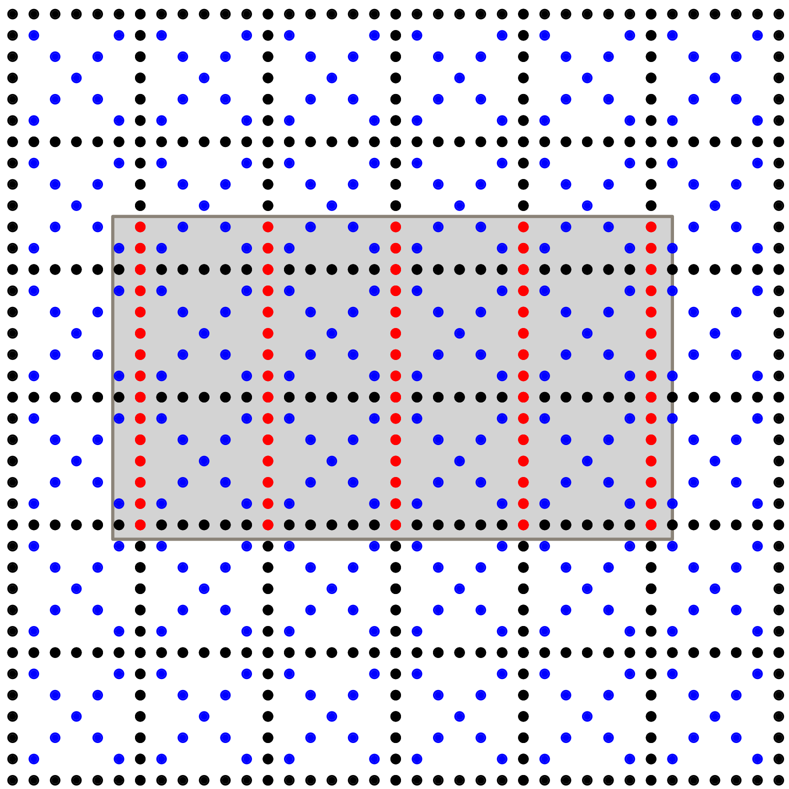


Embedding algorithm

If a "large" rectangle $R(\Delta)$ is allocated to a subgraph lying in a triangle Δ, then we can complete the embedding with the algorithm of de Fraysseix, Pach, & Pollack (1990).

This is possible when k points has to be embedded in a triangle Δ, and the full rows or full columns in the rectangle $R(\Delta)$ form a $k \times k$ grid.





Universal Point Sets: Summary

Problem: Is our point set universal for all planar graphs?

For **all** planar graphs, the currently best bounds are $1.235n - o(n)$ (Kurowski) and $n^2/4$ (Bannister et al.).

Open Problem: Find n-universal point sets of size $o(n^2)$.

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Generalization:
A point set S is **universal for** a family of graphs $\mathcal{G}$ if every graph $G \in \mathcal{G}$ has a geometric realization with $\mathrm{cr}(G)$ crossings such that all vertices are mapped into S.

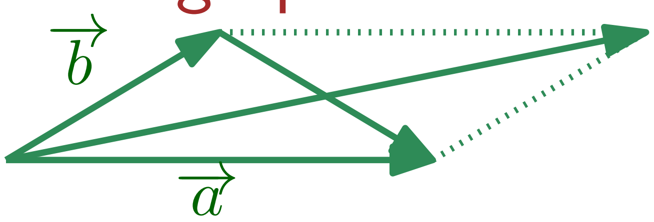
Open Problem: Find n-universal point sets for **all** graphs.

...might be elusive:
—computing the crossing number, $\mathrm{cr}(G)$, is NP-hard,
—no optimal embedding is known for the complete graph K_n.

Universal Slope Sets

Keszegh et al. (2008):

- Every (abstract) graph with maximum degree 3 has a geometric realization with 5 distinct slopes.
- Every graph with vertices of both degree 2 and 3 has a geometric realization with 4 slopes,
- A set S of 4 slopes is universal for all such graphs iff $S = \{\overrightarrow{a}, \overrightarrow{b}, \overrightarrow{a} - \overrightarrow{b}, \overrightarrow{a} + \overrightarrow{b}\}$.



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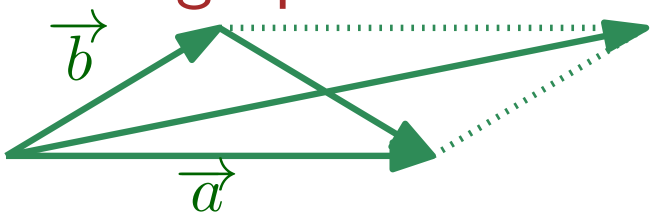

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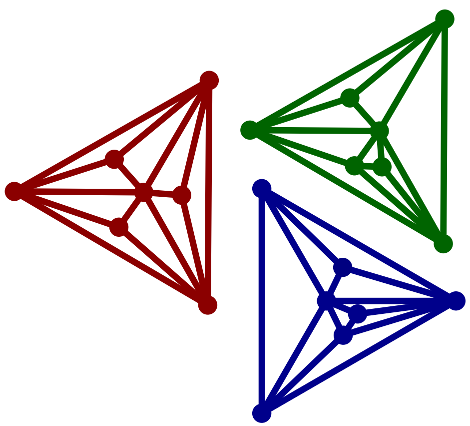
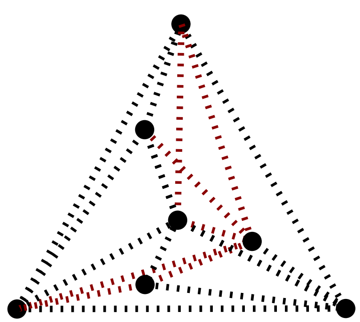
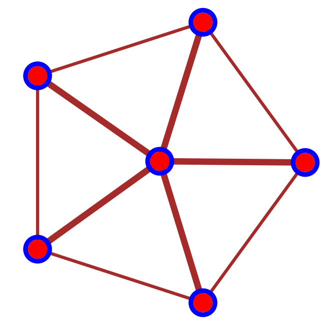
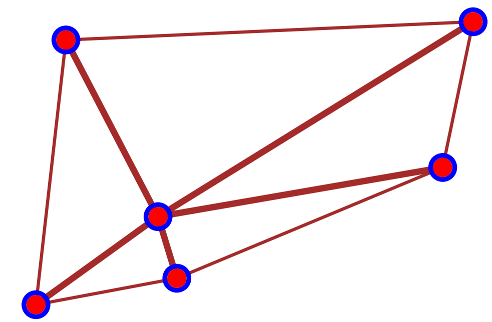
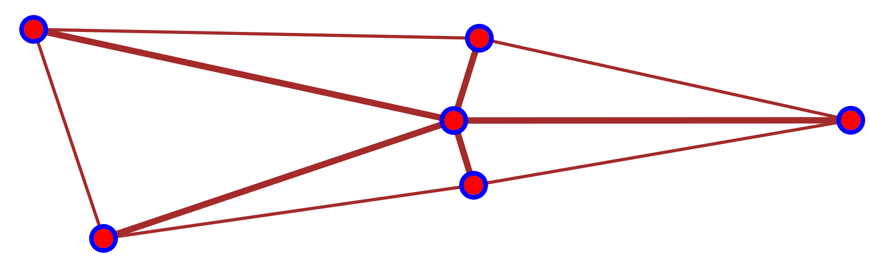
Open Problem: Which slope sets are universal for all *planar* graphs of maximum degree d?

Universality in Geometric Graphs

1. A structute is **universal** if it is “compatible” with every geometric graph from a certain family
   (e.g., universal point sets, universal slopes, etc.)

2. An abstract graph is **universal** if it has a geometric realization for any possible choice of certain parameters
   (e.g., globally globally rigid graphs, length-universal graphs, area universal floorplans).

Globally Rigid Graphs

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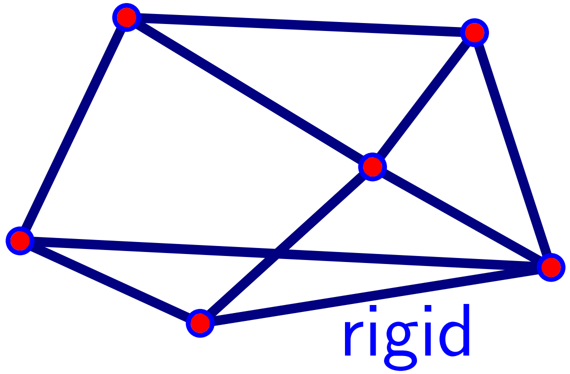

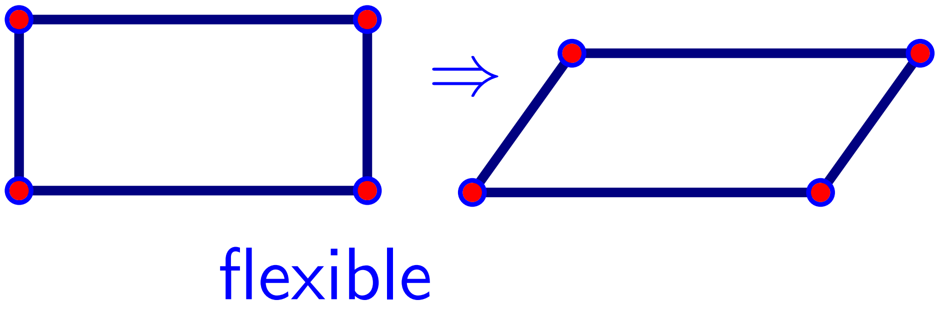

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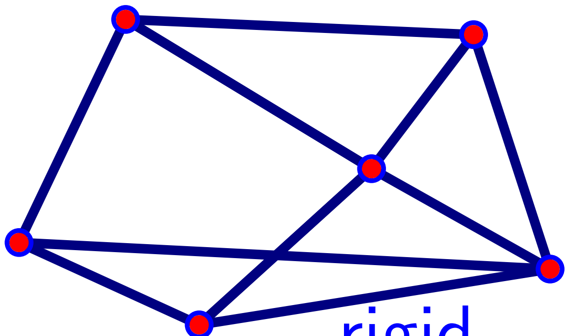

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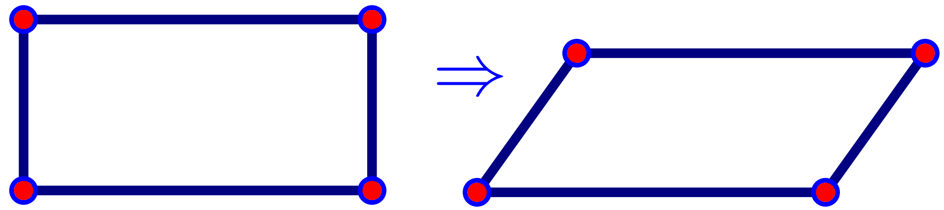

flexible

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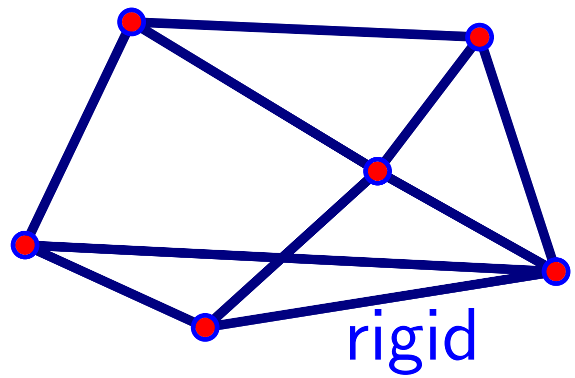

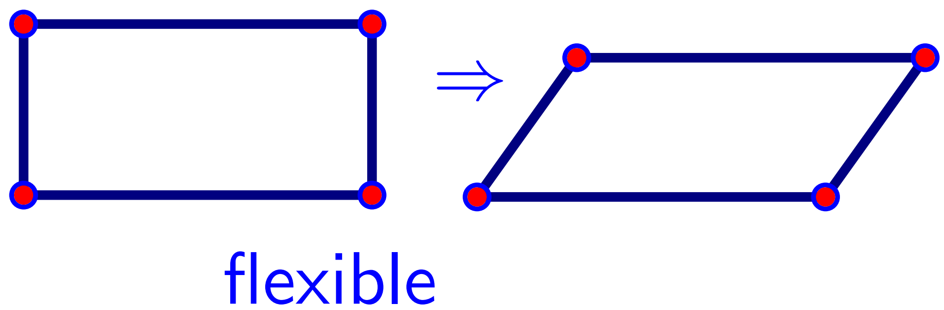

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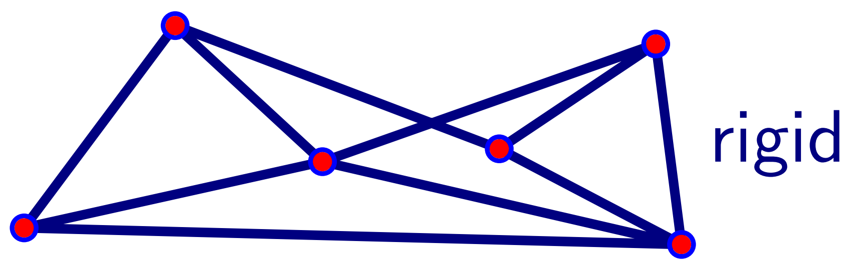

Jackson & Jordán (2005):
A graph G is generically globally rigid iff

- either $G = K_n$, $n \leq 3$,
- or G is 3-connnected and redundantly rigid.

