Optimal investment and contingent claim valuation in illiquid markets

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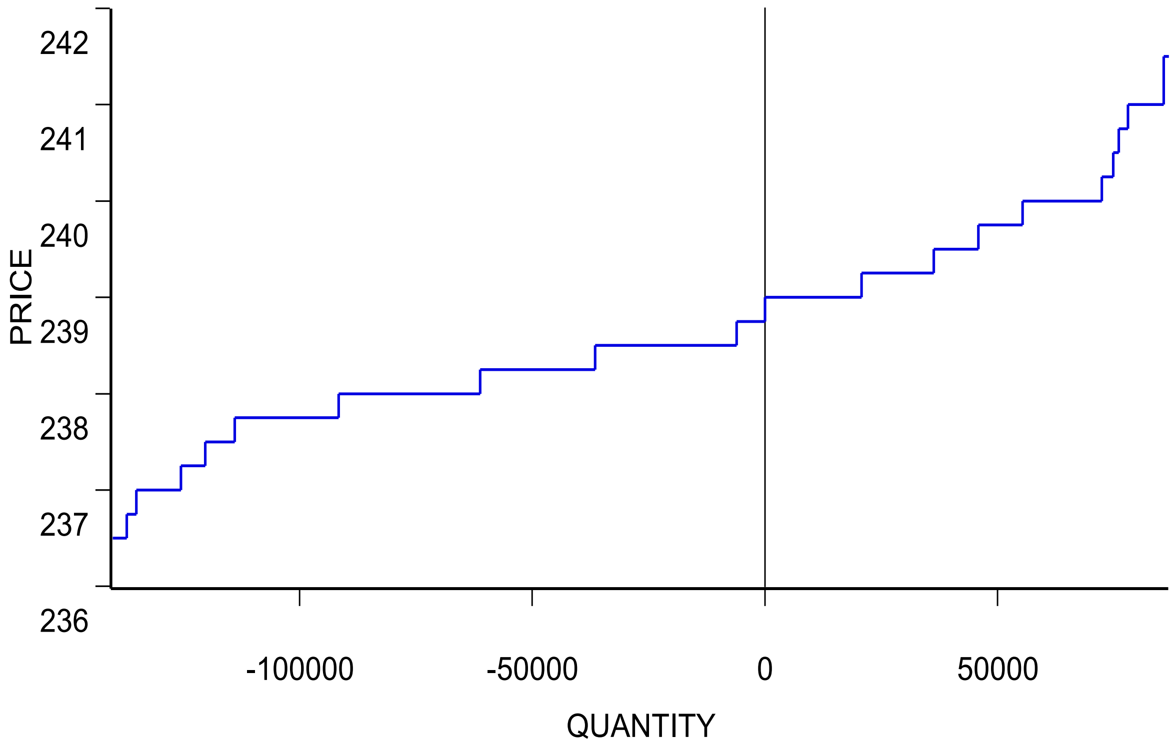
Illiquidity

- The cost of a market orders depends nonlinearly on the traded amount.
- There is no numeraire: much of trading consists of exchanging sequences of cash-flows (swaps, insurance contracts, coupon payments, dividends, . . .)
- We extend basic results on indifference pricing, arbitrage, optimal portfolios and duality to markets with nonlinear illiquidity effects and general swap contracts.

Outline

1. Market model with nonlinear trading costs and portfolio constraints. In particular, existence of a numeraire is not assumed.
2. Optimal investment problem parameterized by a sequence of cash-flows.
3. Indifference pricing extended to general swap contracts.
4. Existence of solutions established under an extended no-arbitrage condition.
5. Dual expressions for the optimal value and swap rates in terms of state price densities that capture uncertainty as well as time-value of money in the absense of a numeraire.

Example 1 (Limit order markets) *The cost of a market order is obtained by integrating the order book.*



Market model

Consider a financial market where a finite set J of assets can be traded at $t = 0, \dots, T$.

- Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t=0}^T, P)$ be a filtered probability space.
- The cost (in cash) of buying a portfolio $x \in \mathbb{R}^J$ at time t in state ω will be denoted by $S_t(x, \omega)$.
- We will assume that
  - $S_t(\cdot, \omega)$ is convex with $S_t(0, \omega) = 0$,
  - $S_t(x, \cdot)$ is $\mathcal{F}_t$-measurable.

