Relaxations of Graph Partitioning and Vertex Separator Problems using Continuous Optimization

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Motivation: Graph Partitioning/Vertex Separator

- Model NP-hard problem using quadratic-quadratic program
- approximate/relax using eigenvalue bounds and semidefinite programming
- bounds: we follow approaches for eigenvalue and projected eigenvalue bounds in:
  Hadley, Rendl, W. 1990 [1, 5]
  Rendl, Lisser, Piacenti, (RLP) 2012 [4]
  and
  Semidefinite bounds in: W., Zhao 1996 [6].

Background/Notation

Given graph G and set sizes m

- $G = (N, E)$ edge-weighted undirected graph
  $N = \{1, 2, \dots, n\}$ node set
  $E_{ij}, ij = 1, 2, \dots, n$ edge weights
  $m = \begin{pmatrix} m_1 \\ \dots \\ m_k \end{pmatrix}$ (pos. integer) set sizes, with $m^T e = n$

Set of all Partitions $P_m =$

$$\{(S_1, \dots, S_k) : \quad S_i \subset N, \ |S_i| = m_i \ \forall i; \quad S_i \cap S_j = \emptyset \ \forall i \neq j; \ \cup_i S_i = N\}$$

Partition matrix $X \in \mathbb{R}^{n \times k}$; col. $X_{:j}$ incidence vector of S_j

$$X_{ij} = \begin{cases} 1 & \text{if } i \in S_j \\ 0 & \text{otherwise.} \end{cases}$$

Partition Matrix Constraints

linear/quadratic constraints (many are redundant)

set of: zero-one; nonnegative; linear equalities; m-diagonal orthogonality type; e-diagonal orthogonality type; and gangster constraints, respectively:

$$
\begin{array}{rcl}
\mathcal{Z} & := & \{X \in \mathbb{R}^{n\times k} : X_{ij} \in \{0,1\}, \forall ij\} \\
 & & = \{X \in \mathbb{R}^{n\times k} : X_{ij}^2 = X_{ij}, \forall ij\} \\
\mathcal{N} & := & \{X \in \mathbb{R}^{n\times k} : X_{ij} \geq 0, \forall ij\} \\
\mathcal{E} & := & \{X \in \mathbb{R}^{n\times k} : Xe = e, X^T e = m\} \\
 & & = \{X \in \mathbb{R}^{n\times k} : \|Xe - e\|^2 + \|X^T e - m\|^2 = 0\} \\
\mathcal{D}_O & := & \{X \in \mathbb{R}^{n\times k} : X^T X = \mathrm{Diag}\,(m)\} \\
\mathcal{D}_e & := & \{X \in \mathbb{R}^{n\times k} : \mathrm{diag}\,(XX^T) = e\} \\
\mathcal{G} & := & \{X \in \mathbb{R}^{n\times k} : X_{:i} \circ X_{:j} = 0, \forall i \neq j\}, \quad \circ \text{ Hadamard prod.}
\end{array}
$$

Equivalent Representations of Partition Matrices

The set of partition matrices in $\mathbb{R}^{nk}$, $\mathcal{M}_m =$

$$
\begin{aligned}
\mathcal{M}_m &= \mathcal{E} \cap \mathcal{Z} \\
&= \mathcal{E} \cap \mathcal{D}_O \cap \mathcal{N} \\
&= \mathcal{E} \cap \mathcal{D}_O \cap \mathcal{D}_e \cap \mathcal{N} \\
&= \mathcal{E} \cap \mathcal{Z} \cap \mathcal{D}_O \cap \mathcal{G} \cap \mathcal{N}
\end{aligned}
$$

Cuts and Separators, Objective

Cut of a partition

- $\delta(S_i, S_j)$ - set of edges between sets S_i, S_j
- $\delta(S) = \cup_{i<j<k}\delta(S_k, S_j)$ - set of edges with endpoints in distinct partition sets $S_1, \dots, S_{k-1}$
- The minimum of the cardinality $|\delta(S)|$ is denoted (objective) $\quad \text{cut}(m) = \min\{|\delta(S)| : S \in P_m\}$

$\mathcal{G}$ has a vertex separator

graph G has a vertex separator if there exists $S \in P_m$ with $\delta(S) = \emptyset$, i.e., $\text{cut}(m) = 0$.
(see (RLP) [4], Hager, Hungerford 2013 [2] for relationship with bandwidth of graph and other applications)