Length Universal Graphs: An Easy Exercise

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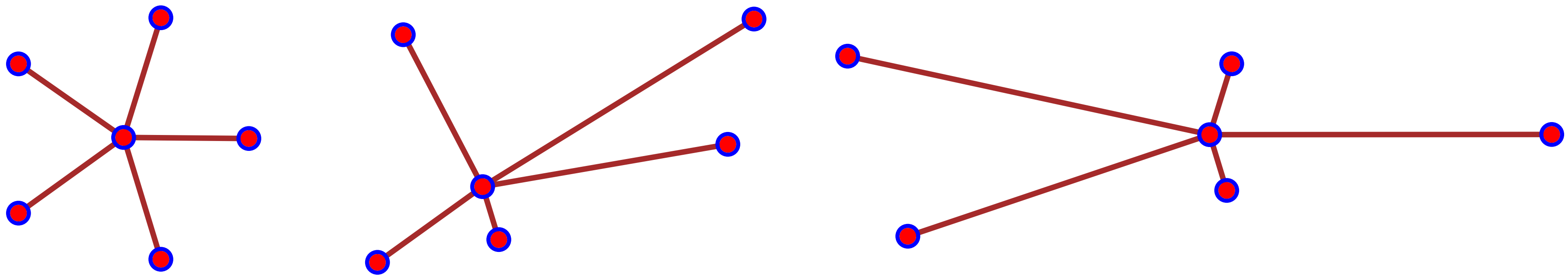

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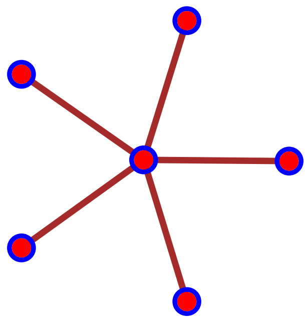
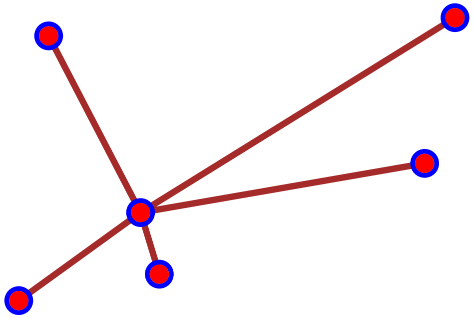
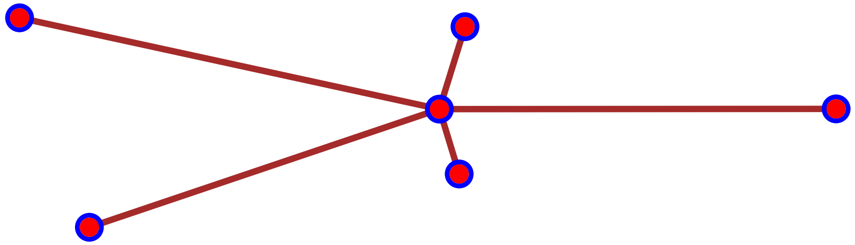
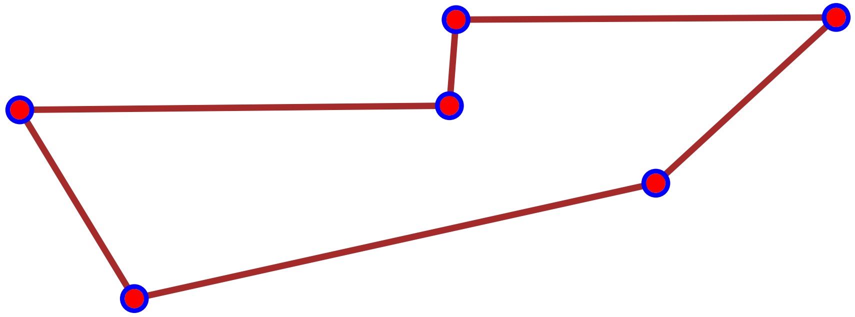






But the edges of a cycle must satisfy the triangle inequality.
The edge lengths cannot be chosen arbitrarily.



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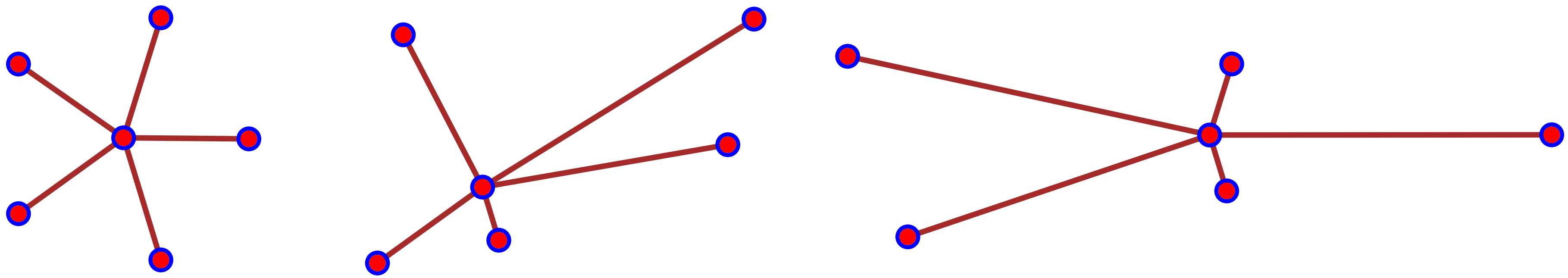

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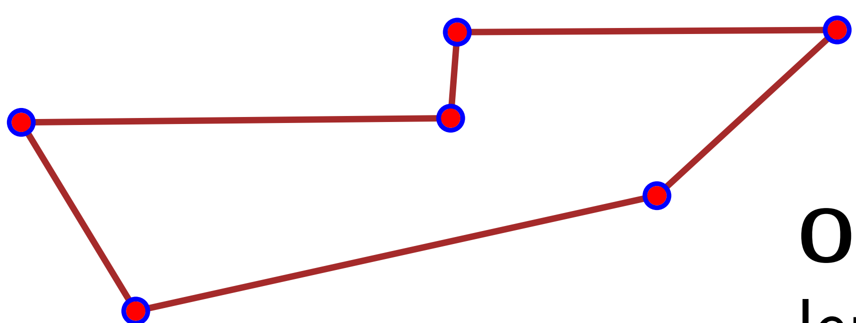

Observation: A graph is length universal iff it is a forest.

Free Graphs

Let $G = (V, E)$ be a subgraph of a planar graph H. Graph G is **free in** H if for every function $\ell : E \to \mathbb{R}^+$, H has a geometric emgedding such that every $e \in E$ has length $\ell(e)$

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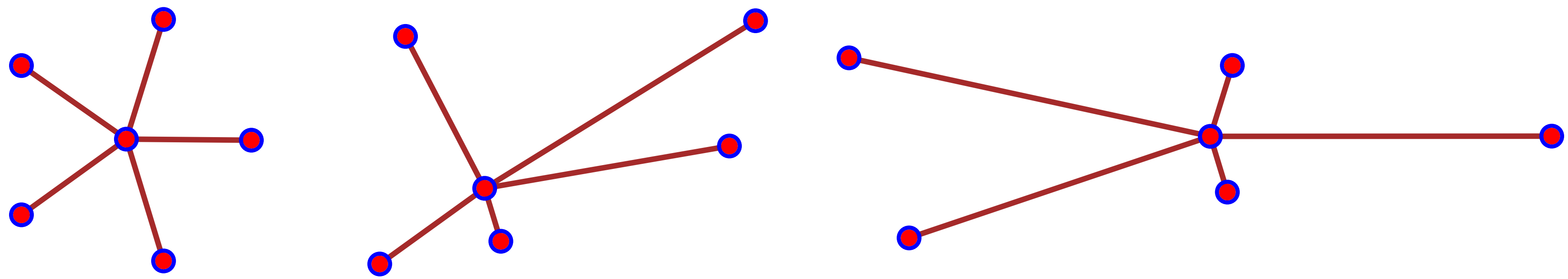

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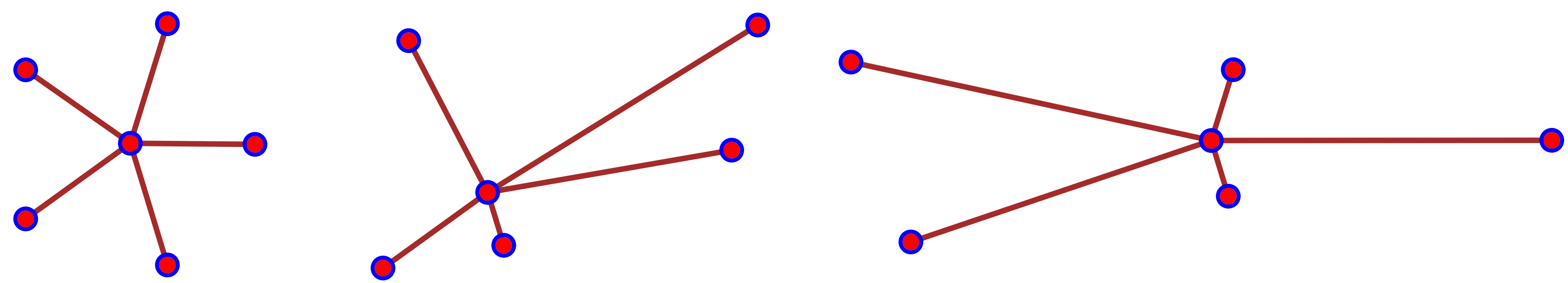
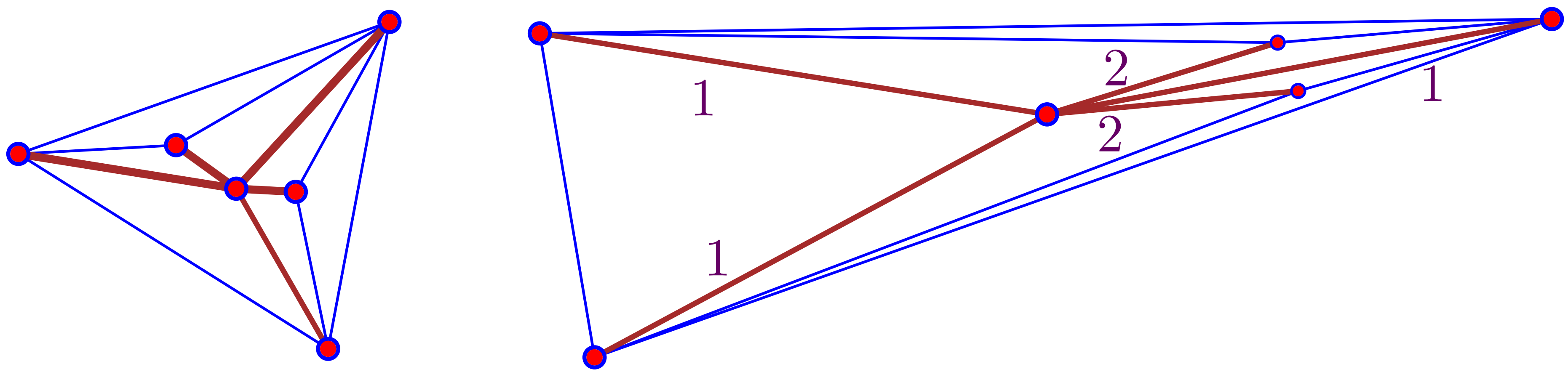


But a star with $n \geq 5$ vertices cannot have arbitrary positive edge lengths *in a triangulation* H.