  (In particular, S_t is a Carathéodory function and thus, $\mathcal{B}(\mathbb{R}^J) \otimes \mathcal{F}_t$-measurable, so $\omega \mapsto S_t(x_t(\omega), \omega)$ is $\mathcal{F}_t$-measurable when x_t is so.)
- Such a sequence (S_t) will be called a convex cost process.

Market model

Example 2 (Liquid markets) *If* $s = (s_t)_{t=0}^T$ *is an* $(\mathcal{F}_t)_{t=0}^T$*-adapted* $\mathbb{R}^J$*-valued price process, then the functions*

$$S_t(x,\omega) = s_t(\omega) \cdot x$$

define a convex cost process.

Example 3 (Jouini and Kallal, 1995) *If* $(s_t^a)_{t=0}^T$ *and* $(s_t^b)_{t=0}^T$ *are* $(\mathcal{F}_t)_{t=0}^T$*-adapted with* $s^b \le s^a$*, then the functions*

$$S_t(x,\omega) = \begin{cases} s_t^a(\omega)x & \text{if } x \ge 0, \\ s_t^b(\omega)x & \text{if } x \le 0 \end{cases}$$

define a convex cost process.

Market model

Example 4 (Çetin and Rogers, 2007) *If $s = (s_t)_{t=0}^T$ is an $(\mathcal{F}_t)_{t=0}^T$-adapted process and ψ is a lower semicontinuous convex function on $\mathbb{R}$ with $\psi(0) = 0$, then the functions*

$$S_t(x, \omega) = x^0 + s_t(\omega)\psi(x^1)$$

define a convex cost process.

Example 5 (Dolinsky and Soner, 2013) *If $s = (s_t)_{t=0}^T$ is $(\mathcal{F}_t)_{t=0}^T$-adapted and $G_t(x, \cdot)$ are $\mathcal{F}_t$-measurable functions such that $G_t(\cdot, \omega)$ are finite and convex, then the functions*

$$S_t(x, \omega) = x^0 + s_t(\omega) \cdot x^1 + G_t(x^1, \omega)$$

define a convex cost process.

Market model

- We allow for portfolio constraints requiring that the portfolio held over $(t, t+1]$ in state ω has to belong to a set $D_t(\omega) \subseteq \mathbb{R}^J$.
- We assume that
  - $D_t(\omega)$ are closed and convex with $0 \in D_t(\omega)$.
  - $\{\omega \in \Omega \,|\, D_t(\omega) \cap U \neq \emptyset\} \in \mathcal{F}_t$ for every open $U \subset \mathbb{R}^J$.

Market model

- Models where $D_t(\omega)$ is independent of (t,ω) have been studied e.g. in [Cvitanić and Karatzas, 1992] and [Jouini and Kallal, 1995].
- In [Napp, 2003],

$$D_t(\omega) = \{x \in \mathbb{R}^d \mid M_t(\omega)x \in K\},$$

  where $K \subset \mathbb{R}^L$ is a closed convex cone and M_t is an $\mathcal{F}_t$-measurable matrix.
- General constraints have been studied in [Evstigneev, Schürger and Taksar, 2004], [Rokhlin, 2005] and [Czichowsky and Schweizer, 2012].

Optimal investment

Let $c \in \mathcal{M} := \{(c_t)_{t=0}^T \,|\, c_t \in L^0(\Omega, \mathcal{F}_t, P)\}$ and consider the problem

$$\text{minimize} \quad \sum_{t=0}^{T} \mathcal{V}_t(S_t(\Delta x_t) + c_t) \quad \text{over} \quad x \in \mathcal{N}_D$$

- $\mathcal{N}_D = \{(x_t)_{t=0}^T \,|\, x_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R}^J),\ x_t \in D_t,\ x_T = 0\}$,
- $\mathcal{V}_t : L^0 \to \overline{\mathbb{R}}$ are convex, nondecreasing and $\mathcal{V}_t(0) = 0$.

