A probabilistic weak formulation of mean field games and applications

Daniel Lacker (joint work with René Carmona)

Department of Operations Research & Financial Engineering
Princeton University

August 27, 2013

Price impact model

- n brokers trade in the same asset and maximize wealth.
- Brokers face identical limit order books.
- Broker i controls his rate of trade α_t^i.
- The asset price is a martingale plus a drift given by price impact. (Almgren-Chriss '01, Carlin et al '09)

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$$dK_t^i = -(\alpha_t^i S_t + c(\alpha_t^i))dt$$
$$dX_t^i = \alpha_t^i dt + \sigma dW_t^i$$

- Asset price:

$$dS_t = \frac{\gamma}{n} \sum_{j=1}^{n} c'(\alpha_t^j)dt + \sigma_0 dB_t$$

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- Broker i's wealth is $V_t^i = V_0^i + X_t^i S_t + K_t^i$, or

$$dV_t^i = \left(\frac{\gamma}{n} \sum_{j=1}^{n} c'(\alpha_t^j) X_t^i - c(\alpha_t^i) \right) dt + \sigma S_t dW_t^i + \sigma_0 X_t^i dB_t$$

Price impact model - optimization

Broker i maximizes expected wealth $\mathbb{E}[V^i_T]$:

$$\sup_{\alpha^i} \mathbb{E} \int_0^T \left(\frac{\gamma}{n} \sum_{j=1}^n c'(\alpha^j_t) X^i_t - c(\alpha^i_t) \right) dt,$$

$$\text{s.t. } dX^i_t = \alpha^i_t dt + \sigma dW^i_t$$

Are there Nash equilibria?

Price impact model - general objectives

Additional objective functions G and F allow for time-T liquidation demands, tracking requirements, etc.

$$\sup_{\alpha^i} \mathbb{E}\left[G(X_T^i) + \int_0^T \left(\frac{\gamma}{n}\sum_{j=1}^n c'(\alpha_t^j)X_t^i - c(\alpha_t^i) - F(t, X_t^i)\right) dt\right],$$

$$\text{s.t. } dX_t^i = \alpha_t^i dt + \sigma dW_t^i$$

The optimization problems are coupled through the empirical distribution of the controls. Limit $n \to \infty$?

Mean field stochastic differential games (MFG)

Player i's state process and objective:

$$\sup_{\alpha^i} \mathbb{E}\left[\int_0^T f(t, X_t^i, \overline{\mu}_t^n, \overline{\nu}_t^n, \alpha_t^i)dt + g(X_T^i, \overline{\mu}_T^n)\right],$$

$$\text{s.t. } dX_t^i = b\left(t, X_t^i, \overline{\mu}_t^n, \alpha_t^i\right) dt + \sigma(t, X_t^i) dW_t^i,$$

$$\overline{\mu}_t^n := \frac{1}{n}\sum_{j=1}^n \delta_{X_t^j}, \quad \overline{\nu}_t^n := \frac{1}{n}\sum_{j=1}^n \delta_{\alpha_t^j}$$

We study the mean field limit, as proposed (with no $\overline{\nu}_t^n$ dependence) by Lasry & Lions and independently by Caines, Huang, & Malhamé in '06.

Limit $n \to \infty$

1. Fix measure flows $t \mapsto (\mu_t, \nu_t)$
2. Solve a standard optimal control problem

$$\sup_{\alpha} \mathbb{E}\left[\int_0^T f(t, X_t, \mu_t, \nu_t, \alpha_t)dt + g(X_T, \mu_T)\right], \text{ s.t.}$$
$$dX_t = b\,(t, X_t, \mu_t, \alpha_t)\,dt + \sigma(t, X_t)dW_t,$$

3. Let μ'_t denote the law of the optimally controlled state process at time t and ν'_t the law of the optimal control at time t.
4. Find a fixed point $(\mu'_t, \nu'_t) = (\mu_t, \nu_t)$.

Existence and uniqueness theory

Any approach to stochastic optimal control may be applied in step 2.