Trace Representation of Cut Problem

- $B := \begin{bmatrix} ee^T - I_{k-1} & 0 \\ 0 & 0 \end{bmatrix} \in \mathbb{S}^k$,
  $\mathbb{S}^k$ - $k \times k$ symm. matrices with trace inner-product.
- $A = (a_{ij})$ - adjacency matrix, $a_{ij} = \begin{cases} 1 & \text{if } E_{ij} \neq 0 \\ 0 & \text{otherwise} \end{cases}$
- $L := \mathrm{Diag}\,(Ae) - A = \sum_{ij \in E(G)} (e_i - e_j)(e_i - e_j)^T$ - Laplacian (e_i unit vectors)

Quadratic objective for cut(m)

Proposition RLP [4, Prop. 2] For partition $S \in P_m$, and associated partition matrix $X \in \mathcal{M}_m$, the cardinality of the partition is
$|\delta(S)| = \frac{1}{2}\,\mathrm{trace}\,AXBX^T = \frac{1}{2}\,\mathrm{trace}(-L)XBX^T$ □

Basic Eigenvalue Bound

Relaxed problem

$$\begin{array}{rcll} \mathrm{cut}(m) & \geq & p^*_{eig}(m) & \\ & := & \min & \frac{1}{2}\,\mathrm{trace}\, AXBX^T \quad (A \text{ or } -L) \\ & & \text{s.t.} & X \in \mathcal{D}_O \end{array}$$

$\mathcal{D}_O = \{X \in \mathbb{R}^{n \times k} : X^T X = M := \mathrm{Diag}\,(m)\}$
(orthogonal type cols for X)

Hoffman-Wielandt '53 [3] bound/Theorem

C, D symmetric order n, k, resp., $k \leq n$. Then
$\min \left\{\mathrm{trace}\, CXDX^T : X^T X = I_k\right\} =$
$\min \left\{\sum_{i=1}^k \lambda_i(D)\lambda_{\phi(i)}(C) : \phi : N \to \{1, \ldots, k\} \text{ is an injection}\right\}.$
minimum attained for $X = \left(p_{\phi(1)}, \ldots, p_{\phi(k)}\right) Q^T$, where $p_{\phi(i)}$ normalized eigenvector to $\lambda_{\phi(k)}(C)$ and cols of $Q = \left[q_1 \quad \ldots \quad q_k\right]$ contains normalized eigenvectors q_i of $\lambda_i(D)$.

Basic Eigenvalue Bound II

Lemma (RLP)

k-ordered eigs of $\tilde{B} := M^{1/2}BM^{1/2}$ satisfy

$$\lambda_1(\tilde{B}) \leq \lambda_2(\tilde{B}) \leq \ldots \leq \lambda_{k-2}(\tilde{B}) < \lambda_{k-1}(\tilde{B}) = 0 < \lambda_k(\tilde{B}).$$

Basic Eigenvalue Bound, apply Hoffman-Wielandt **Theorem**

Let $-\lambda_1(L) \geq -\lambda_2(L) \geq -\lambda_n(L)$ denote ordered n eigenvalues of $-L$; $-\lambda(L)$ denotes corresponding vector of eigenvalues. Pad the 0 eigenvalue of $\tilde{B}$ with further zeros to get an ordered vector of length n and denote it by $\hat{\lambda}(\tilde{B})$. Then

$$\text{cut}(m) \geq 0 > p^*_{eig} = -\lambda(L)^T \hat{\lambda}(\tilde{B})$$

Two Projected Eigenvalue Bound

Relaxed problem

$$\begin{array}{rclcl} \text{cut}(m) & \geq & p^*_{projeig}(m) & & \\ & := & \min & \frac{1}{2}\,\text{trace}\, AXBX^T & (A \text{ or } -L) \\ & & \text{s.t.} & X \in \mathcal{D}_O \cap \mathcal{E} & \end{array}$$

$\mathcal{D}_O = \{X \in \mathbb{R}^{n\times k} : X^TX = M := \mathrm{Diag}\,(m)\}$ (orthog type)

$\mathcal{E} = \{X \in \mathbb{R}^{n\times k} : Xe = e, X^Te = m\}$ (linear row/col sums)