Example 6 *If $\mathcal{V}_t = \delta_{L^0_-}$ for $t < T$, the problem can be written*

$$\begin{array}{lll} \text{minimize} & \mathcal{V}_T(S_T(\Delta x_T) + c_T) \quad \text{over} \quad x \in \mathcal{N}_D \\ \text{subject to} & S_t(\Delta x_t) + c_t \le 0, \quad t = 0, \ldots, T-1. \end{array}$$

Example 7 (Markets with a numeraire) *When*

$$S_t(x,\omega) = x^0 + \tilde{S}_t(\tilde{x},\omega) \quad \textit{and} \quad D_t(\omega) = \mathbb{R} \times \tilde{D}_t(\omega),$$

the problem can be written as

$$\text{minimize} \qquad \mathcal{V}_T\left(\sum_{t=0}^{T} \tilde{S}_t(\Delta\tilde{x}_t) + \sum_{t=0}^{T} c_t\right) \quad \text{over} \quad x \in \mathcal{N}_D.$$

When $\tilde{S}_t(\tilde{x},\omega) = \tilde{s}_t(\omega)\cdot\tilde{x}$*,*

$$\sum_{t=0}^{T} \tilde{S}_t(\Delta\tilde{x}_t) = \sum_{t=0}^{T} \tilde{s}_t \cdot \Delta\tilde{x}_t = -\sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta\tilde{s}_{t+1}.$$

Optimal investment

We denote the optimal value function by

$$\varphi(c) = \inf_{x \in \mathcal{N}_D} \sum_{t=0}^{T} \mathcal{V}_t(S_t(\Delta x_t) + c_t).$$

- When $\mathcal{V}_t = \delta_{L^0_-}$ for $t = 0, \ldots, T$, we have $\varphi = \delta_{\mathcal{C}}$ where

$$\mathcal{C} = \{c \in \mathcal{M} \,|\, \exists x \in \mathcal{N}_D : \; S_t(\Delta x_t) + c_t \leq 0 \quad \forall t\}.$$

  is the set of claims that can be superhedged for free.
- In the classical linear model,

$$\mathcal{C} = \{c \in \mathcal{M} \,|\, \exists x \in \mathcal{N}_D : \; \sum_{t=0}^{T} c_t \leq \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1}\}.$$

- We always have, $\varphi(c) = \inf_{d \in \mathcal{C}} \sum_{t=0}^{T} \mathcal{V}_t(c_t - d_t)$.

Optimal investment

Lemma 8 *The value function φ is convex and*

$$\varphi(\bar{c}+c) \leq \varphi(\bar{c}) \quad \forall \bar{c} \in \mathcal{M},\ c \in \mathcal{C}^\infty.$$

where $\mathcal{C}^\infty = \{c \in \mathcal{M} \,|\, \bar{c} + \alpha c \in \mathcal{C} \quad \forall \bar{c} \in \mathcal{C},\ \forall \alpha > 0\}$.

- In particular, φ is constant with respect to the linear space $\mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$.
- If S_t are positively homogeneous and D_t are conical, then $\mathcal{C}$ is a cone and $\mathcal{C}^\infty = \mathcal{C}$.

Swap contracts

- In a swap contract, an agent receives a sequence $p \in \mathcal{M}$ of premiums and delivers a sequence $c \in \mathcal{M}$ of claims.
- Examples:
  - Swaps with a "fixed leg": $p = (1, \dots, 1)$, random c.
  - In credit derivatives (CDS, CDO, ...) and other insurance contracts both p and c are random.
  - Traditionally in mathematical finance:

$$p = (1, 0, \dots, 0) \quad \text{and} \quad c = (0, \dots, 0, c_T).$$

- Claims and premiums live in the same space

$$\mathcal{M} = \{(c_t)_{t=0}^T \,|\, c_t \in L^0(\Omega, \mathcal{F}_t, P; \mathbb{R})\}.$$

Swap contracts

- If we already have liabilities $\bar{c} \in \mathcal{M}$, then

$$\pi(\bar{c}, p; c) := \inf\{\alpha \in \mathbb{R} \,|\, \varphi(\bar{c} + c - \alpha p) \leq \varphi(\bar{c})\}$$

gives the least swap rate that would allow us to enter a swap contract without worsening our financial position.