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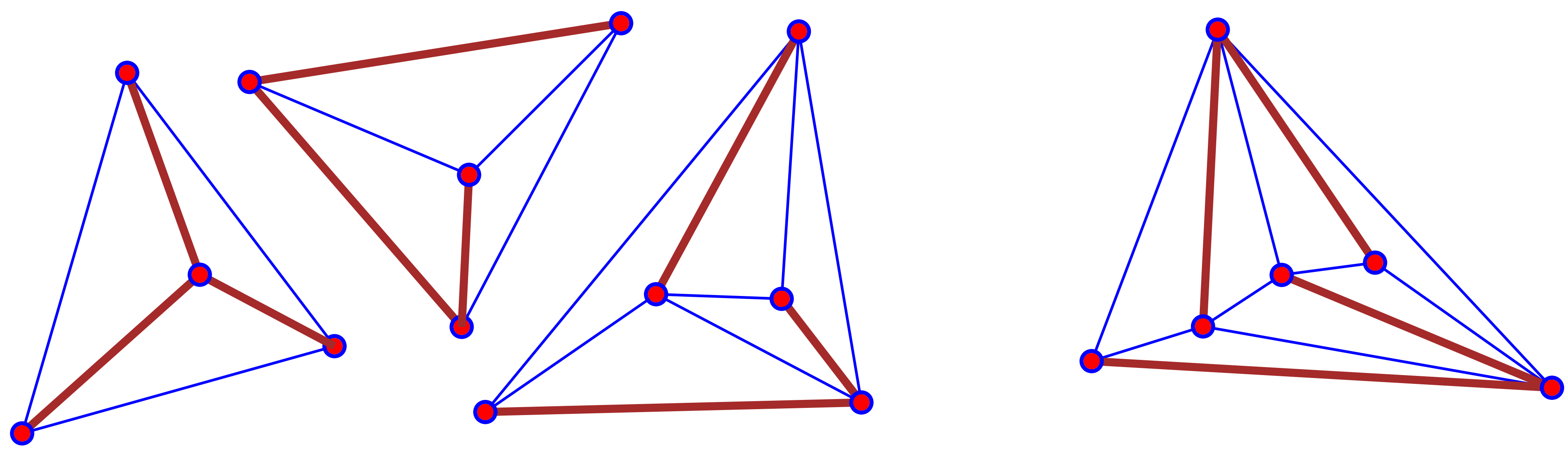

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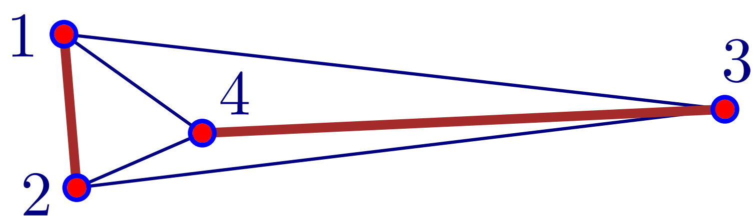

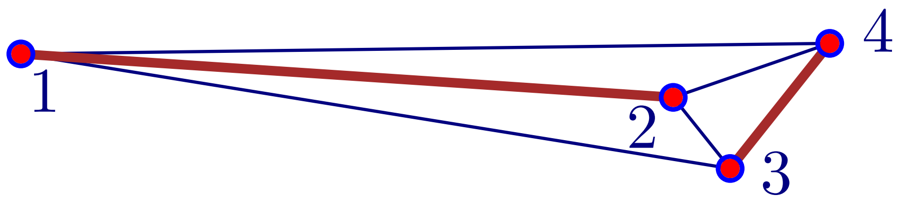

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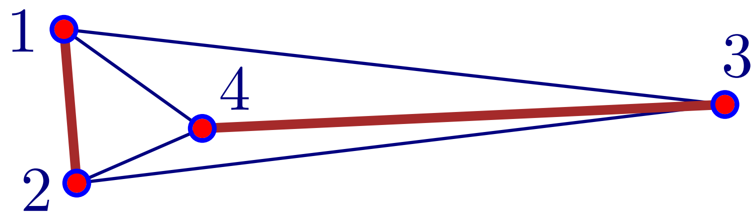

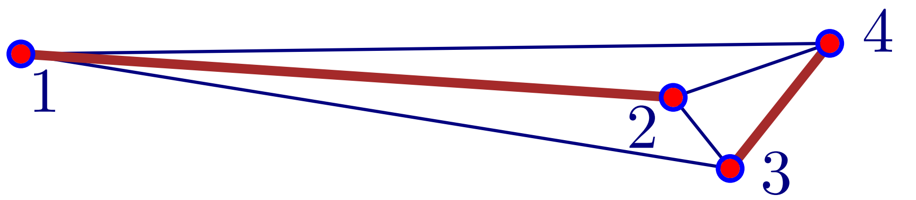

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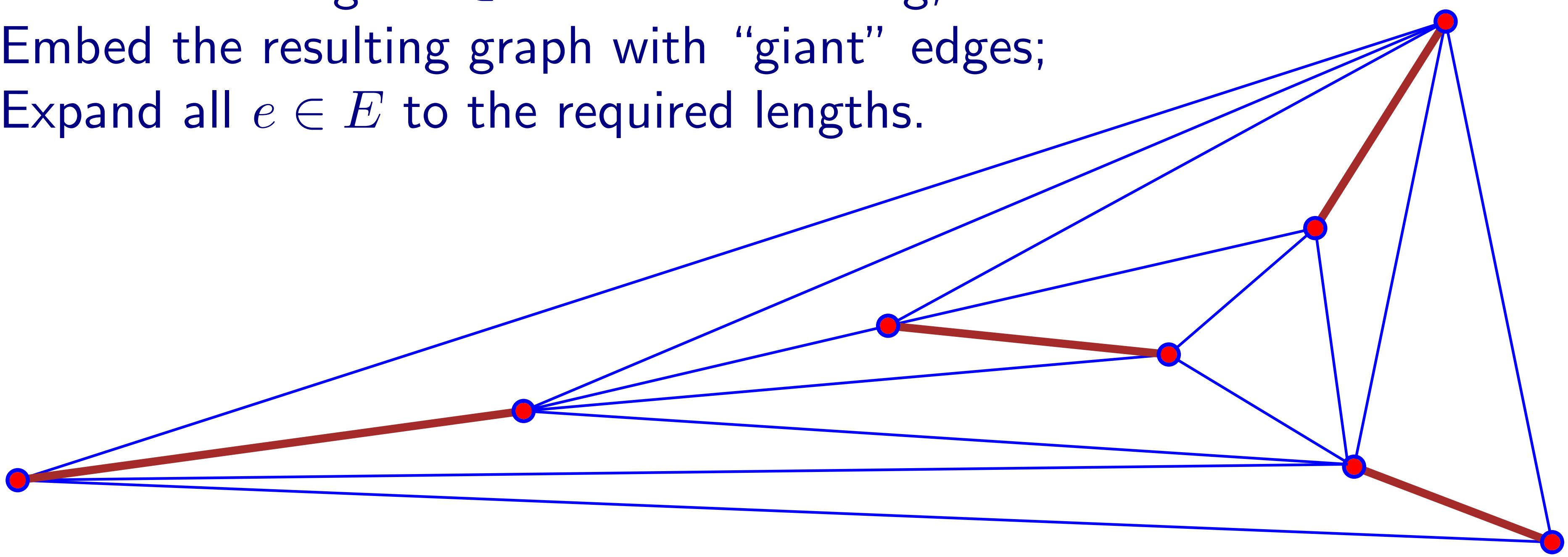

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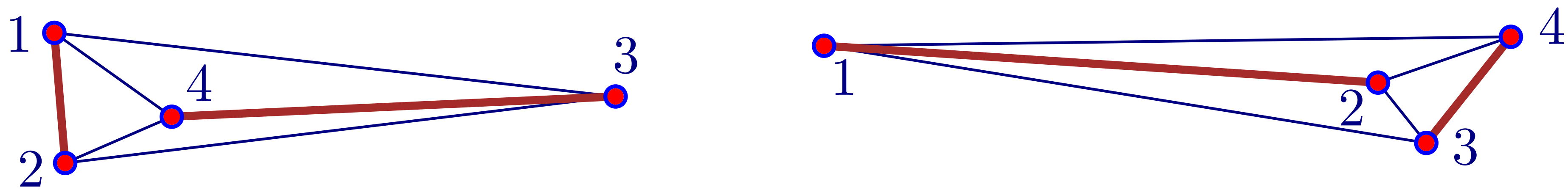

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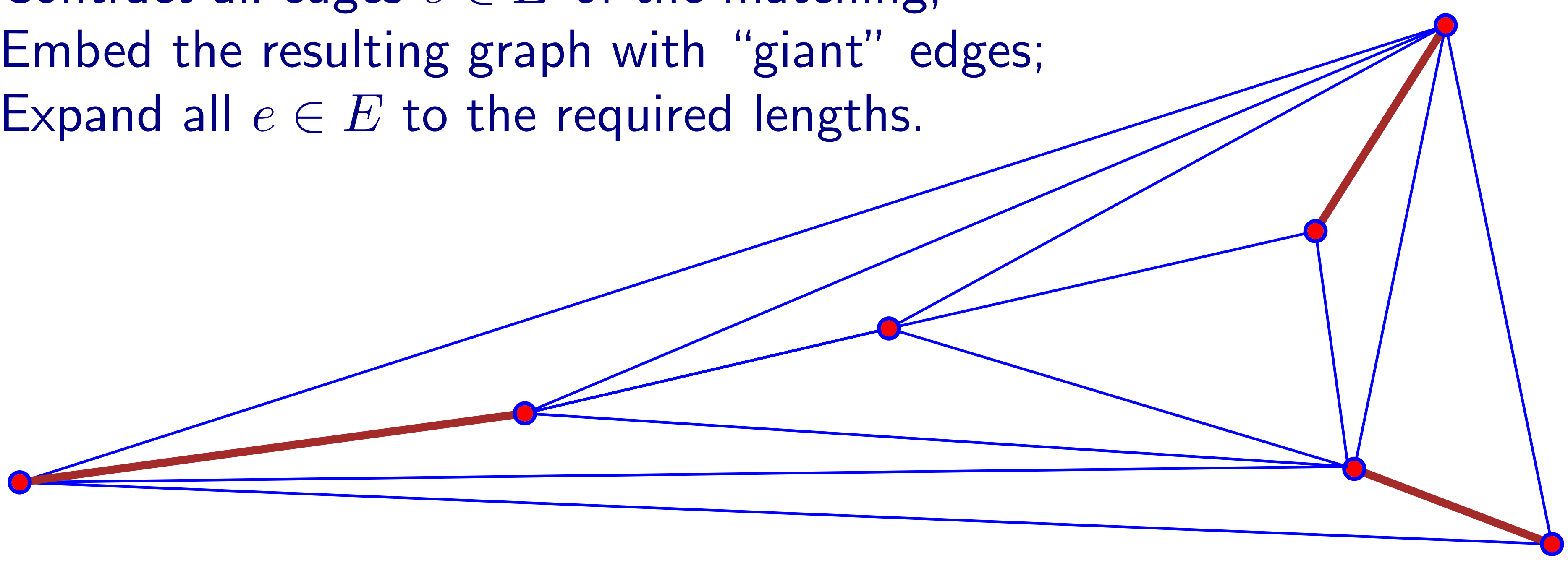

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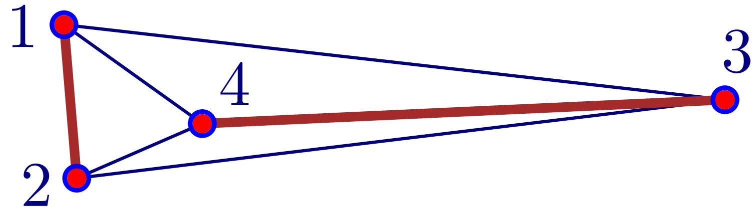