1. PDEs - Lasry & Lions, Caines et al.
2. Stochastic maximum principle - Carmona & Delarue, Bensoussan et al.
3. Weak formulation

Assumptions

- compact convex control space A
- admissible controls $\mathbb{A}$ = progressively measurable A-valued processes
- b, f, g jointly measurable and continuous in (μ, ν, a), at points where $\mu \sim$ Lebesgue
- σ is measurable and bounded away from zero, and there exists a unique strong solution to $dX_t = \sigma(t, X_t)dW_t$
- b, σ bounded
- some growth assumptions for f and g

Setup

On $(\Omega, \mathcal{F}, P)$, solve $dX_t = \sigma(t, X_t)dW_t$. For (μ, α) fixed,

$$\frac{dP^{\mu,\alpha}}{dP} := \exp\left[\int_0^T \sigma^{-1}b(t, X_t, \mu_t, \alpha_t)dW_t - \frac{1}{2}\int_0^T |\sigma^{-1}b(t, X_t, \mu_t, \alpha_t)|^2 dt\right]$$

Under $P^{\mu,\alpha}$, X is a weak solution of the state equation,

$$dX_t = b(t, X_t, \mu_t, \alpha_t)dt + \sigma(t, X_t)dW_t^{\mu,\alpha}$$

Let $\Phi(\mu, \alpha)_t = (P^{\mu,\alpha} \circ X_t^{-1}, P^{\mu,\alpha} \circ \alpha_t^{-1})$.

Setup

For (μ, ν) fixed, define the value function

$$V_t^{\mu,\nu} := \operatorname*{ess\,sup}_{\alpha \in \mathbb{A}} \mathbb{E}^{P^{\mu,\alpha}} \left[\int_t^T f(t, X_t, \mu_t, \nu_t, \alpha_t) dt + g(X_T, \mu_T) \middle| \mathcal{F}_t \right]$$

Hamiltonian and maximized Hamiltonian:

$$h(t, x, \mu, \nu, z, a) := f(t, x, \mu, \nu, a) + z \cdot \sigma^{-1} b(t, x, \mu, a),$$
$$H(t, x, \mu, \nu, z) := \sup_{a \in A} h(t, x, \mu, \nu, z, a)$$

Can show $V_t^{\mu,\nu}$ solves the BSDE

$$\begin{cases} dV_t^{\mu,\nu} & = -H(t, X_t, \mu_t, \nu_t, Z_t^{\mu,\nu}) dt + Z_t^{\mu,\nu} dW_t, \\ V_T^{\mu,\nu} & = g(X_T, \mu_T) \end{cases}$$

By comparison principle, the set of optimal controls is exactly

$$\mathbb{A}(\mu, \nu) := \{\alpha \in \mathbb{A} : \alpha_t \in A(t, X_t, \mu_t, \nu_t, Z_t^{\mu,\nu}) \; dt \times dP - a.e.\}$$
$$A(t, x, \mu, \nu, z) := \arg\max_{a \in A} h(t, x, \mu, \nu, z, a)$$

Existence and uniqueness

A MFG solution is a fixed point

$$(\mu,\nu) \in \Phi(\mu, \mathbb{A}(\mu,\nu)) = \{\Phi(\mu,\alpha) : \alpha \in \mathbb{A}(\mu,\nu)\}$$

Theorem

- *Assume the Hamiltonian h is concave in a and* $f = f_1(t,x,\mu,a) + f_2(t,x,\mu,\nu)$*. Then there exists a fixed point.*
- *Assume the Hamiltonian h is strictly concave in a,* $f = f_1(t,\mu,\nu) + f_2(t,x,a)$*,* $g = g(x)$*, and* $b = b(t,x,a)$*. Then the fixed point is unique.*

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- *Assume the Hamiltonian h is strictly concave in a, $f = f_1(t, \mu, \nu) + f_2(t, x, a)$, $g = g(x)$, and $b = b(t, x, a)$. Then the fixed point is unique.*

Proof.

1. Kakutani's fixed point theorem. 2. Translate Lasry & Lions' proof into probabilistic language. □

Approximate equilibria for the finite-player game

Theorem

If $\alpha = \alpha(t, X_\cdot)$ is an optimal feedback control for the MFG problem, then the strategy profiles $\alpha(t, X^i_\cdot)$ form an approximate Nash equilbrium for the finite-player game - for some $\epsilon_n \downarrow 0$, for each n, no player in the n-player game can increase his expected reward by more than ϵ_n by unilaterally changing strategy.

Price impact model, revisited

Price impact model corresponds to:

- $b(t, x, \mu, \alpha) = \alpha$
- σ constant
- $g(x, \mu) = G(x)$
- $f(t, x, \mu, \nu, \alpha) = \gamma x \int c' d\nu - c(\alpha) - F(t, x)$.

Theorem

For a bounded order book, with c' continuous, the mean field price impact model has a solution. Moreover, the errors ϵ_n are $O(1/\sqrt{n})$.

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- Sub-populations or Types
  - $f(t, x, \mu, \nu, a) = \hat{f}(t, x, F_1(\mu), \ldots, F_m(\mu), \nu, a)$, where $F_i(\mu)(B) := 1_{\{\mu(B_i)>0\}}\mu(B \cap B_i)/\mu(B_i)$ and $B_i \subset \mathbb{R}^d$ have positive Lebesgue measure
  - e.g. income brackets