- Similarly,

$$\pi^b(\bar{c}, p; c) := \sup\{\alpha \in \mathbb{R} \,|\, \varphi(\bar{c} - c + \alpha p) \leq \varphi(\bar{c})\} = -\pi(\bar{c}, p; -c)$$

gives the greatest swap rate we would need on the opposite side of the trade.

- When $p = (1, 0, \ldots, 0)$ and $c = (0, \ldots, 0, c_T)$, we get a nonlinear version of the indifference price of [Hodges and Neuberger, 1989].

Swap contracts

Define the super- and subhedging swap rates,

$$\pi_{\sup}(c) = \inf\{\alpha \,|\, c-\alpha p \in \mathcal{C}^\infty\},\ \pi_{\inf}(c) = \sup\{\alpha \,|\, \alpha p-c \in \mathcal{C}^\infty\}.$$

In the classical model with $p = (1, 0, \ldots, 0)$, we recover the usual super- and subhedging costs.

Theorem 9 *If* $\pi(\bar{c}, p; 0) \geq 0$, *then*

$$\pi_{\inf}(c) \leq \pi_b(\bar{c}, p; c) \leq \pi(\bar{c}, p; c) \leq \pi_{\sup}(c)$$

with equalities if $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ *for some* $\alpha \in \mathbb{R}$.

- Agents with identical views P, preferences $\mathcal{V}$ and financial position $\bar{c}$ have no reason to trade with each other.
- Prices are independent of such subjective factors when $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ for some $\alpha \in \mathbb{R}$.

Swap contracts

Example 10 (Linear models) *When $S_t(x) = s_t \cdot x$ and $D_t = \mathbb{R}^J$, we have $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ if there is an $x \in \mathcal{N}_D$ such that $s_t \cdot \Delta x_t + c_t = \alpha p_t$. The converse holds under the no-arbitrage condition $\mathcal{C} \cap \mathcal{M}_+ = \{0\}$.*

Example 11 (The classical model) *When $D_t = \mathbb{R}^J$, $S_t(x) = x_0 + \tilde{s}_t \cdot \tilde{x}$ and $p = (1, 0, \dots, 0)$, we have $c - \alpha p \in \mathcal{C}^\infty \cap (-\mathcal{C}^\infty)$ if $\sum_{t=0}^T c_t$ is attainable in the sense that*

$$\sum_{t=0}^{T} c_t = \alpha + \sum_{t=0}^{T-1} \tilde{x}_t \cdot \Delta \tilde{s}_{t+1}$$

for some $\alpha \in \mathbb{R}$ and $x \in \mathcal{N}_D$. The converse holds under the no-arbitrage condition.

Existence of solutions

Given a market model (S, D), let

$$S_t^\infty(x,\omega) = \sup_{\alpha>0} \frac{S_t(\alpha x,\omega)}{\alpha} \quad \text{and} \quad D_t^\infty(\omega) = \bigcap_{\alpha>0} \alpha D_t(\omega).$$

If S is sublinear and D is conical, then $S^\infty = S$ and $D^\infty = D$

Theorem 12 *Assume that $\mathcal{V}_t(c_t) = Ev_t(c_t)$, where v_t are bounded from below. If the cone*

$$\mathcal{L} := \{x \in \mathcal{N}_{D^\infty} \mid S_t^\infty(\Delta x_t) \le 0\}$$

is a linear space, then φ is proper and lower semicontinuous in L^0 and the infimum is attained for every $c \in \mathcal{M}$.

Existence of solutions

Example 13 *In the classical perfectly liquid market model*

$$\mathcal{L} = \{x \in \mathcal{N} \mid s_t \cdot \Delta x_t \leq 0,\ x_T = 0\},$$

so the linearity condition coincides with the no-arbitrage condition. When $v_t = \delta_{\mathbb{R}_-}$*, we have* $\varphi = \delta_{\mathcal{C}}$ *so we recover the key lemma from [Schachermayer, 1992].*

Example 14 *In unconstrained models with proportional transactions costs, the linearity condition becomes the robust no-arbitrage condition introduced by [Schachermayer, 2004] (for claims with physical delivery).*

Existence of solutions

Example 15 *If* $S_t^\infty(x,\omega) > 0$ *for* $x \notin \mathbb{R}^J_-$, *we have* $\mathcal{L} = \{0\}$.