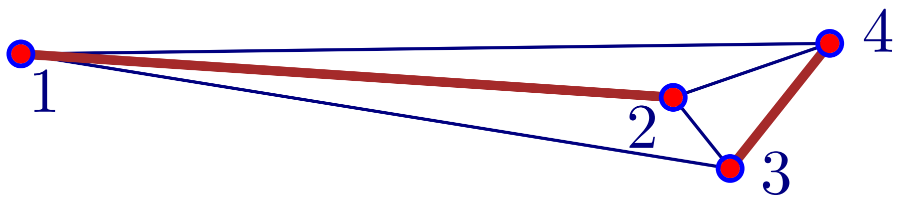

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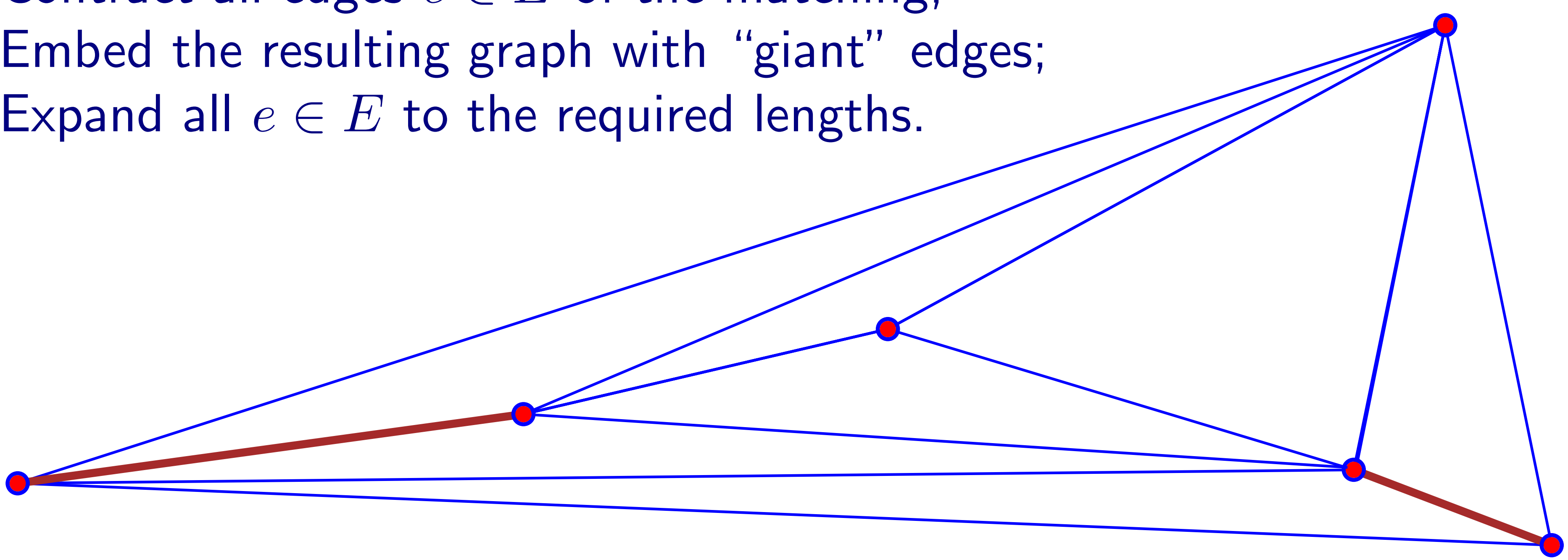

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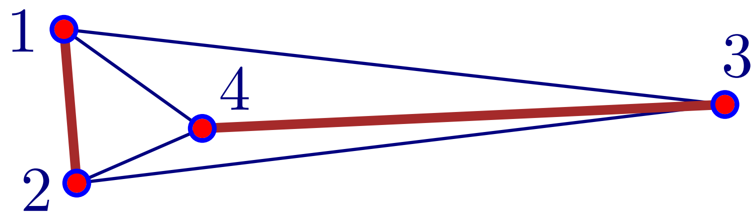

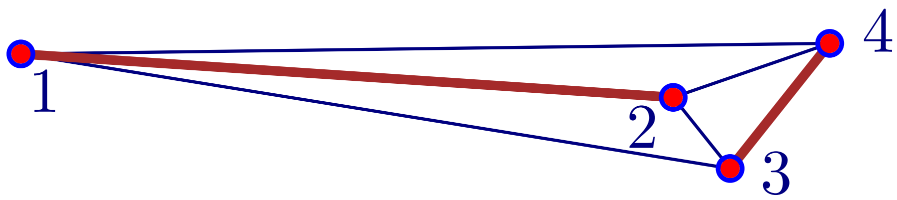

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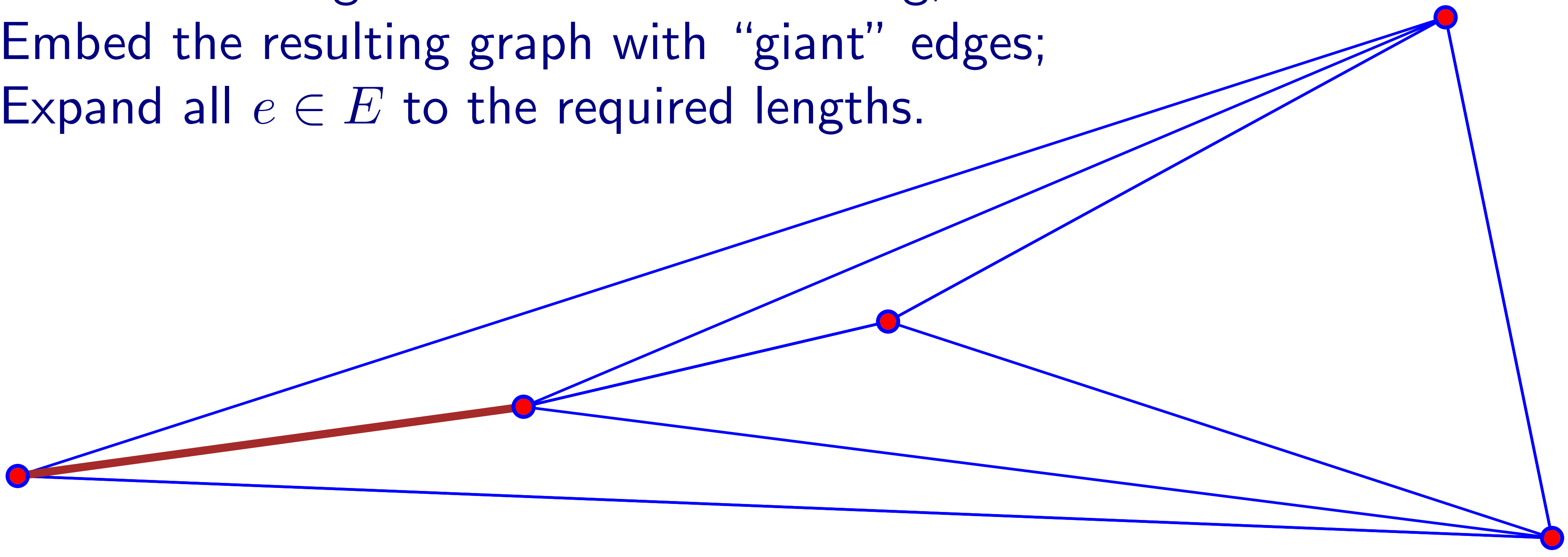

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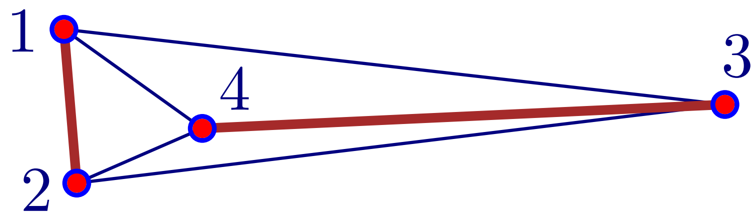

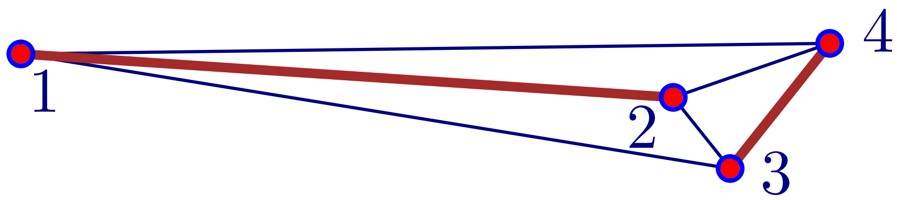

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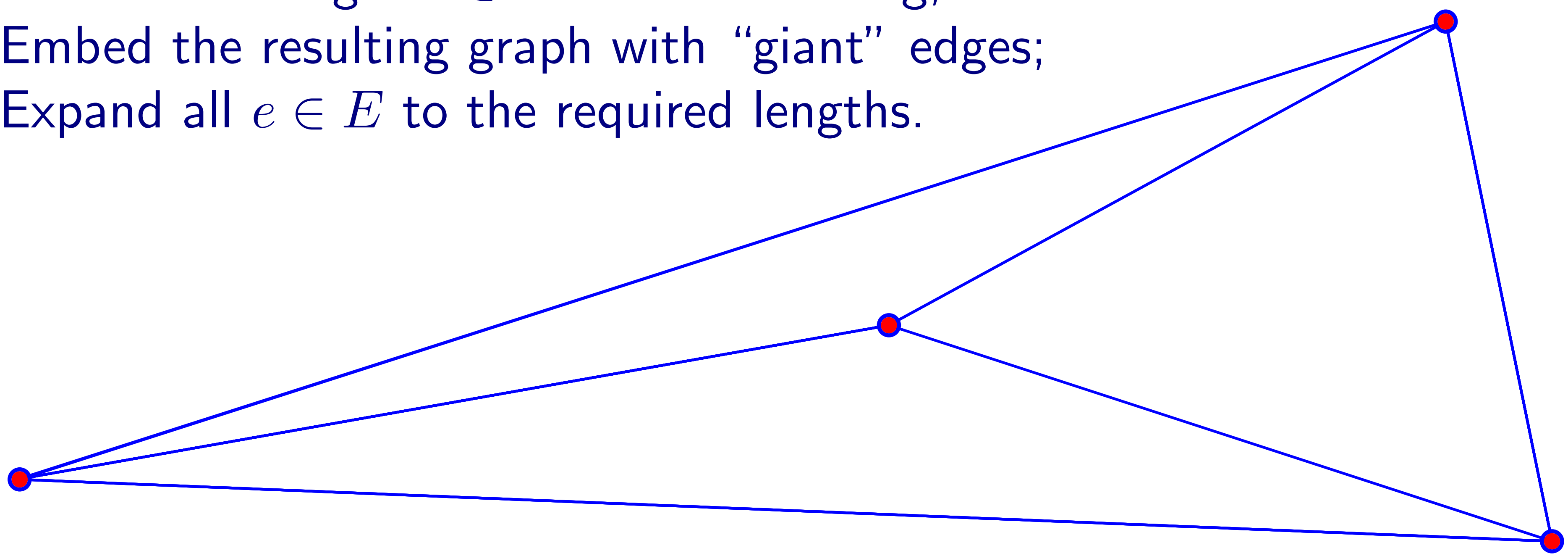

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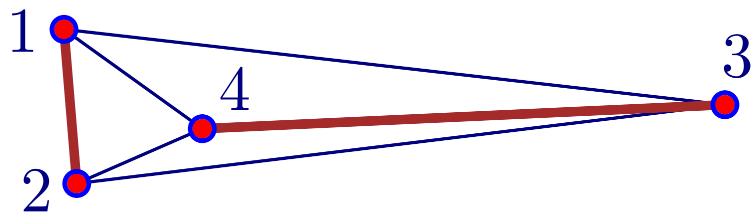