Special Parametrization of $X \in \mathcal{E}$

$\tilde{m} = \sqrt{m}$; $n \times n, k \times k$ orthogonal matrices P, Q

$$P = \begin{bmatrix} \frac{1}{\sqrt{n}} e & V \end{bmatrix} \in \mathcal{O}_n, \quad Q = \begin{bmatrix} \frac{1}{\sqrt{n}} \tilde{m} & W \end{bmatrix} \in \mathcal{O}_k. \qquad (*)$$

LEMMA: Rendl and W. 1990 [5]

Let $\tilde{M} = \mathrm{Diag}(\tilde{m})$. Suppose that $X \in \mathbb{R}^{n \times k}$ and $Z \in \mathbb{R}^{(n-1)\times(k-1)}$ are related by

$$X = P \begin{bmatrix} 1 & 0 \\ 0 & Z \end{bmatrix} Q^T \tilde{M}. \qquad (*)$$

Then the following holds:

1. $X \in \mathcal{E}$.
2. $X \in \mathcal{N} \Leftrightarrow VZW^T \geq -\frac{1}{n} e \tilde{m}^T$
3. $X \in \mathcal{D}_O \Leftrightarrow Z \in \mathcal{O}_{(n-1)\times(k-1)}$

Conversely, if $X \in \mathcal{E}$, then there exists Z such that the representation (*) holds. □

Projected Eigenvalue Lower Bounds/THEOREM

THEOREM

V, W as above, $\hat{X} := \frac{1}{n} e m^T \in \mathbb{R}^{n\times k}$

$\mathcal{Q} : \mathbb{R}^{(n-1)\times(k-1)} \to \mathbb{R}^{n\times k}$, $\mathcal{Q}(Z) = VZW^T\tilde{M}$

Then:

$\hat{X} \in \mathcal{E}$, and $\mathcal{Q}$ is invertible $\mathbb{R}^{(n-1)\times(k-1)} \leftrightarrow \mathcal{E} - \hat{X}$

Equivalently, $\mathcal{E}$ can be parametrized using $\hat{X} + VZW^T\tilde{M}$.

Thus, two objective functions

$\frac{1}{2}\,\text{trace}\, AXBX^T =$

$\frac{1}{2}\,\text{trace}(A\hat{X}B\hat{X}^T + (V^TAV)Z(W^T\tilde{M}B\tilde{M}W)Z^T + 2V^TA\hat{X}B\tilde{M}WZ^T)$

and

$\frac{1}{2}\,\text{trace}((-L)XBX^T) = \frac{1}{2}\,\text{trace}(V^T(-L)V)Z(W^T\tilde{M}B\tilde{M}W)Z^T.$

Two Projected Eigenvalue Bounds

Let
$\hat{A} = V^T A V$, $\hat{L} = V^T(-L)V$, $\hat{B} = W^T \tilde{M} B \tilde{M} W$,
$\alpha = \operatorname{trace} A \hat{X} B \hat{X}^T$, $C = 2V^T A \hat{X} B \tilde{M} W$.
Then:

$$
\begin{aligned}
\operatorname{cut}(m) \geq p^*_{projeig,A} &= \frac{1}{2}\left\{\alpha + \min_{\phi \text{ injective}} \left\{\Sigma_{i=1}^k \lambda_i(\hat{B})\lambda_{\phi(i)}(\tilde{A})\right\} + \min_{0 \leq \hat{X} + VZW^T\tilde{M}} \operatorname{trace} CZ^T\right\} \\
&\geq p^*_{eig}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{cut}(m) \geq p^*_{projeig,L} &= \frac{1}{2} \min_{\phi \text{ injective}} \left\{\Sigma_{i=1}^k \lambda_i(\hat{B})\lambda_{\phi(i)}(\tilde{L})\right\} \\
&\geq p^*_{eig},
\end{aligned}
$$

and note eigenvalues of $V^T L V$ are $n-1$ nonzero eigenvalues of L.