Example 16 *In [Çetin and Rogers, 2007] with*

$$S_t(x,\omega) = x^0 + s_t(\omega)\psi(x^1)$$

one has $S_t^\infty(x,\omega) = x^0 + s_t(\omega)\psi^\infty(x^1)$. *When* $\inf \psi' = 0$ *and* $\sup \psi' = \infty$ *we have* $\psi^\infty = \delta_{\mathbb{R}_-}$, *so the condition in Example 15 holds.*

Example 17 *If* $S_t(\cdot,\omega) = s_t(\omega) \cdot x$ *for a componentwise strictly positive price process* s *and* $D_t^\infty(\omega) \subseteq \mathbb{R}^J_+$ *(infinite short selling is prohibited), we have* $\mathcal{L} = \{0\}$.

Existence of solutions

Proposition 18 *Assume that φ is proper and lower semicontinuous. Then, for every $\bar{c} \in \operatorname{dom}\varphi$ and $p \in \mathcal{M}$, the conditions*

- $\sup_{\alpha>0} \varphi(\alpha p) > \varphi(0)$,
- $\pi(\bar{c}, p; 0) > -\infty$,
- $\pi(\bar{c}, p; c) > -\infty$ *for all* $c \in \mathcal{M}$,

are equivalent and imply that $\pi(\bar{c}, p; \cdot)$ is proper and lower semicontinuous on $\mathcal{M}$ and that the infimum

$$\pi(\bar{c}, p; c) = \inf\{\alpha \,|\, \varphi(\bar{c} + c - \alpha p) \leq \varphi(\bar{c})\}$$

is attained for every $c \in \mathcal{M}$.

Duality

- Let $\mathcal{M}^p = \{c \in \mathcal{M} \,|\, c_t \in L^p(\Omega, \mathcal{F}_t, P; \mathbb{R})\}$.
- The bilinear form

$$\langle c, y \rangle := E \sum_{t=0}^{T} c_t y_t$$

puts $\mathcal{M}^1$ and $\mathcal{M}^\infty$ in separating duality.

- The conjugate of a function f on $\mathcal{M}^1$ is defined by

$$f^*(y) = \sup_{c \in \mathcal{M}^1} \{\langle c, y \rangle - f(c)\}.$$

- If f is proper, convex and lower semicontinuous, then

$$f(y) = \sup_{y \in \mathcal{M}^\infty} \{\langle c, y \rangle - f^*(y)\}.$$

Duality

Lemma 19 *The conjugate of φ can be expressed in terms of the support function $\sigma_{\mathcal{C}}(y) = \sup_{c\in\mathcal{C}}\langle c, y\rangle$ of $\mathcal{C}$ as*

$$\varphi^*(y) = E\sum_{t=0}^{T} v_t^*(y_t) + \sigma_{\mathcal{C}}(y).$$

Theorem 20 *If φ is lower semicontinuous, we have*

$$\varphi(c) = \sup_{y\in\mathcal{M}^\infty}\left\{\langle c, y\rangle - \sigma_{\mathcal{C}}(y) - E\sum_{t=0}^{T} v_t^*(y_t)\right\}.$$

In particular, when $\mathcal{C}$ is a cone,

$$\varphi(c) = \sup_{y\in\mathcal{C}^*}\left\{\langle c, y\rangle - E\sum_{t=0}^{T} v_t^*(y_t)\right\},$$

where $\mathcal{C}^* := \{y \in \mathcal{M}^\infty \mid \langle c, y\rangle \le 0\ \forall c \in \mathcal{C}\cap\mathcal{M}^1\}$ is the polar cone of $\mathcal{C}$.

Duality

Lemma 21 *If $S_t(x,\cdot)$ are integrable, then for $y \in \mathcal{M}^\infty_+$,*

$$\sigma_{\mathcal{C}}(y) = \inf_{v\in\mathcal{N}^1}\left\{\sum_{t=0}^{T} E(y_t S_t)^*(v_t) + \sum_{t=0}^{T-1} E\sigma_{D_t}(E[\Delta v_{t+1}|\mathcal{F}_t])\right\},$$

while $\sigma_{\mathcal{C}^1}(y) = +\infty$ for $y \notin \mathcal{M}^\infty_+$. The infimum is attained.