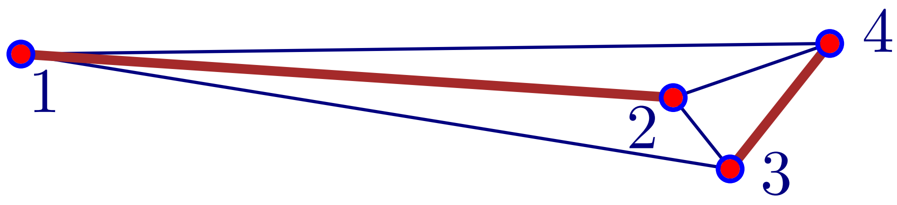

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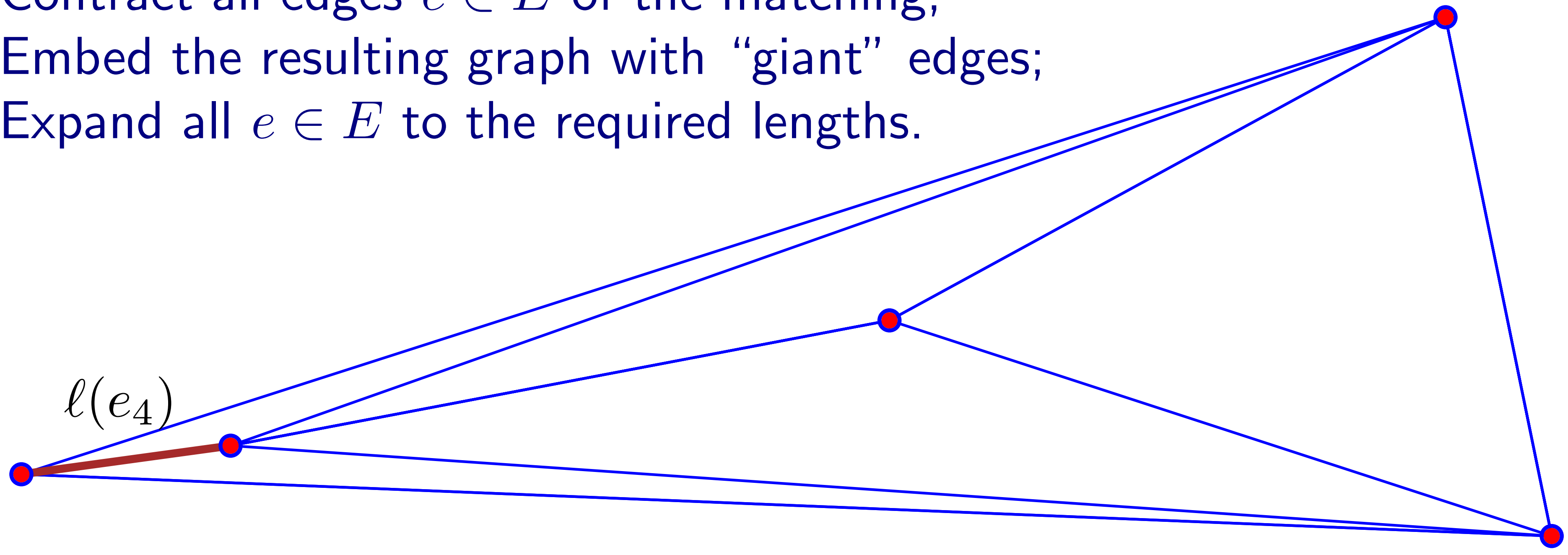

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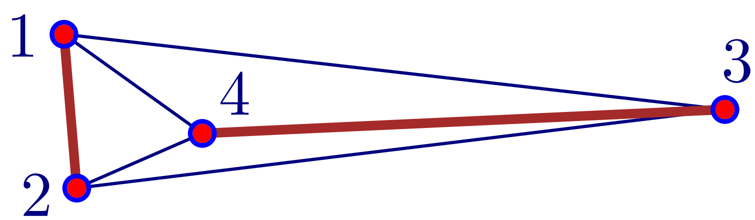

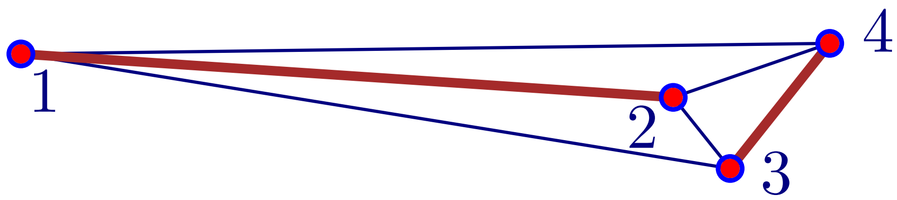

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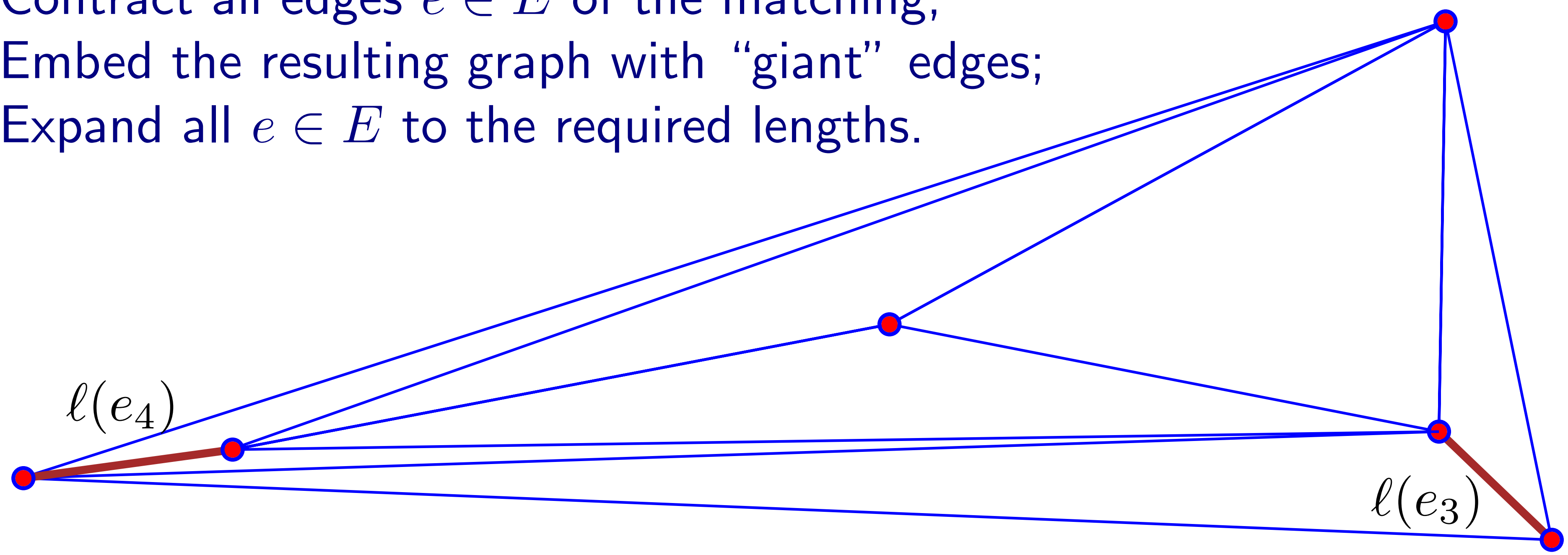

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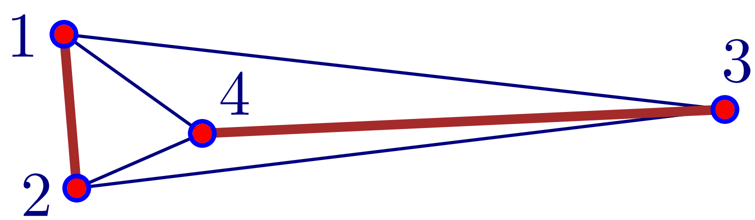

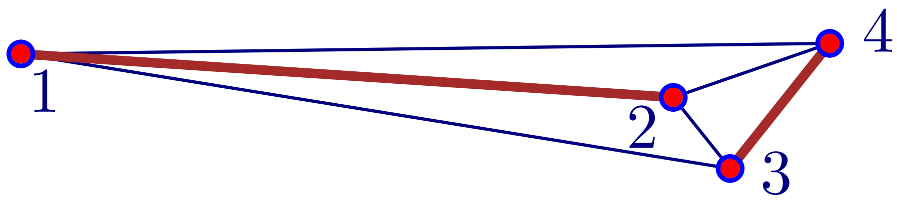

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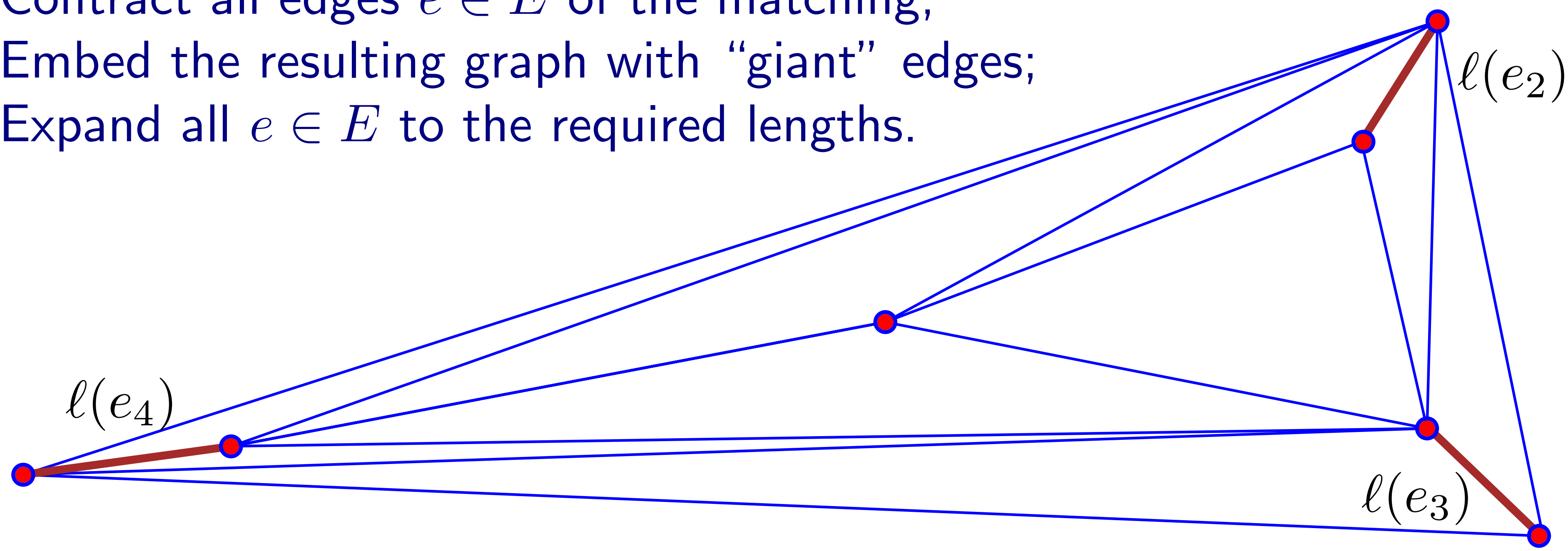

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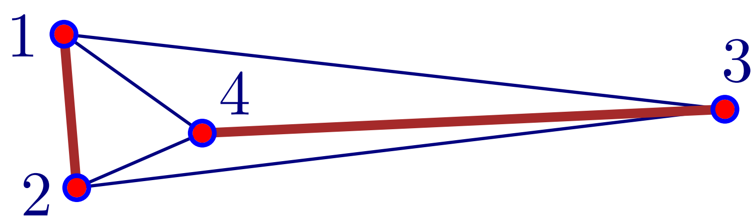

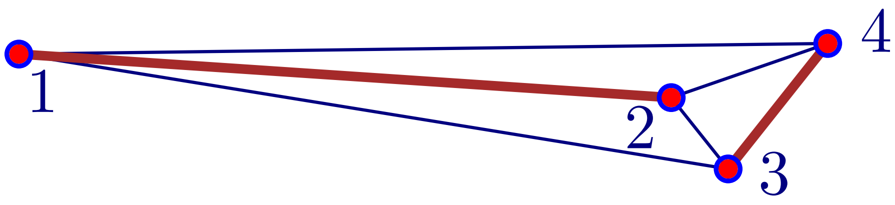

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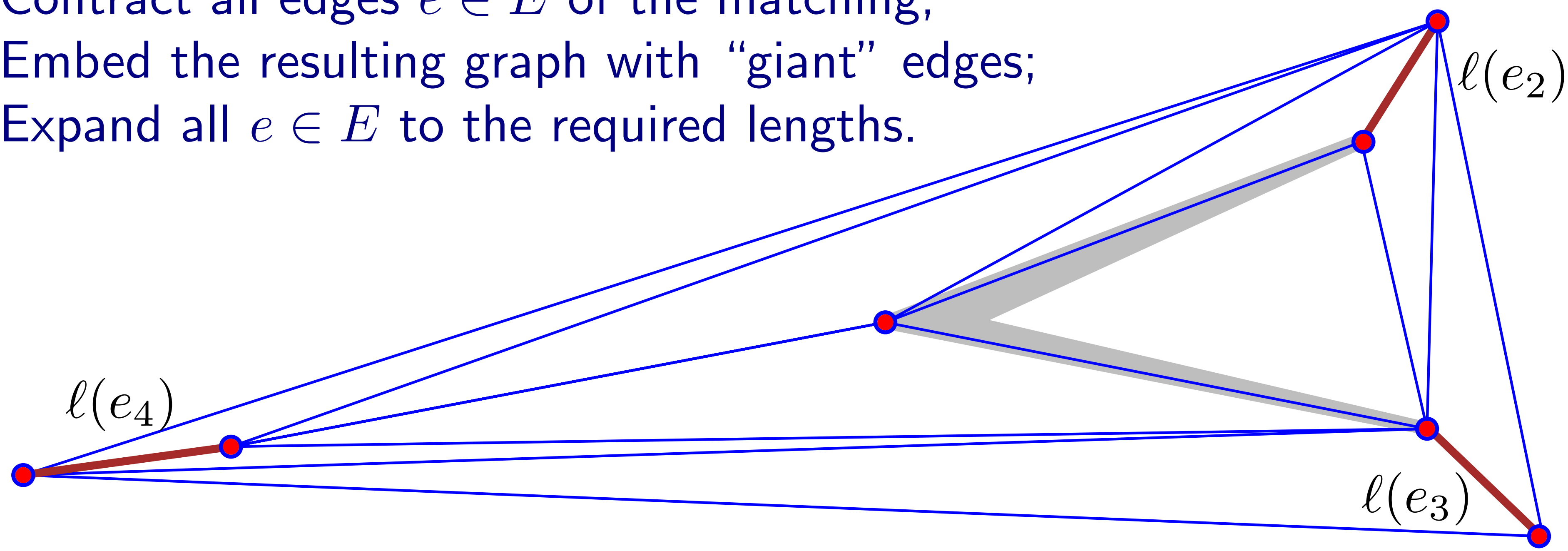