Attainment for Quadratic Terms

let $Q \in \mathbb{R}^{k-1 \times k-1}$ be orthog. with cols consisting of eigenvectors of $\hat{B}$ corresponding to eigenvalues of $\hat{B}$ in nondecreasing order;
let $P_A, P_L \in \mathbb{R}^{n-1 \times k-1}$ have orthonormal cols consisting of $k-1$ eigenvectors of $\hat{A}, \hat{L}$, respectively, corresponding to eigenvalues in nonincreasing order where the columns correspond to the largest $k-2$ followed by the smallest. Then the minimal scalar product terms in $p^*_{projeig,A}, p^*_{projeig,L}$ are attained by resp.

$$Z_A = P_A Q^T, Z_L = P_L Q^T.$$

Get two approx. solutions using $\mathcal{Q}$:

$$X_A = \hat{X} + V Z_A W^T \tilde{M}, \quad X_L = \hat{X} + V Z_L W^T \tilde{M},$$

Feasible Solutions; Upper Bounds

Using an approx. solution $\bar{X}$

Find nearest (Frobenius norm) feas. soln (use strong polytime LP)
Recall: $X \in \mathcal{E} \cap \mathcal{Z}$ implies that $Xe = e, X^T e = m$, and $X^T X = \mathrm{Diag}\,(m)$. Therefore:

$$\begin{aligned} \|\bar{X} - X\|_F^2 &= \operatorname{trace}\left(\bar{X}^T \bar{X} + X^T X - 2\bar{X}^T X\right) \\ &= n + n + 2\operatorname{trace}\left(-\bar{X}^T X\right). \end{aligned}$$

Finding nearest feasible solution; a strong polytime LP

Solve the transportation problem:

$$\begin{array}{ll} \max & = \operatorname{trace} \bar{X}^T X \\ \text{s.t.} & Xe = e \\ & X^T e = m \\ & X \geq 0 \end{array}$$

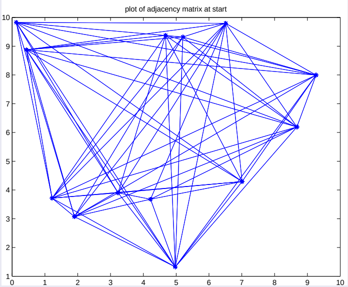
Node-Arcs for a Random Adjacency Matrix

node *i*											
1	2	3	4	5	7	8	9	10	11	12	13
2	3	4	8	9	10	11	12	13	14		
3	6	7	8	9	10	11	12	13	14		
4	7	8	9	11	13	14					
5	6	7	9	10	12	13					
6	7	9	10	12	13						
7	8	10	12	13							
8	9	10	11	12	14						
9	10	13	14								
10	11	12	14								
11	12										
12	13	14									

Table: Existing edges node *i* to node *j*

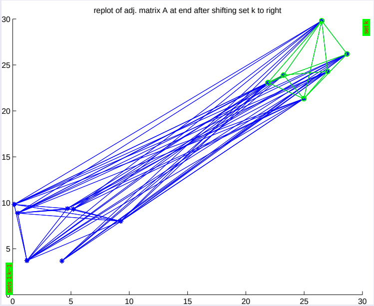
Random Ex.; Proj. Eigenvalue Lower Bound

Adjacency Matrix, $m = \begin{pmatrix} 4 & 2 & 1 & 6 \end{pmatrix}, k = 4, n = 13$

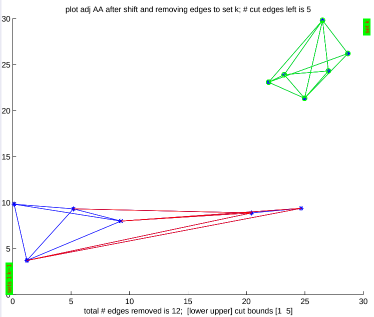


total edges: 61

Adj. after shift set k right



Adj. after delet. edges;
([low up] bnds: [0.76067 5])



Random Problems

$imax = 35$; $k = 6$

n is 144 and m is [28 17 28 32 34 5]
best projection lower and upper bounds are: 5092 5495
relative gap is: 0.076131

n is 94 and m is [3 17 14 32 19 9]
best projection lower and upper bounds are 1672 1890
relative gap is 0.1224

$imax = 35$; $k = 8$

n is 188 and m is [31 27 26 34 7 6 35 22]
best projection lower and upper bounds are 7558 8285
relative gap is 0.091776

Quadratic-Quadratic Model

An equivalent quadratically constrained quadratic problem

$$\begin{array}{rll} \text{cut}(m) \geq p^*_{SDP} = & \min & \frac{1}{2}\,\text{trace}\, AXBX^T \qquad\qquad (A \text{ or } (-L)) \\ & \text{s.t.} & X \circ X = X \\ & & \|Xe - e\|^2 = 0 \\ & & \|X^T e - m\|^2 = 0 \\ & & X_{:i} \circ X_{:j} = 0 \;\forall i \neq j. \end{array}$$

where $\circ$ is the Hadamard (elementwise) product

Quadratic Model

We can use the various equality (quadratic) constraints in the representation and use the quadratic objective function. The Lagrangian relaxation for this quadratic-quadraitc problem is equivalent to a semidefinite program, SDP. The dual of this is the SDP relaxation. Adding redundant constraints can help.