Example 22 *If $S_t(\omega,x) = s_t(\omega)\cdot x$ and $D_t(\omega)$ is a cone,*

$$\mathcal{C}^* = \{y \in \mathcal{M}^\infty \,|\, E[\Delta(y_{t+1}s_{t+1})\,|\mathcal{F}_t] \in D_t^*\}.$$

Example 23 *If $S_t(\omega,x) = \sup\{s\cdot x \,|\, s \in [s_t^b(\omega), s_t^a(\omega)]\}$ and $D_t(\omega) = \mathbb{R}^J$, then*

$$\mathcal{C}^* = \{y \in \mathcal{M}^\infty \,|\, ys \text{ is a martingale for some } s \in [s^b, s^a]\}.$$

Example 24 *In the classical model, $\mathcal{C}^*$ consists of positive multiples of martingale densities.*

Duality

Theorem 25 *Let* $\bar{c} \in \mathcal{M}^1$, $\mathcal{A}(\bar{c}) = \{c \,|\, \varphi(\bar{c}+c) \leq \varphi(\bar{c})\}$ *and assume that* φ *is proper and lower semicontinuous. Then*

1. $\sup_{\alpha>0} \varphi(\alpha p) > \varphi(0)$,
2. $\pi(\bar{c}, p; 0) > -\infty$,
3. $\pi(\bar{c}, p; c) > -\infty$ *for all* $c \in \mathcal{M}$,
4. $\langle p, y \rangle = 1$ *for some* $y \in \operatorname{dom} \sigma_{\mathcal{A}(\bar{c})}$

are equivalent and imply that

$$\pi(\bar{c}, p; c) = \sup_{y \in \mathcal{M}^\infty} \left\{ \langle c, y \rangle - \sigma_{\mathcal{A}(\bar{c})}(y) \,\middle|\, \langle p, y \rangle = 1 \right\}.$$

Moreover, if $\inf \varphi < \varphi(\bar{c})$, *then*

$$\sigma_{\mathcal{A}(\bar{c})} = \sigma_{\mathcal{B}(\bar{c})} + \sigma_{\mathcal{C}},$$

where $\mathcal{B}(\bar{c}) = \{c \in \mathcal{M}^1 \,|\, \mathcal{V}(\bar{c}+c) \leq \varphi(\bar{c})\}$.

Duality

Example 26 *In the classical model, with* $p = (1, 0, \dots, 0)$ *and* $v_t = \delta_{\mathbb{R}_-}$ *for* $t < T$*, we get*

$$\begin{aligned}
\pi(\bar{c}, p; c) &= \sup_{y \in \mathcal{M}^\infty} \left\{ \langle c, y \rangle - \sigma_{\mathcal{A}(\bar{c})}(y) \,\middle|\, \langle p, y \rangle = 1 \right\} \\
&= \sup_{Q \in \mathcal{Q}} \left\{ E^Q \sum_{t=0}^{T} (\bar{c}_t + c_t) - \sigma_{\mathcal{B}(\bar{c})} \left(E_t \frac{dQ}{dP} \right) \right\} \\
&= \sup_{Q \in \mathcal{Q}} \sup_{\alpha > 0} E^Q \left\{ \sum_{t=0}^{T} (\bar{c}_t + c_t) - \alpha \left[v_T^* (\frac{dQ}{dP} / \alpha) - \varphi(\bar{c}) \right] \right\}
\end{aligned}$$

where $\mathcal{Q}$ is the set of absolutely continuous martingale measures; see [Biagini, Frittelli, Grasselli, 2011] for a continuous time version.

Summary

- Financial contracts often involve sequences of cash-flows.
- The adequacy of swap rates/prices is subjective (views, risk preferences, the current financial position).
- Much of classical asset pricing theory can be extended to convex models of illiquid markets.
- In the absence of numeraire, martingale measures have to be replaced by more general dual variables that capture uncertainty as well as time value of money.