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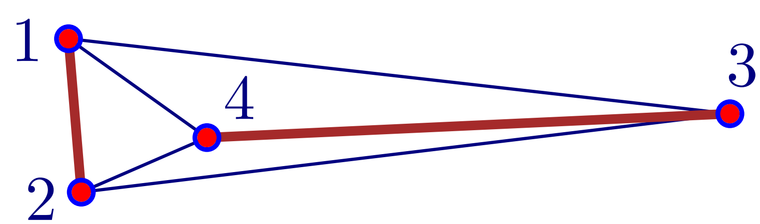

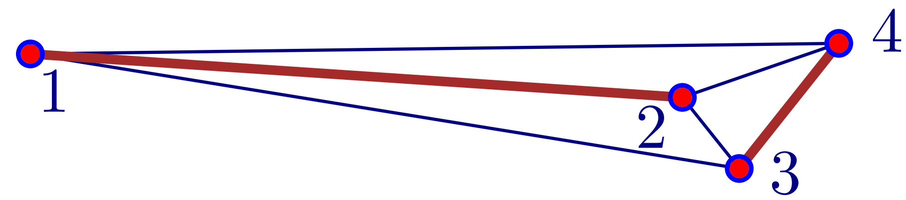

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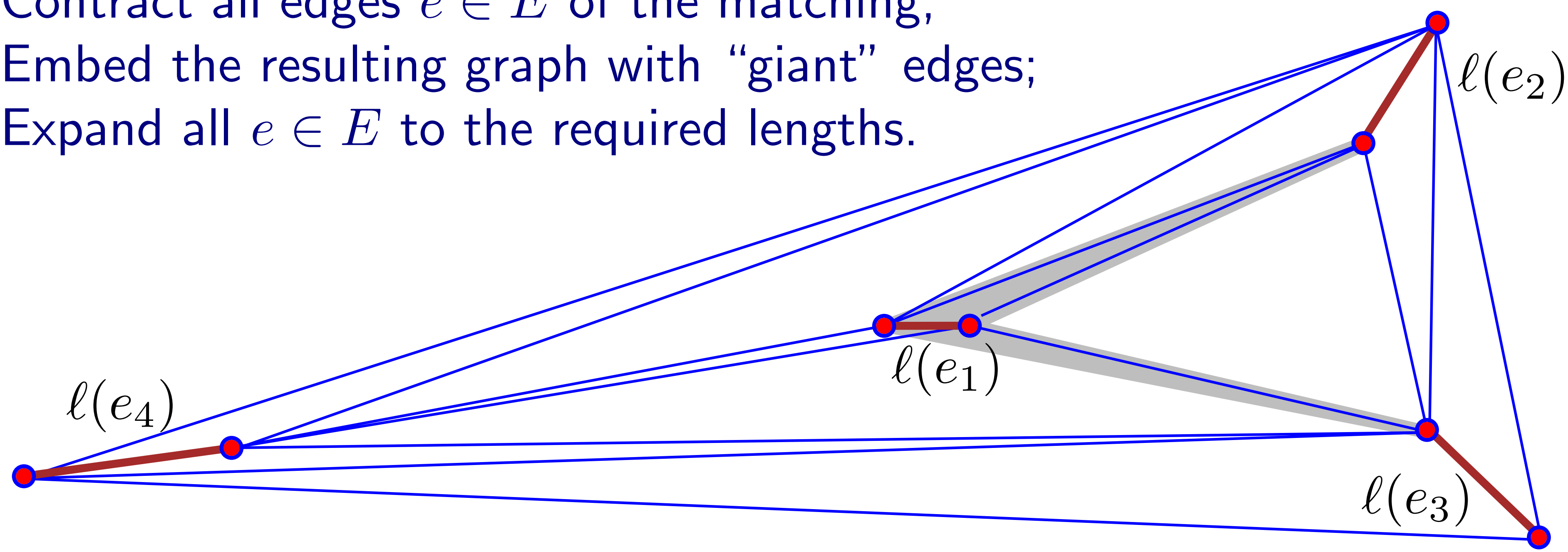

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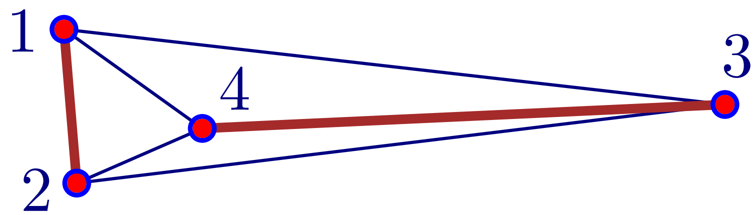

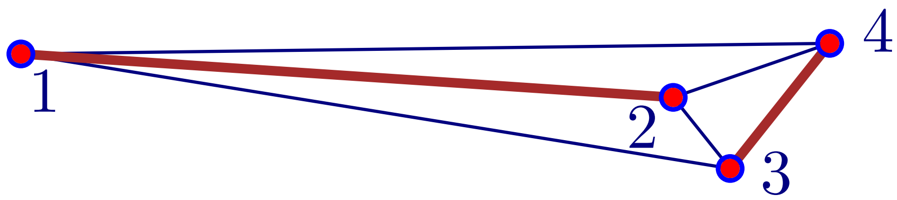

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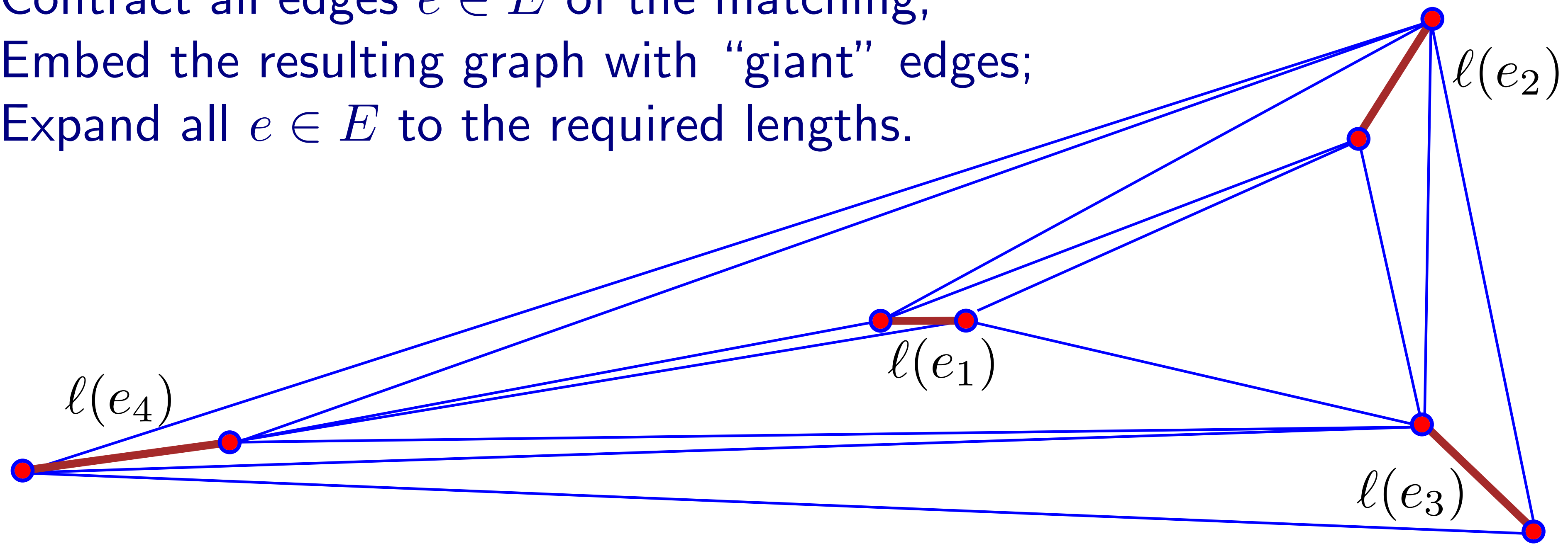

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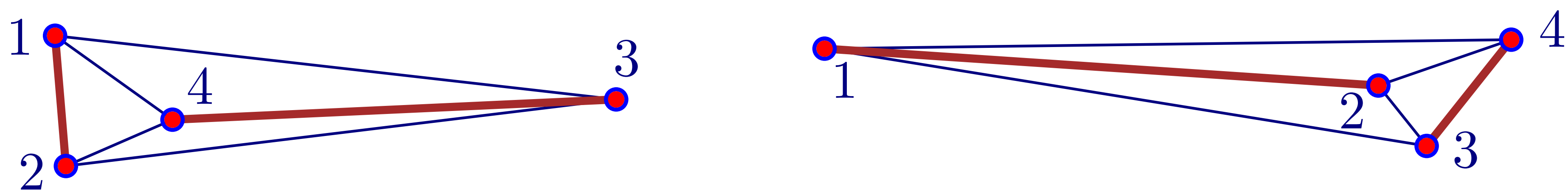

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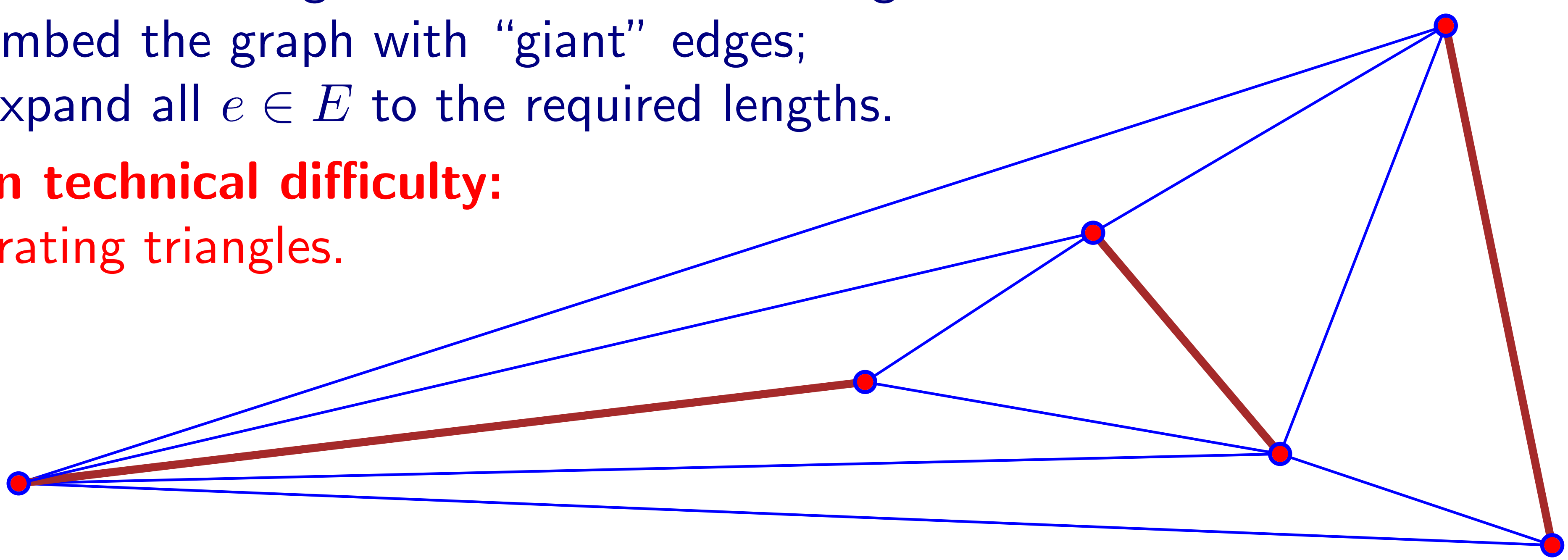