Alternatively: directly by lifting process

linearize quadratic terms using the matrix

$$Y_X := \begin{pmatrix} 1 \\ \mathrm{vec}\,(X) \end{pmatrix} (1 \ \ \mathrm{vec}\,(X)^T),$$

$\mathrm{vec}\,(X)$ is vector formed from the columns of X.
$Y_X \succeq 0$ and is rank one, the hard constraint that is relaxed.

SDP Relaxation

From direct lifting (can use A or $-L$?)

$\operatorname{trace} AXBX^T = \langle AXB, X\rangle = \operatorname{vec}(X)^T(\operatorname{vec} AXB) =$
$\operatorname{vec}(X)^T(B \otimes A)\operatorname{vec}(X) = \operatorname{trace}(B \otimes A)(\operatorname{vec}(X)\operatorname{vec}(X)^T)$

The objective function becomes $\operatorname{trace} AXBX^T = \operatorname{trace} L_A Y_X$,

$$L_A := \begin{bmatrix} 0 & 0 \\ 0 & B \otimes A \end{bmatrix}$$

$B \otimes A$ is the Kronecker product

Relax the rank one restriction

$$(RGP) \qquad \begin{array}{rll} \operatorname{cut}(m) \geq p^*_{SDP} := \min & \operatorname{trace} L_A Y & \\ \text{s.t.} & \operatorname{arrow}(Y) = e_0 & \\ & \operatorname{trace} D_1 Y = 0 & \\ & \operatorname{trace} D_2 Y = 0 & \\ & \mathcal{G}_J(Y) = 0 & \\ & Y_{00} = 1 & \\ & Y \succeq 0, & \end{array}$$

arrow operator

acting $(kn+1) \times (kn+1)$ matrix Y

$$\text{arrow}(Y) := \operatorname{diag}(Y) - (0, Y_{0,1:kn})^T$$

represents the $0, 1$ constraints; guarantees diagonal and 0-th row (or column) are identical;

Gangster operator $\mathcal{G}_J : \mathcal{S}_{kn+1} \to \mathcal{S}_{kn+1}$

shoots "holes" in a matrix

$$(\mathcal{G}_J(Y))_{ij} := \begin{cases} Y_{ij} & \text{if } (i,j) \text{ or } (j,i) \in J \\ 0 & \text{otherwise,} \end{cases}$$

$$J := \left\{ (i,j) : i = (p-1)n + q,\ j = (r-1)n + q, \quad \text{for} \begin{array}{l} p < r,\ p,r \in \{1,\dots,k\} \\ q \in \{1,\dots,n\} \end{array} \right\}$$

represents the (Hadamard) orthogonality of the cols

The norm constraints

represented by the $(kn+1)\times(kn+1)$ matrices

$$D_1 := \begin{bmatrix} n & -e_k^T \otimes e_n^T \\ -e_k \otimes e_n & (e_k e_k^T) \otimes I_n \end{bmatrix}$$

and

$$D_2 := \begin{bmatrix} \bar{m}^T \bar{m} & -\bar{m}^T \otimes e_n^T \\ -\bar{m} \otimes e_n & I_k \otimes (e_n e_n^T) \end{bmatrix}.$$

Loss of Slater's condition

all $D_1, D_2, Y \succeq 0$,

both $\text{trace } YD_1 = 0, \text{trace } YD_2 = 0$; therefore, range of Y subset intersection of nullspaces of D_1, D_2.

feasible set of (RGP) has no strictly feasible points; implies numerical difficulties for interior-point methods.

Fix: apply facial reduction.