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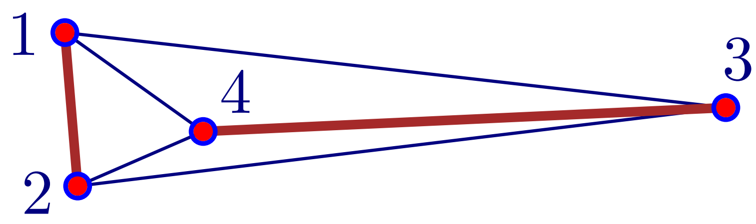

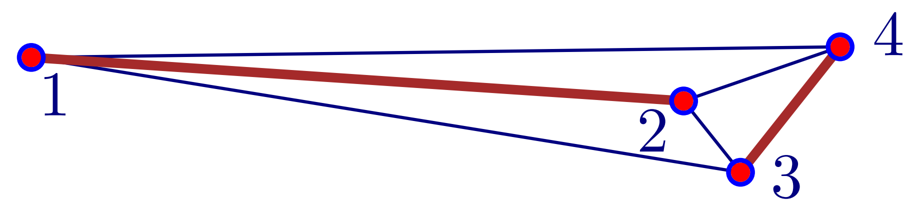

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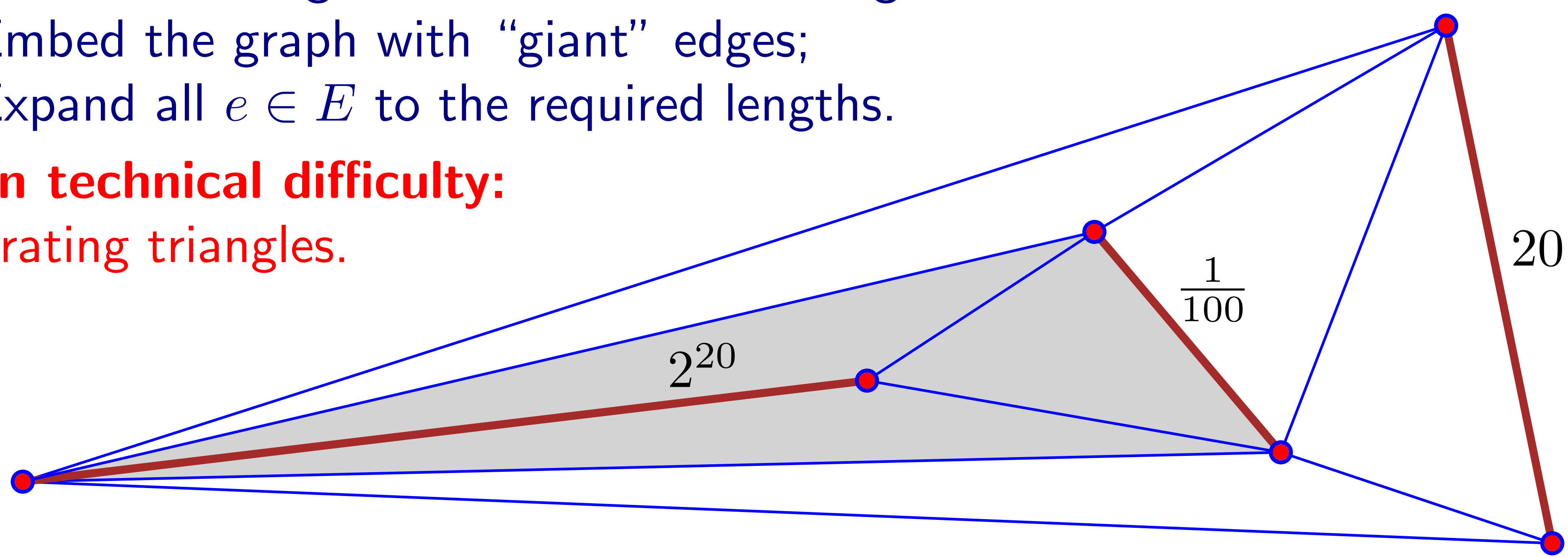

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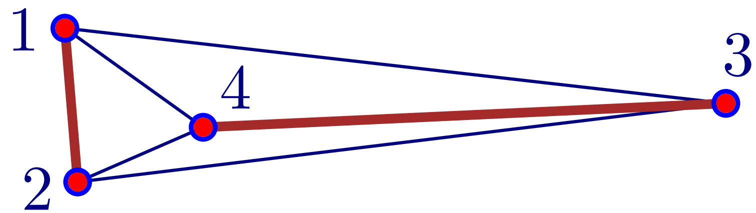

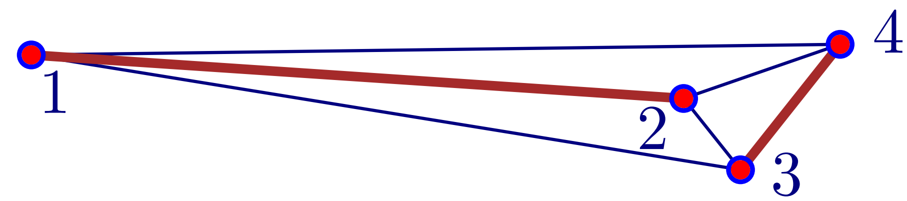

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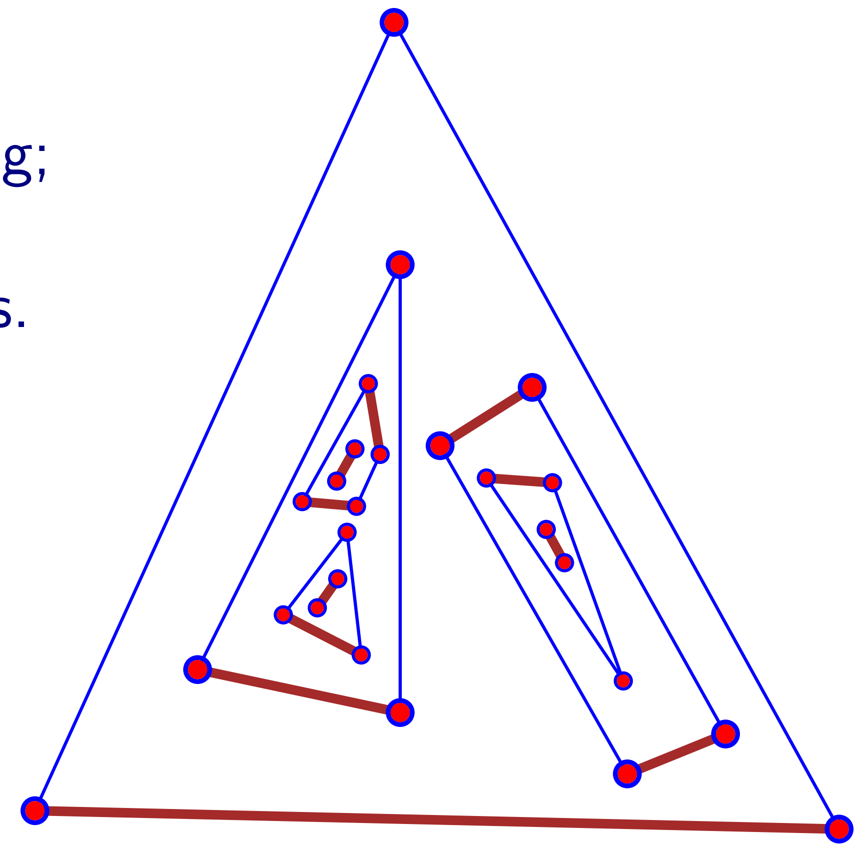
A recursion on the hierarchy of separating triangles *and separating 4-cycles* works, using an appropriate linear transformation in each step.



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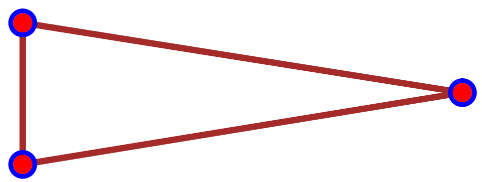

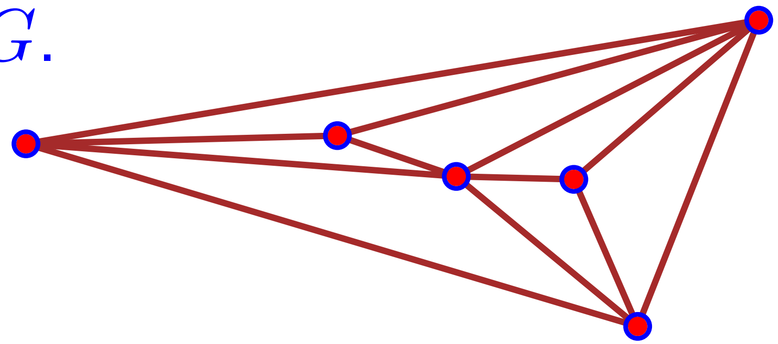

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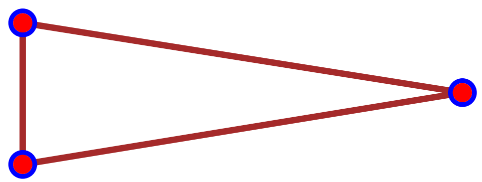
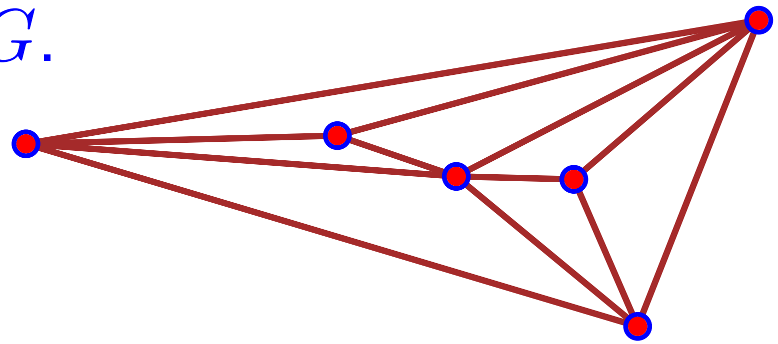
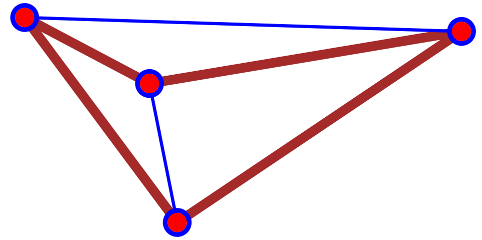
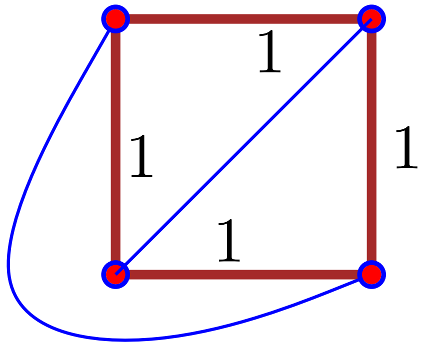




The 4-cycle C_4 is not extrisically free: if all four edges have unit length, then C_4 is a rhombus (i.e., convex), and cannot have an external diagonal.





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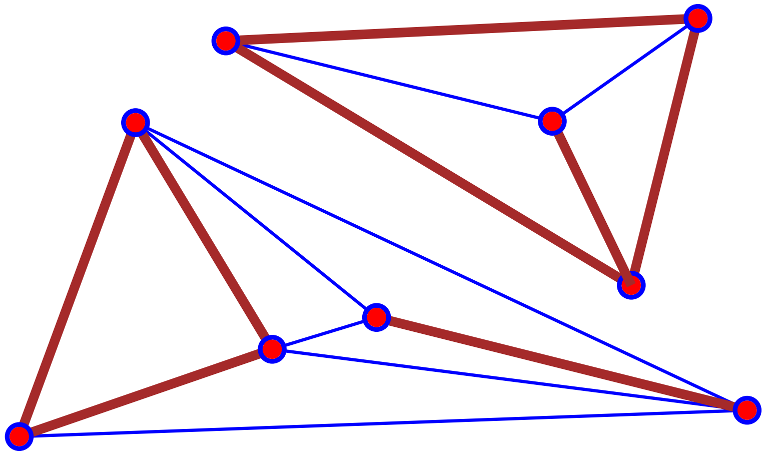

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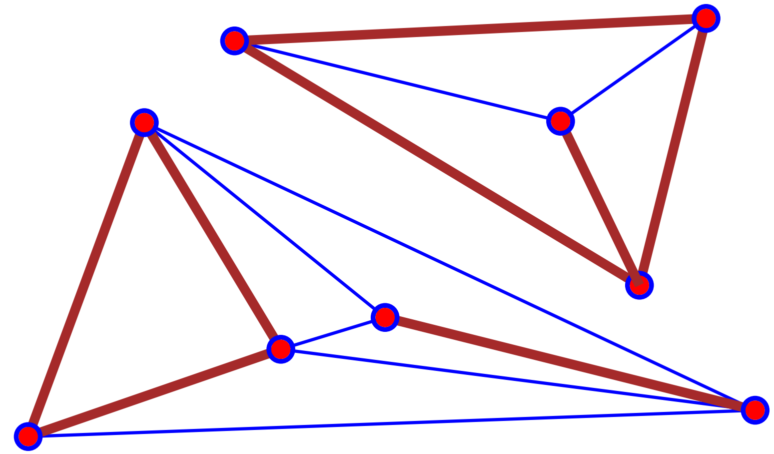
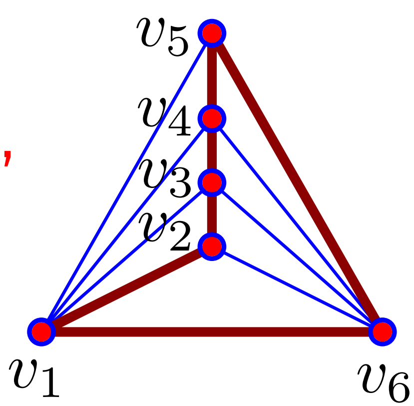
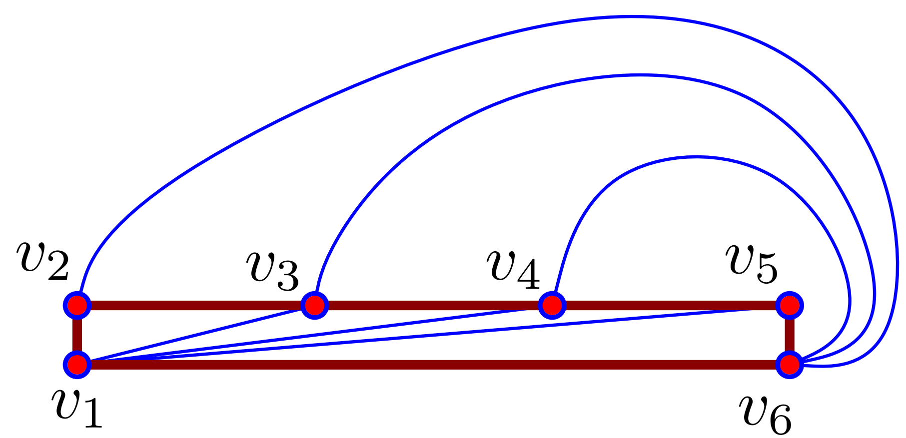


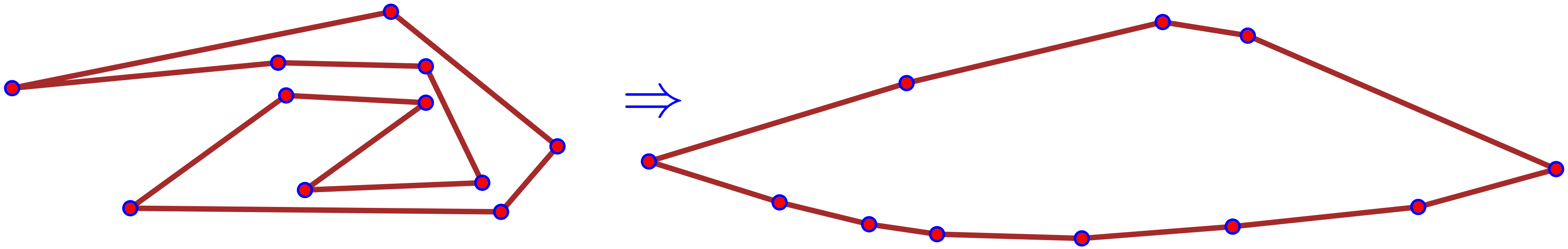
No cycle C_k, $k \geq 4$, is extrinsically free:





“Triangulated” Carpenter’s Rule

Connelly et al. (2003): Every simple polygonal cycle (with fixed edge lengths) can be continuously unfolded into convex position (i.e., its configuration space is connected).



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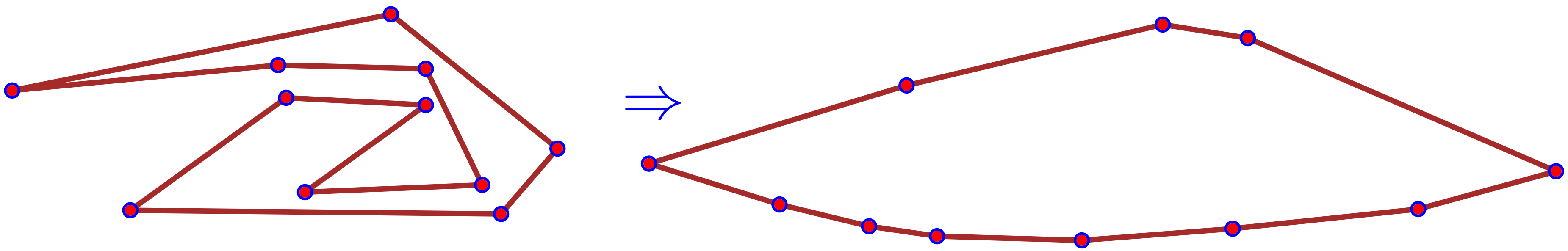
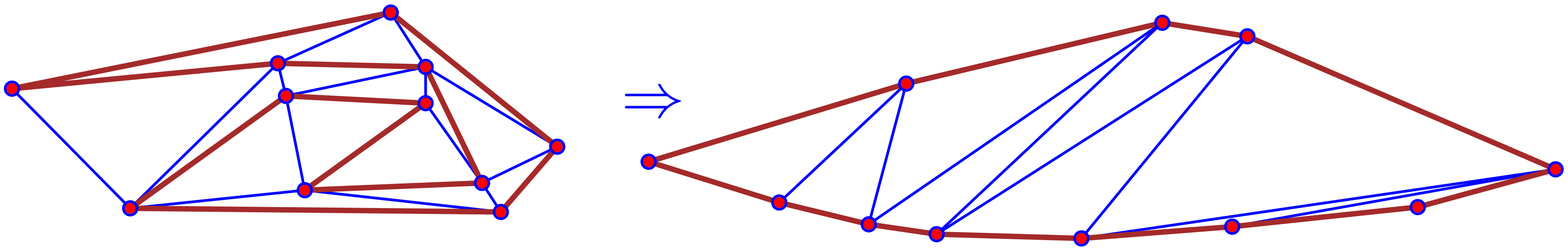
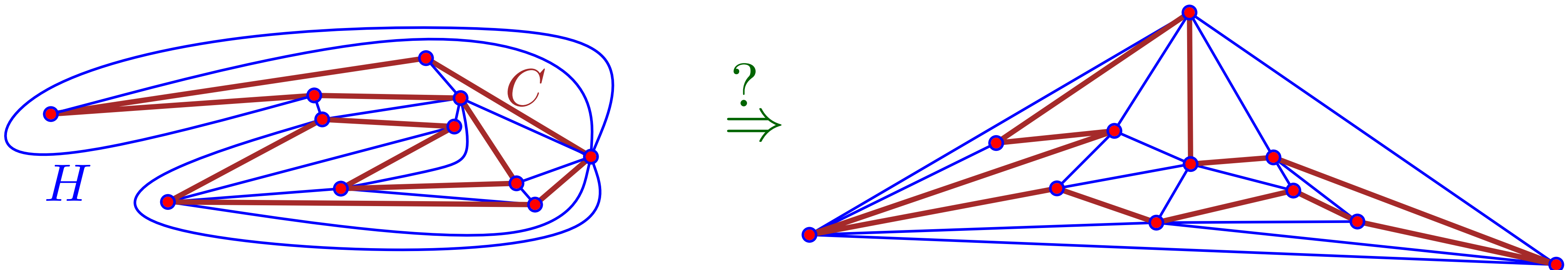


The unfolding algorithm by Streinu maintains a triangulation of C: The edges of the interior triangulation are preserved, and the edges of the exterior triangulation vanish.



“Triangulated” Carpenter’s Rule

Given a **simple polygonal cycle** C and an arbitrary **curvilinear triangulation** H, does H admit a straight-line embedding such that the cycle C keeps its given edge lengths?



"Triangulated" Carpenter's Rule

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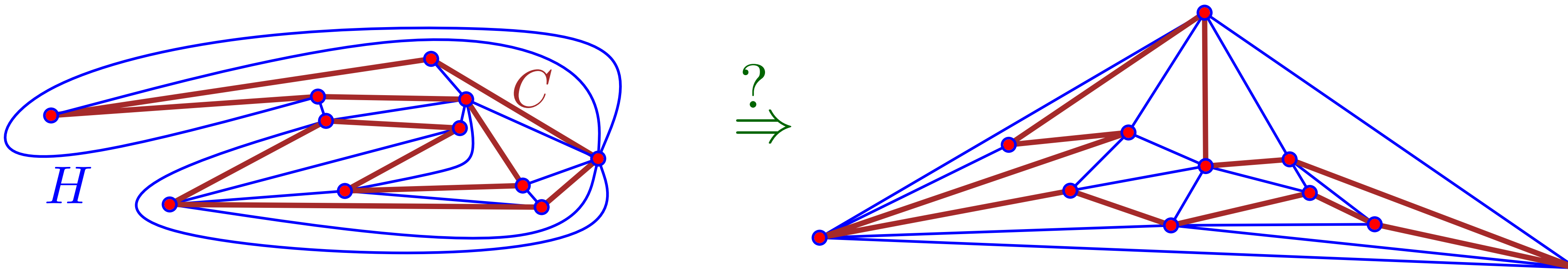
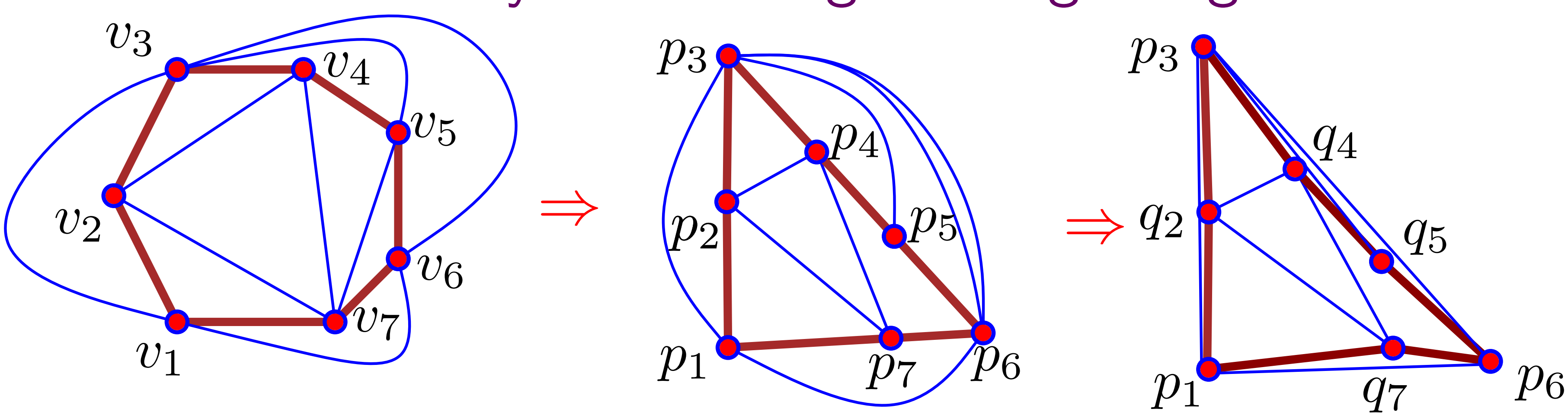


Thm. (Abel et al., 2013): "Yes" if the edge lengths are *nondegenerate*, that is, if the cycle cannot be "flattened" into 1D in two different ways with the given edge lengths.



Thank you!