Facial Reduction; $Y = \hat{V} Z \hat{V}^T \in \mathbb{S}^{kn+1}$, $Z \in \mathbb{S}^{(n-1)(k-1)+1}$

$V_j \in \mathbb{R}^{j \times j-1}$

$V_j e = 0, V_j^T V_j = \mathrm{Diag}(w) \succ 0$, e.g.,

$$V_j := \begin{bmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \dots & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & 1 \\ -1 & \dots & \dots & 0 & -1 \end{bmatrix}_{j \times (j-1)} .$$

Range of $\hat{V}$ forms basis for range (any) $\hat{Y} \in \operatorname{relint} F$

$$\hat{V} := \begin{bmatrix} 1 & 0 \\ \frac{1}{n} m \otimes e_n & V_k \otimes V_n \end{bmatrix}$$

Facially Reduced SDP

Constraints for $X \in \mathcal{E}$ eliminated; $Z \in \mathbb{S}^{(n-1)(k-1)+1}$

$$
\begin{array}{rl}
\min & \operatorname{trace} \hat{V}^T L_A \hat{V} Z \\
\text{s.t.} & \operatorname{arrow}(\hat{V} Z \hat{V}^T) = 0 \\
& \mathcal{G}_J(\hat{V} Z \hat{V}^T) = 0 \\
& (\hat{V} Z \hat{V}^T)_{00} = 1 \\
& Z \succeq 0
\end{array}
$$

Slater's CQ now holds (strict feasibility).
But are we done? Are the constraints onto?

Final SDP; Slater and Onto (range of $\mathcal{G}$) Constraints

Projected onto range of gangster; $\bar{J} = J \cup (0,0)$

$$\begin{array}{ll} \min & \operatorname{trace}\left(\hat{V}^T L_A \hat{V}\right) Z \\ \text{s.t.} & \mathcal{G}_{\bar{J}}(\hat{V} Z \hat{V}^T) = \mathcal{G}_{\bar{J}}(E_{00}) \\ & Z \succeq 0 \end{array}$$

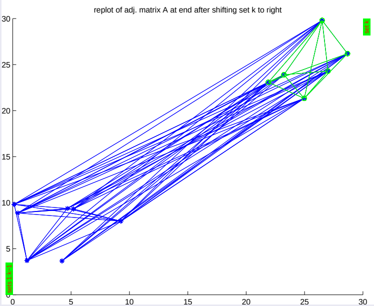
Dual program (also satisfies Slater)

$$\begin{array}{ll} \max & W_{00} \\ \text{s.t.} & \hat{V}^T \mathcal{G}_{\bar{J}}(W) \hat{V} \preceq \hat{V}^T L_A \hat{V} \end{array}$$

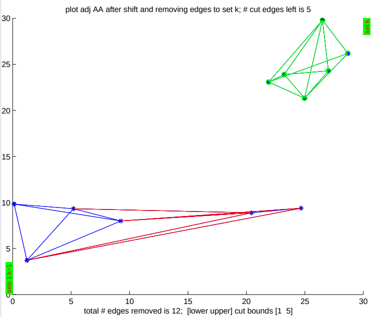
Doubly Nonnegative

A stronger relaxation adds the nonnegativity elementwise: $\hat{V} Z \hat{V}^T \geq 0$.

Adj. after shift set k right



Adj. after delet. edges;
([low up] bnds: [0.76067 5])



lower bnds: [Proj L and A; SDP; Doubly Nonneg.]

$\begin{bmatrix} -0.52065 & 0.76067 & 2.9057 & 4.8603 \end{bmatrix}$

rounded up: $\begin{bmatrix} 0 & 1 & 3 & 5 \end{bmatrix}$.

Therefore, 5 is optimal value.

Random Ex; $n = 85, k = 6, m = [18\ 20\ 11\ 18\ 11\ 7]$

Proj. Eig. Bounds

n is 85 and m is 18 20 11 18 11 7
best projection lower and upper bounds are 1518 1714
relative gap is 0.12129

SDP Bounds

sdp lower and upper bounds are 1556 1726
current best lower/upper bounds are: 1556 1714
relative gap is 0.096636

Summary

- Model NP hard problems using quadratic-quadratic models
- First Relaxations lead to eigenvalue problems
- Lagrangian Relaxation leads to SDP problem and the dual is the SDP (strong) relaxation
- The Slater condition typically fails for SDP relaxations (facial reduction is needed for stability)

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Thanks for your attention!

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